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SPECTRUMS OF WEIGHTED LEFT REGULAR ISOMETRIES OF A STRONGLY PERFORATED SEMIGROUP

S.Y. Jang*, B. J. Kim, T. W. Lee, Y. J. Kang and S.H. Jeon

Abstract. We compute spectrums of left regular isometries and weighted left regular isometries of a strongly perforated semigroup $P = \{0, 2, 3, 4, \cdots\}$.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, A a bounded linear operator on $\mathcal{H}$, and $C^*(A)$ denote the C^*-algebra generated by A and the identity operator I. The operator A is called GCR, or postliminal, if $C^*(A)$ is a GCR C^*-algebra. Recall that a C^*-algebra $\mathcal{A}$ is called CCR, or liminal, if for every irreducible representation π of $\mathcal{A}$ on a Hilbert space and for every $A \in \mathcal{A}$, $\pi(A)$ is compact [4, 4.2.1]. A C^*-algebra $\mathcal{A}$ is called GCR if $\mathcal{A}$ has an increasing family of closed two-sided ideals $(\mathcal{I}_\rho)_{0\le\rho\le\alpha}$ satisfying $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_\alpha = \mathcal{A}$, if $\rho \le \alpha$ is a limit ordinal, then $\mathcal{I}_\rho$ is the uniform closure of $\cup\{\mathcal{I}'_\rho : \rho' < \rho\}$ and $\mathcal{I}_{\rho+1}/\mathcal{I}_\rho$ is CCR [4, 4.3.4]. Equivalently, $\mathcal{A}$ is GCR if every irreducible representation of $\mathcal{A}$ contains a nonzero compact operator [16, 4.6.4]. It has been known that this is equivalent to requiring that for every representation π on a Hilbert space, $\pi(\mathcal{A})$ generates a type I W^*-algebra [16].

The basic examples of CCR algebras are commutative C^*-algebra, the algebras M_n of $n \times n$ complex matrices, and the algebra $\mathcal{K}(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$. Also C^*-subalgebras of GCR algebras are GCR [4, 4.3.5].

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*Corresponding author.

An operator A on a Hilbert space $\mathcal{H}$ is called n-normal [14] if

$$\sum sgn(\sigma)A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(2n)}=0$$

where $A_1, A_2, \cdots, A_{2n}$ are arbitrary elements of the C^*-algebra generated by A, and the summation is taken over all permutations σ of $(1, 2, \cdots, 2n)$. It is clear that if A is n-normal, then every operator in $C^*(A)$ is also n-normal. Every n-normal operator A can be written as the direct sum of $\{A_k\}_{k=1}^{n}$ where each A_k is a $k\times k$ operator-valued matrix whose entries belong to a commutative C^*-algebra [14]. Thus n-normal operators are CCR, since every irreducible representation is of dimension less than or equal to n. Let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators in $\mathcal{B}(\mathcal{H})$, and let ϕ denote the canonical homomorphism for $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. We call an operator A essentially n-normal if $\phi(A)$ is n-normal, or equivalent if

$$\sum sgn(\sigma)A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(2n)}\in\mathcal{K}(\mathcal{H})$$

where $A_1, A_2, \cdots, A_{2n}$ are arbitrary elements of the C^*-algebra generated by A. We remark that essentially n-normal operators are GCR since $C^*(A)/(C^*(A)\cap\mathcal{K}(\mathcal{H}))$ is an n-normal algebra and hence CCR. A large class of essentially n-normal operators are those operators which can be written as direct sum of $k\times k$ operator-valued matrices ($k\le n$) with entries in a C^*-algebra $\mathcal{A}_k$ such that $\phi(\mathcal{A}_k)$ is commutative. As an important example we mention those $n\times n$ operator-valued matrices whose entries are Toeplitz operators with continuous symbol [6].

Now fix a separable Hilbert space $l^2(\mathbb{N})$ and an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $l^2(\mathbb{N})$. A bounded linear operator S on $l^2(\mathbb{N})$ is called a *weighted shift* with weights $\{\alpha_n\}_{n=1}^{\infty}\in l^{\infty}(\mathbb{N})$ if

