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ISOMETRIC IMMERSIONS OF FINSLER MANIFOLDS

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Abstract. For an isometric immersion $f : M \to \overline{M}$ of Finsler manifolds M into $\overline{M}$, we compare the intrinsic Chern connection on M and the induced connection on M. We find the conditions for them to coincide and generalize the equations of Gauss, Ricci and Codazzi to Finsler submanifolds. In case the ambient space is a locally Minkowskian Finsler manifold, we simplify the above equations.

1. Introduction

The equations of Gauss and Codazzi for hypersurfaces of Finsler manifolds had been studied in the monographs by H. Rund [16] and M. Mastumoto [15]. Following the connection theories of Rund and Mastumoto, there have been simplifications and generalizations to submanifolds of arbitrary codimensions, e.g., in [17, 3, 14, 1].

The connection introduced by S. S. Chern in [7] did not appear in [16, 15] at all. Following the pull-back formalism in [12], Chern himself reinterpreted his connection in a modern fashion [8, 5]. In [4], M. Anastasiei established that the Chern connection coincided with the Rund connection. Now the equations in the modern language seem to be more tractable and more applicable in the pursuit of geometric consequences.

In this paper, we generalize the problem of isometric immersions of the Riemannian manifolds to the class of Finsler manifolds. For ambient spaces, we are eventually interested in the locally Minkowskian Finsler manifolds. A Finsler metric is locally Minkowskian if the metric

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depends only on the space variables. Such a metric has computational advantage because its Riemann curvature tensor and Chern curvature tensor vanish.

In §2, we set up the notations for Finsler manifolds and introduce the pull-back formalism. Under this setting, we can apply the usual techniques in Riemannian geometry to the class of Finsler manifolds. The Chern connection in [7, 8, 5] is the one we used in §3 and §4 to deduce the equations. This connection is torsion free but is not metric compatible. Because a Finsler manifold admitting a connection which is both torsion free and metric compatible must be Riemannian, a weaker assumptions on the connection is necessary. Though the Chern connection is not metric compatible, almost metric compatibility gives us enough information to carry out reasonable computations. To make the equations as simple as we can, throughout this paper, we use the Einstein summation convention: a repeated upper and lower index is summed over all possible values.

In §3, we define isometric immersions for Finsler manifolds. Given an isometric immersion, we produce various vector bundles by the usual operations on the vector bundles and end up with various connections. Once we have connections, we can naturally define the second fundamental form of the immersion.

In §4, we generalize the equations of Gauss and Ricci to Finsler submanifolds. And we further carry out simplification in case the ambient space is locally Minkowskian. The Riemann curvature tensor and the Chern curvature tensor of the locally Minkowskian Finsler manifold vanish so that it seems reasonable that the locally Minkowskian Finsler manifold plays the role of the Euclidean space in the problem of the isometric embeddings of Riemannian manifolds.

2. Preliminaries

2.1. Notations. Let M be a n-dimensional differentiable manifold with a local coordinate system $(x^1, \cdots, x^n)$. Then we have a canonical local coordinate system $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ of the tangent bundle TM of M induced by $(x^1, \cdots, x^n)$.

Definition 2.1. A Finsler metric F on the manifold M is a function $F : TM \to \mathbb{R}$ satisfying

(F1) Regularity: F is smooth away from the zero section of TM ,
(F2) Nonnegativity: $F(x,y) \geq 0$ and $F(x,y) = 0$ if and only if $y = 0$,
(F3) Absolute homogeneity: $F(x, \lambda y) = |\lambda| F(x,y)$ for all $\lambda \in \mathbb{R}$,
(F4) Strongly convexity: $\left[\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right]$ is positive definite.

A Finsler manifold (M, F) is a differentiable manifold with a Finsler metric F on M. Thus by definition, we have a Banach norm on each tangent space $T_x M$ of M.