$$S(e_n)=\alpha_{n+1}e_{n+1}$$

for $n=0,1,2,\cdots$. Since the weighted shift with weights $\{|\alpha_n|\}_{n=1}^{\infty}$, we assume that $\alpha_n\ge 0$. When $\alpha_n=1$ for all n, we obtain the unilateral shift U defined by $U(e_n)=e_{n+1}$. Notice that U is a pure isometry that is essentially normal hence GCR. In fact, it is known that $\mathcal{K}(l^2(\mathbb{N}))\subset C^*(U)$ and $C^*(U)/\mathcal{K}(l^2(\mathbb{N}))$ is *-isomorphic to $C(\mathbb{T})$, the continuous functions on the unit circle $\mathbb{T}$. If S is any weighted shift, then $S=UD$ where D is the diagonal operator, $D=Diag(\alpha_1,\alpha_2,\alpha_3,\cdots)$, defined by $De_n=\alpha_{n+1}e_n$. Since we assume that $\alpha_n\ge 0$, we have that $D=(S^*S)^{1/2}\in C^*(S)$, and that $S=UD$ is the polar decomposition of S if $\alpha_n>0$ for all n. A weighted shift S with weights $\{\alpha_n\}_{n=1}^{\infty}$ is called *periodic* if there exists an integer p such that $\alpha_n = \alpha_{n+p}$ for all n. In this case S is said to be periodic of period p. It is known that a weighted shift with weights $\{\alpha_n\}$ such that $\alpha_n - \beta_n \to 0$ as $n \to \infty$ for some periodic sequence $\{\beta_n\}$ is GCR.

In this paper we will study shifts on a Hilbert space $l^2(P)$ when $P = \{0, 2, 3, \cdots\}$. Though the natural number semigroup $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ is a totally and well ordered semigroup, the semigroup $P = \{0, 2, 3, \cdots\}$ is partially ordered semigroup and strongly perforated semigroup. That is, the order structure of P is very different from $\mathbb{N}$. Hence the shift on $l^2(P)$ acts differently from the shift on $l^2(\mathbb{N})$.

We define an isometric representation of a semigroup M and also the left regular isometry $\mathcal{L}_x$ as a generalized shift on a Hilbert space for each $x \in M$. We compute spectrums of left regular isometries and weighted left regular isometries on $l^2(P)$. And also we show that the operator $\mathcal{L}_3\mathcal{L}_2^*$ can be perturbed as a GCR element for $2, 3 \in P$.

2. Isometric representation

Let M denote a semigroup with unit e and let B be a unital C^*−algebra with unit I_B. We call a map $W : M \to B,\ x \to W_x$ an isometric homomorphism if each W_x is an isometry and $W_{xy} = W_x W_y$ for all $x, y \in M$ and $W_e = 1_B$. If $B = B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, $(\mathcal{H}, W)$ is called an isometric representation of M.

If M is left cancellative , we can have a specific isometric representation. Let $\mathcal{H}$ be a non-zero Hilbert space and $l^2(M, \mathcal{H})$ denote the Hilbert space of all norm-square summable maps f from M to $\mathcal{H}$, i.e.,

$$l^2(M, \mathcal{H}) = \{f \mid f : M \to \mathcal{H},\ \Sigma_{x\in M} \mid f(x) \mid^2 < \infty\}.$$

The norm and scalar product on $l^2(M, \mathcal{H})$ are given as follows;

$$\| f \|^2 = \Sigma_{x\in M} \mid f(x) \mid^2 < \infty$$

$$< f, g >= \Sigma_{x\in M} < f(x), g(x) > .$$

For each $x \in M$, we define a map $\mathcal{L}_x$ on $l^2(M, \mathcal{H})$ by the equation ;

$$(\mathcal{L}_x f)(z) = \begin{cases} f(y), & \text{if } z = xy \text{ for some } y \in M \\ 0, & \text{if } z \notin xM. \end{cases}$$

By the definition of adjoint operator $(\mathcal{L}_x^* f)(z) = f(zx)$ for $x, z \in M$, we have

$$(\mathcal{L}_x^*\mathcal{L}_x f)(z) = \begin{cases} \mathcal{L}_x^* f(y), & \text{if } z = xy \text{ for some } y \in M \\ 0, & \text{if } z \notin xM, \end{cases}$$

so $\mathcal{L}_x^*\mathcal{L}_x$ is the identity operator on $l^2(M, H)$. And

$$(\mathcal{L}_x\mathcal{L}_x^* f)(z) = f(zx) = \begin{cases} \mathcal{L}_x f(y), & \text{if } z = xy \text{ for some } y \in M \\ 0, & \text{if } z \notin xM. \end{cases}$$

Thus $\mathcal{L}_x\mathcal{L}_x^*$ is the orthogonal projection onto the subspace generated by $\{z \in M \mid z \in xM\}$, so $\mathcal{L}_x$ is a non-unitary isometry on $l^2(M, \mathcal{H})$ when $x \neq e \in M$.

Since $\mathcal{L}_x$ is an isometry and $\mathcal{L}_x\mathcal{L}_y = \mathcal{L}_{xy}$ for each $x, y \in M$, the map $\mathcal{L} : M \to B(l^2(M, \mathcal{H}))$, $x \to \mathcal{L}_x$ is an isometric representation. The map $\mathcal{L}$ is called a *left regular isometric representation*.

If the Hilbert space $\mathcal{H}$ is the complex field $\mathbb{C}$, then we have $l^2(M, \mathcal{H}) = l^2(M)$. In this case we can see more explicitly how $\mathcal{L}_x$ acts for each $x \in M$. Let $\{\delta_x \mid x \in M\}$ be the orthonormal basis of $l^2(M)$ defined by

$$\delta_x(y) = \begin{cases} 1, & x = y \\ 0, & otherwise. \end{cases}$$

$\mathcal{L}_x$ acts like a shift and translates the elements of the orthonormal basis of $l^2(M, \mathcal{H})$ as follows:

$$(\mathcal{L}_x(\delta_y))(z) = \begin{cases} \delta_y(t), & z = xt \\ 0, & otherwise \end{cases} = \begin{cases} 1, & z = xy \\ 0, & otherwise. \end{cases}$$

Hence $\mathcal{L}_x(\delta_y) = \delta_{xy}$ for each $x, y \in M$. Clearly $\mathcal{L}_x^*\mathcal{L}_x$ is the identity because $\mathcal{L}_x^*\mathcal{L}_x(\delta_y) = \mathcal{L}_x^*(\delta_{xy}) = \delta_y$ for all $y \in M$. Furthermore, since

$$(\mathcal{L}_x\mathcal{L}_x^*(\delta_y)) = \begin{cases} \mathcal{L}_x(\delta_y), & y = xz \\ 0, & otherwise \end{cases} = \begin{cases} \delta_{xz} = \delta_y, & z = xy \\ 0, & otherwise, \end{cases}$$

$\mathcal{L}_x\mathcal{L}_x^*$ is the projection onto the sub-Hilbert space generated by $\{y \in M \mid y \in xM\}$.

If $M = \mathbb{N}$, the semigroup of natural numbers, $\mathcal{L}_1$ is the unilateral shift on $l^2(\mathbb{N})$ with respect to the orthonormal basis $\{\delta_n \mid n \in \mathbb{N}\}$ because $\mathcal{L}_1(\delta_n) = \delta_{n+1}$ for all $n \in \mathbb{N}$. This is why the left regular isometry can be called as a generalized shifts.

Assume that $x \in M$ is invertible in M. Then we have $(\mathcal{L}_x f)(z) = f(x^{-1}z)$ for $x, z \in M$, $\mathcal{L}_x$ is an unitary.

The C^*-algebra generated by $\{\mathcal{L}_x \mid x \in M\}$ is denoted by $C^*_{red}(\mathcal{L}_M)$ [7]. The C^*-algebras generated by isometries is one of the interesting area of C^*-algebras [2, 3, 9,10,11,12, 13]

Let $\alpha = (\alpha_x)_{x\in M}$, and $\alpha \in l^\infty(M)$. Let D_α be the diagonal operator acting as follows:

$$D_\alpha(\delta_x) = \alpha_x \delta_x$$

for all $x \in M$. Let $W^\alpha_x = \mathcal{L}_x D_\alpha$ be the *weighted left regular isometry* for each $x \in M$

3. Left regular isometric representation of $P = \{0, 2, 3, \cdots\}$

Let M be a countable discrete semigroup. We can give an order on M as follows: if an element x in M is contained in yM for some element $y \in M$, then x and y are comparable and we denote this by $y \leq x$. This relation makes M a pre-ordered semigroup.

If M is abelian, M can be equipped with the algebraic order $y \leq x$ if and only if $x = y + z$ for some $z \in M$. An element $x \in M$ is called positive if $y \leq y + x$ for all $y \in M$, and M is positive if all elements in M are positive. If M has a zero element 0, then M is positive if and only if $0 \leq x$ for all $x \in M$.

A positive ordered abelian semigroup W is said to be *almost unperforated* if for all $x, y \in M$ and all $n, m \in M$, with $nx \leq my$ and $n > m$, one has $x \leq y$. A partially ordered abelian group G with the positive cone M is said to be *almost unperforated* if the statement that $x \in G$ and $n \in \mathbb{N}$ with $nx, (n+1)x \in M$ implies that $x \in M$. It is known that G is almost unperforated if and only if the positive semigroup M is almost unperforated for a partially ordered abelian group (G, M) [17].

If the condition that $n \in \mathbb{N}$ and $x \in G$ with $nx \in M$ implies that $x \in M$, then the partially ordered abelian group (G, M) is *weakly unperforated.* Any weakly unperforated group is almost unperforated, but the converse is not true. The negation of almost unperforated property is *strongly perforated.*

The semigroup $P = \{0, 2, 3, 4, 5, 6, \cdots\}$ is strongly perforated. In [8] we show that the reduced semigroup C^*-algebra $C^*_{red}(P)$ of the semigroup P is isomorphic to the classical Toeplitz algebra and $C^*_{red}(P)$ is not isomorphic to the semigroup C^*-algebra $C^*(P)$.

Let $A \in B(H)$ be any bounded linear operator on a Hilbert space H. We write $\sigma(A)$, $\sigma_p(A)$, and $\sigma_{ap}(A)$ for the spectrum, point spectrum, and approximate point spectrum of A.

THEOREM 3.1. *Let $\mathcal{L}_2$ be a left regular isometry on $l^2(P)$ and $\mathcal{L}_2^*$ be the adjoint operator of $\mathcal{L}_2$. Then we have the following results on the spectrums:*

1. $\sigma(\mathcal{L}_2) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$;
2. $\sigma_{ap}(\mathcal{L}_2) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$;
3. $\sigma_p(\mathcal{L}_2) = \emptyset$;
4. $\sigma(\mathcal{L}_2^*) = \sigma_{ap}(\mathcal{L}_2^*) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$;
5. $\sigma_p(\mathcal{L}_2^*) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$.

Proof. We consider a non-zero element $\mathbf{x}_\lambda = (x_n)_{n\in P}$ such that $\mathcal{L}_2\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$. Then we have

$$\mathcal{L}_2(\mathbf{x}_\lambda) = (0, x_0, 0, x_2, x_3, x_4, \cdots) = (\lambda x_0, \lambda x_2, \lambda x_3, \lambda x_4 \cdots).$$

So we have $0 = \lambda x_0,\ x_0 = \lambda x_2,\ 0 = \lambda x_3, \cdots$. If $0 \neq \lambda$, $x_0 = x_2 = x_3 = \cdots = 0$. This contradicts to the fact $\mathbf{x}_\lambda$ is non-zero vector. Since $\mathcal{L}_2$ is isometry, $Ker\mathcal{L}_2 = \{\mathbf{0}\}$. Hence $\lambda = 0 \notin \sigma_p(\mathcal{L}_2)$. Therefore $\sigma_p(\mathcal{L}_2) = \emptyset$.

Next, we will show that $\sigma_p(\mathcal{L}_2^*) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$. We assume that $|\lambda| < 1$. We consider a non-zero element $\mathbf{x}_\lambda = (x_n)$ such that $\mathcal{L}_2^*\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$. Then we have

$$\mathcal{L}_2^*(\mathbf{x}_\lambda) = (x_2, x_4, x_5, x_6, x_7 \cdots) = (\lambda x_0, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5 \cdots).$$

So we have $x_2 = \lambda x_0,\ x_4 = \lambda x_2 = \lambda^2 x_0,\ x_5 = \lambda x_3, x_6 = \lambda x_4 = \lambda^3 x_0, x_7 = \lambda x_5 = \lambda^2 x_3 \cdots$. Since $|\lambda| < 1$,

$$\mathbf{x}_\lambda = (x_n) = (x_0, \lambda x_0, x_3, \lambda^2 x_0, \lambda x_3, \lambda^3 x_0, \lambda^2 x_3, \cdots) \in l^2(P).$$

Hence $\mathbf{x}_\lambda = (x_n) = (x_0, \lambda x_0, x_3, \lambda^2 x_0, \lambda x_3, \lambda^3 x_0, \lambda^2 x_3, \cdots)$ is an eigenvector of $\mathcal{L}_2^*$. So we have that $\sigma_p(\mathcal{L}_2^*) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$.

Since $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\} = \sigma_p(\mathcal{L}_2^*) \subset \sigma_{ap}(\mathcal{L}_2^*) \subset \sigma(\mathcal{L}_2^*)$, $||\mathcal{L}_2|| = 1$, and both $\sigma_{ap}(\mathcal{L}_2^*)$ and $\sigma(\mathcal{L}_2^*)$ are closed, $\sigma(\mathcal{L}_2^*) = \sigma_{ap}(\mathcal{L}_2^*) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. And furthermore since $\sigma(\mathcal{L}_2^*) = \sigma(\mathcal{L}_2)^* = \{\bar{\lambda} \mid \lambda \in \sigma(\mathcal{L}_2^*)\}$, we have $\sigma(\mathcal{L}_2) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.

Assume that $|\lambda| < 1$.

$$||(\mathcal{L}_2 - \lambda I)(\mathbf{x})|| \geq ||\mathcal{L}_2(\mathbf{x})|| - |\lambda|||\mathbf{x}|| \geq (1 - |\lambda|)||\mathbf{x}|| > 0$$

for any $\mathbf{x} \in l^2(P)$. So if $|\lambda| < 1$, then $\lambda \notin \sigma_{ap}(\mathcal{L}_2)$. Therefore $\sigma_{ap}(\mathcal{L}_2) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. □

Corollary 3.2. *Let $\mathcal{L}_3$ be a left regular isometry on $l^2(P)$ and $\mathcal{L}_3^*$ be the adjoint operator of $\mathcal{L}_3$. Then we have the following results on the spectrums:*

1. $\sigma(\mathcal{L}_3) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$;
2. $\sigma_{ap}(\mathcal{L}_3) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$;
3. $\sigma_p(\mathcal{L}_3) = \emptyset$;
4. $\sigma(\mathcal{L}_3^*) = \sigma_{ap}(\mathcal{L}_3^*) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$;
5. $\sigma_p(\mathcal{L}_3^*) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$.

Proof. First, we will show that $\sigma_p(\mathcal{L}_3^*) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$. We assume that $|\lambda| < 1$. We consider a non-zero elemenet $\mathbf{x}_\lambda = (x_n)$ such that $\mathcal{L}_3^*\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$. Then we have

$$\mathcal{L}_3^*(\mathbf{x}_\lambda) = (x_3, x_5, x_6, x_7, x_8, \cdots) = (\lambda x_0, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5, \cdots).$$

So we have $x_3 = \lambda x_0,\ x_5 = \lambda x_2,\ x_6 = \lambda^2 x_0, x_7 = \lambda x_4, x_8 = \lambda x_5 = \lambda^2 x_2 \cdots$. Since $|\lambda| < 1$, $(x_0, x_2, \lambda x_0, x_4, \lambda x_2, \lambda^2 x_0, \lambda x_4, \lambda^2 x_2, \cdots) \in l^2(P)$. Hence $\mathbf{x}_\lambda = (x_n) = (x_0, x_2, \lambda x_0, x_4, \lambda x_2, \lambda^2 x_0, \lambda x_4, \lambda^2 x_2, \cdots)$ is an eigenvector of $\mathcal{L}_3^*$. So we have that $\sigma_p(\mathcal{L}_3^*) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$.

Since $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\} = \sigma_p(\mathcal{L}_3^*) \subset \sigma_{ap}(\mathcal{L}_3^*) \subset \sigma(\mathcal{L}_3^*)$, $||\mathcal{L}_2|| = 1$, and both $\sigma_{ap}(\mathcal{L}_3^*)$ and $\sigma(\mathcal{L}_3^*)$ are closed, $\sigma(\mathcal{L}_3^*) = \sigma_{ap}(\mathcal{L}_3^*) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. And furthermore since $\sigma(\mathcal{L}_3^*) = \sigma(\mathcal{L}_3)^*$, we have $\sigma(\mathcal{L}_3) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.

We consider a non-zero elemenet $\mathbf{x}_\lambda = (x_n)$ such that $\mathcal{L}_3\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$. Then we have

$$\mathcal{L}_3(\mathbf{x}_\lambda) = (0, 0, x_0, 0, x_2, x_3, x_4, \cdots) = (\lambda x_0, \lambda x_2, \lambda x_3, \lambda x_4 \cdots).$$

If $0 \neq \lambda$, $x_0 = x_2 = x_3 = \cdots = 0$. This contradicts to the fact $\mathbf{x}_\lambda \neq \mathbf{0}$. Since $ker\mathcal{L}_3 = \{\mathbf{0}\}$, $0 \notin \sigma_p(\mathcal{L}_3)$ So, we have that $\sigma_p(\mathcal{L}_3) = \emptyset$. By the similar computation of the proof of the Theorem 3.1 we will see that if $|\lambda| < 1$, then $\lambda \notin \sigma_{ap}(\mathcal{L}_2)$. Therefore $\sigma_{ap}(\mathcal{L}_2) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. □

Theorem 3.3. *Suppose that $0 < |\alpha_0| \leq |\alpha_1| \leq |\alpha_2| \leq \cdots$ and $r = sup|\alpha_n| < \infty$. Let $W_2^\alpha = \mathcal{L}_2 M_\alpha$ be a weighted left regular isometry on $l^2(P)$ and $W_2^{\alpha^*}$ be the adjoint operator of W_2^α. Then we have the following results on the spectrums:*

1. $\sigma(W_2^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$;
2. $\sigma_{ap}(W_2^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$;
3. $\sigma_p(W_2^\alpha) = \emptyset$;
4. $\sigma(W_2^{\alpha^*}) = \sigma_{ap}(W_2^*) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$;
5. $\sigma_p(W_2^{\alpha^*}) = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$.

Proof. First, we will show that $\sigma_p(W_2^{\alpha *}) = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$. We assume that $|\lambda| < r$. We consider a non-zero elemenet $\mathbf{x}_\lambda = (x_n)$ such that $W_2^{\alpha *}\mathbf{x}_\lambda = \lambda \mathbf{x}_\lambda$. Then we have

$$W_2^\alpha(\mathbf{x}_\lambda) = (\alpha_0 x_2, \alpha_2 x_4, \alpha_3 x_5, \alpha_4 x_6, \cdots) = (\lambda x_0, \lambda x_2, \lambda x_3, \lambda x_4 \cdots).$$

So

$$\mathbf{x}_\lambda = (x_0, \frac{\lambda}{\alpha_0}x_0, x_3, \ \frac{\lambda^2}{\alpha_0\alpha_2}x_0, \ \frac{\lambda}{\alpha_3}x_3, \ \frac{\lambda^3}{\alpha_0\alpha_2\alpha_4}x_0, \ \frac{\lambda^2}{\alpha_3\alpha_5}x_3, \ \cdots).$$

Since $|\lambda| < r = sup\alpha_n$, we can see that $\mathbf{x}_\lambda = (x_n) \in l^2(P)$. Hence $\sigma_p(W_2^{\alpha^*}) = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$. Since $\{\lambda \in \mathbb{C} \mid |\lambda| < r\} = \sigma_p(W_2^{\alpha^*}) \subset \sigma_{ap}(W_2^{\alpha^*}) \subset \sigma(W_2^{\alpha^*})$, and both $\sigma_{ap}(W_2^{\alpha^*})$ and $\sigma(W_2^{\alpha^*})$ are closed, $\sigma(W_2^{\alpha^*}) = \sigma_{ap}(W_2^{\alpha^*}) = \{\lambda \in \mathbb{C} \mid |\lambda| \le r\}$. And furthermore since $\sigma(W_2^{\alpha^*}) = \sigma(W_2)^*$, we have $\sigma(W_2^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| \le r\}$.

Suppose that $\mathbf{x} = (x_0, x_1, x_2, \cdots) \in l^2(P)$ and $\lambda \neq 0$. Assume that $W_2\mathbf{x} = \lambda\mathbf{x}$. Then

$$(\lambda x_0, \lambda x_2, \lambda x_3, \cdots) = (0, \alpha_0 x_0, 0, 0, \alpha_2 x_2, \alpha_3 x_3 \cdots).$$

So $0 = \lambda x_0$, $x_0 = \lambda x_2$, $\cdots$. Hence $0 = x_0 = x_2 = x_3 = \cdots$. Therefore $\lambda \notin \sigma_p(W_2^\alpha)$. Since W_2^α is isometry, $KerW_2^\alpha = \{\mathbf{0}\}$. Hence $\lambda = 0 \notin \sigma_p(W_2^\alpha)$. Hence $\sigma_p(W_2^\alpha) = \emptyset$.

Suppose that $|\lambda| < r$. Then there exists a real number q such that $|\lambda| < q < r$. Since $r = sup|\alpha_n| < \infty$, there exists a integer number n_0 such that $\alpha_n \ge q$ for all $n \ge n_0$. Thus for any $\mathbf{x} \in l^2(P)$

$$\begin{aligned}&||W_2^\alpha(\mathbf{x})||\\ &= |\alpha_0|^2|x_0|^2 + |\alpha_2|^2|x_2|^2 + \cdots + |\alpha_{n_0}|^2|x_{n_0}|^2 + |\alpha_{n_0+1}|^2|x_{n_0+1}|^2 + \cdots\\ &> |\alpha_0|^2|x_0|^2 + |\alpha_2|^2|x_2|^2 + \cdots + |q|^2|x_{n_0}|^2 + |q|^2|x_{n_0+1}|^2 + \cdots .\end{aligned}$$

Hence we can say that $||W_2^\alpha(\mathbf{x})|| \ge \mathbf{q}||\mathbf{x}||$ essentially. Thus $||(W_2^\alpha - \lambda I)(\mathbf{x}_n)||$ dose not converge to 0 for any sequence $\{\mathbf{x}_n\}$ with $||\mathbf{x}_n|| = 1$. So if $|\lambda| < r$, then $\lambda \notin \sigma_{ap}(W_2^\alpha)$. Therefore $\sigma_{ap}(W_2^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$. □

Corollary 3.4. *Suppose that $\alpha = (\alpha_0, \alpha_2, \cdots)$ and $0 < |\alpha_0| \le |\alpha_2| \le |\alpha_3| \le \cdots$, and $r = sup|\alpha_n| < \infty$. Let $W_3^\alpha = \mathcal{L}_3 M_\alpha$ be a weighted left regular isometry on $l^2(P)$ and $W_3^{\alpha^*}$ be the adjoint operator of W_3^α on $l^2(P)$. Then we have the following results on the spectrums:*

1. *$\sigma(W_3^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| \le r\}$;*
2. *$\sigma_{ap}(W_3^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$;*

3. $\sigma_p(W_3^\alpha) = \emptyset$;
4. $\sigma(W_3^{\alpha^*}) = \sigma_{ap}(W_3^{\alpha^*}) = \{\lambda \in \mathbb{C} \mid |\lambda| \le r\}$;
5. $\sigma_p(W_3^{\alpha^*}) = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$.

Proof. We assume that $|\lambda| < r$. We consider a non-zero elemenet $\mathbf{x}_\lambda = (x_n)$ such that $W_3^{\alpha^*}\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$. Then we have

$$W_3^\alpha(\mathbf{x}_\lambda) = (\alpha_0 x_3, \alpha_2 x_5, \alpha_3 x_6, \alpha_4 x_7, \cdots) = (\lambda x_0, \lambda x_2, \lambda x_3, \lambda x_4 \cdots \}.$$

So we get $\lambda x_0 = \alpha_0 x_3$, $\lambda x_2 = \alpha_2 x_5$, $\lambda x_3 = \alpha_3 x_6$, $\lambda x_4 = \alpha_4 x_7 \cdots$ and

$$\begin{aligned}&\mathbf{x}_\lambda \\ &= (x_0, x_2, \frac{\lambda}{\alpha_0}x_0,\ \frac{\lambda}{\alpha_2}x_2,\ \frac{\lambda^2}{\alpha_0\alpha_3}x_0,\ \frac{\lambda}{\alpha_4}x_4,\ \frac{\lambda^2}{\alpha_2\alpha_5}x_2,\ \frac{\lambda^3}{\alpha_0\alpha_3\alpha_6}x_0,\ \frac{\lambda^2}{\alpha_4\alpha_7}x_4,\\ &\frac{\lambda^3}{\alpha_2\alpha_5\alpha_8}x_2,\ \cdots).\end{aligned}$$

Since $|\lambda| < r = \sup\, \alpha_n$, we can see that $\mathbf{x}_\lambda = (x_n) \in l^2(P)$. Hence $\sigma_p(W_3^{\alpha^*}) = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$. Since $\{\lambda \in \mathbb{C} \mid |\lambda| < r\} = \sigma_p(W_3^{\alpha^*}) \subset \sigma_{ap}(W_3^{\alpha^*}) \subset \sigma(W_3^{\alpha^*})$ and $\sigma_{ap}(W_3^{\alpha^*})$, $\sigma(W_3^{\alpha^*})$ are closed, $\sigma(W_3^{\alpha^*}) = \sigma_{ap}(W_3^{\alpha^*}) = \{\lambda \in \mathbb{C} \mid |\lambda| \le r\}$. And furthermore since $\sigma(W_3^{\alpha^*}) = \sigma(W_3^\alpha)^*$, we have $\sigma(W_3^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| \le r\}$.

Suppose that $\mathbf{x} = (x_0, x_2, x_3, \cdots) \in l^2(P)$ and $\lambda \ne 0$. Assume that $W_3^\alpha\mathbf{x} = \lambda\mathbf{x}$. Then

$$(\lambda x_0, \lambda x_2, \lambda x_3, \cdots) = (0, 0, \alpha_0 x_0, 0, \alpha_2 x_2, \alpha_3 x_3 \cdots).$$

So $0 = \lambda x_0$, $0 = \lambda x_2$, $\cdots$. Hence $0 = x_0 = x_2 = x_3 = \cdots$. Since $\mathbf{x}_\lambda$ is not zero-vector, $\lambda \notin \sigma_p(W_3^\alpha)$. Furthermore the fact $KerW_3^\alpha = \{\mathbf{0}\}$ implies that $\lambda = 0 \notin \sigma_p(W_3^\alpha)$. Hence $\sigma_p(W_3^\alpha) = \emptyset$.

Suppose that $|\lambda| < r$. Then there exists a real number q such that $|\lambda| < q < r$. Since $r = sup|\alpha_n| < \infty$, there exists a integer number n_0 such that $\alpha_n \ge q$ for all $n \ge n_0$. Thus for any $\mathbf{x} \in l^2(P)$

$$\begin{aligned}&||W_2^\alpha(\mathbf{x})|| \\ &= |\alpha_0|^2|x_0|^2 + |\alpha_2|^2|x_2|^2 + \cdots + |\alpha_{n_0}|^2|x_{n_0}|^2 + |\alpha_{n_0+1}|^2|x_{n_0+1}|^2 + \cdots \\ &> |\alpha_0|^2|x_0|^2 + |\alpha_2|^2|x_2|^2 + \cdots + |q|^2|x_{n_0}|^2 + |q|^2|x_{n_0+1}|^2 + \cdots .\end{aligned}$$

Hence we can say that $||W_3^\alpha(\mathbf{x})|| \ge q||\mathbf{x}||$ essentially. Thus $||(W_3^\alpha - \lambda I)(\mathbf{x}_n)||$ does not converge to 0 for any sequence $\{\mathbf{x}_n\}$ with $||\mathbf{x}_n|| = 1$. So if $|\lambda| < r$, then $\lambda \notin \sigma_{ap}(W_3^\alpha)$. Therefore $\sigma_{ap}(W_3^\alpha) = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$. $\square$

The semigroup $P = \{0, 2, 3, \cdots\}$ is generated by 2 and 3 . Next we are going to consider a C^*-algebra $C^*(\mathcal{L}_P)$, which is generated by $\{\mathcal{L}_x \mid x \in P\}$. Then the C^*-algebra $C^*(\mathcal{L}_P)$ is generated by $\mathcal{L}_2$ and $\mathcal{L}_3$ because every element in P is generated by 2 and 3. For any $n, m \in P$,

$$\mathcal{L}_n(\delta_m) = \delta_{n+m}$$

where $\{\delta_m \mid m \in P\}$ is the canonical orthonormal basis of $l^2(M)$ defined by

$$\delta_m(l) = \begin{cases} 1, & \text{if } m = l, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{L}^*{}_n(\delta_m) = \begin{cases} \delta_{m-n}, & \text{if } m \in P + n, \\ 0, & \text{otherwise.} \end{cases}$$

If we consider an operator $\mathcal{L}_3\mathcal{L}_2^*$, then we have

$$\mathcal{L}_3\mathcal{L}^*{}_2(\delta_m) = \begin{cases} \delta_{m+1}, & \text{if } m \in P + 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mathcal{L}_3\mathcal{L}^*{}_2(\delta_0) = 0$ and $\mathcal{L}_3\mathcal{L}^*{}_2(\delta_m) = \delta_{m+1}$ for all $m \in P - \{0\}$. That is, $\mathcal{L}_3\mathcal{L}^*{}_2$ acts like a unilateral shift on the Hilbert $l^2(P)$ except δ_0 with respect to the canonical basis $\{\delta_m \mid m \in P\}$. To exclude the gap of δ_0 we consider a rank one operator K_0 defined by

$$K_0(\delta_n) = \begin{cases} \delta_2, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{L}_3\mathcal{L}^*{}_2 + K_0)(\delta_m) = \delta_{m+1}$ for all $m \in P$. Put $\mathcal{S} = \mathcal{L}_3\mathcal{L}^*{}_2 + K_0$. Then $\mathcal{S}$ acts like a unilateral shift on the Hilbert $l^2(P)$ with respect to the canonical basis $\{\delta_m \mid m \in P\}$.

And $\mathcal{L}_2\mathcal{L}_2^*$ is an orthogonal projection on the sub-Hilbert generated by $\{\delta_2, \delta_4, \delta_5 \cdots\}$, so $I - \mathcal{L}_2\mathcal{L}_2^*$ is the orthogonal projection on the subspace generated by $\{\delta_0, \delta_2\}$. We denote $\mathcal{L}_n\mathcal{L}_n^*$ and $I - \mathcal{L}_n\mathcal{L}_n^*$ by P_n and Q_n, respectively. Since by [8] the C^*-algebra $C^*(\mathcal{L}_2, \mathcal{L}_3)$ acts irreducibly on $l^2(P)$ and Q_2 is the compact operator of rank two, the compact operator algebra $\mathcal{K}(l^2(H))$ is contained in the C^*-algebra $C^*(\mathcal{L}_3, \mathcal{L}_2)$ Hence $K_0 \in C^*(\mathcal{L}_3, \mathcal{L}_2)$ and $\mathcal{S} \in C^*(\mathcal{L}_3, \mathcal{L}_2)$. And we can see that $\mathcal{L}_2$ and $\mathcal{L}_3$ can be made by $\mathcal{L}_3\mathcal{L}_2^*$ and some compact operators[8].

THEOREM 3.5. *[8] The C^*-algebra $C^*(\mathcal{L}_P)$ is generated by $\mathcal{S}$.*

Since Coburn proved [1] that the C^*-algebra generated by single non-unitary isometry is isomorphic to the Toeplitz algebra, the C^*-algebra $C^*(\mathcal{L}_3, \mathcal{L}_2) = C^*(\mathcal{L}_P)$ is isomorphic to the Toeplitz algebra.

THEOREM 3.6. *The operator $\mathcal{S}$ in $\mathcal{B}(l^2(P))$ is GCR.*

Proof. Since $C^*(\mathcal{S}) = C^*(\mathcal{L}_3, \mathcal{L}_2^*) = C^*(\mathcal{L}_P)$ is isomorphic to the Toeplitz algebra, $\mathcal{S}$ is GCR. □

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Department of Mathematics
University of Ulsan
Ulsan 680-749, Republic of Korea
E-mail: jsym@uou.ulsan.ac.kr

Byeong Jun Kim,
Ulsan Science High School

Tae Woo Lee,
Ulsan Science High School

Yeong Joon Kang,
Ulsan Science High School

Seong Hoon Jeon,
Ulsan Science High School