In order to apply well-developed techniques of Riemannian geometry, we need to recall the pull-back formalism of S. Kobayashi[12, 13]: let $\tilde{\pi} : p^*TM \to \widetilde{TM}$ be the pull-back bundle of the tangent bundle $\pi : TM \to M$ by the projection $p : \widetilde{TM} \to M$. Here $\widetilde{TM} = TM \backslash \{$zero section of $\pi : TM \to M\}$ is the slit tangent bundle of M.

$$\begin{array}{ccc} p^*TM & \xrightarrow{\tilde{p}} & TM \\ {\scriptstyle \tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \widetilde{TM} & \xrightarrow[p]{} & M \end{array}$$

A section of $\tilde{\pi} : p^*TM \to \widetilde{TM}$ is called a Finsler vector field. And the set of all sections of this bundle is denoted by $\Gamma(p^*TM)$. Let

$$g_{ij} = \tfrac{1}{2} \partial F^2 / \partial y^i \partial y^j .$$

on the projectivised slit tangent bundle $P\widetilde{TM}$ of M . The strong convexity condition (F4) on F induces a Riemannian structure g_{ij} on $\tilde{\pi} : p^*TM \to \widetilde{TM}$. So we can define the fundamental tensor g of F : for $U = U^i(x,y) \frac{\partial}{\partial x^i}|_{(x,y)}, V = V^i(x,y) \frac{\partial}{\partial x^i}|_{(x,y)}$,

$$g(U, V) = g_{ij} U^i V^j .$$

Now the Cartan coefficients C_{ijk} of a Finsler metric F is

$$C_{ijk} = \frac{1}{4} \frac{\partial^3 F}{\partial y^i \partial y^j \partial y^k}$$

and the Cartan tensor is

$$C(U, V, W) = C_{ijk} U^i V^j W^k ,$$

where U, V, W are as above. It is known that a Finsler metric F is a priori a Riemannian metric if and only if the Cartan coefficients vanish. In this paper, we are only interested in the class of proper Finsler metrics.

Now consider the geodesic coefficient G^i and non-linear connection N^i_j defined by

$$G^i = \frac{1}{4} g^{il} \left\{ 2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k ,$$

$$N^i_j = \frac{\partial G^i}{\partial y^j} .$$

Let $\mathcal{V}$ be the vertical subspace of $T\widetilde{TM}$ with the basis $\left\{ \frac{\partial}{\partial y^i} \right\}_{1=1}^n$. Then the non-linear connection N^i_j defines the decomposition $T\widetilde{TM} = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal subspace of $T\widetilde{TM}$ with the basis $\left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} \right\}_{i=1}^n$. These subspaces $\mathcal{H}$ and $\mathcal{V}$ can be identified with p^*TM by the isomorphisms $\mathfrak{h} : p^*TM \to \mathcal{H}$ and $\mathfrak{v} : p^*TM \to \mathcal{V}$ defined by $\mathfrak{h}\left(\frac{\partial}{\partial x^i}\right) = \frac{\delta}{\delta x^i}$ and $\mathfrak{v}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$. If we let

$$\delta y^i = dy^i + N^i_j dx^j,$$

then $\{dx^i, \delta y^i\}$ is a coframe of $\widetilde{TM}$ dual to $\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}$.

2.2. Chern connection. Given a local frame $\{e_i\}$ of p^*TM, we can define a coframe $\{\omega^i, \omega^{n+i}\}$ of $\widetilde{TM}$ dual to $\{\mathfrak{h}(e_i), \mathfrak{v}(e_i)\}$. And we let

$$\begin{aligned} \mathfrak{g}_{ij} &= g(e_i, e_j) , \\ \mathfrak{C}_{ijk} &= C(e_i, e_j, e_k) . \end{aligned}$$

Then there exists a unique set $\{\omega^j_i\}$ of local 1-forms on $\widetilde{TM}$ satisfying

(TF) $\quad d\omega^i = \omega^j \wedge \omega^i_j ,$

(AMC) $\quad d\mathfrak{g}_{ij} = \mathfrak{g}_{kj}\omega^k_i + \mathfrak{g}_{ik}\omega^k_j + 2\mathfrak{C}_{ijk}\omega^{n+k} .$

This set $\{\omega^j_i\}$ of local 1-forms defines a connection on the pull-back bundle $\tilde{\pi} : p^*TM \to \widetilde{TM}$. It is called the Chern connection.

3. Isometric immersions

3.1. Isometries. Let (M, F) be a Finsler manifold of dimension n. We will consider an ambient manifold $\overline{M}$ of dimension $m = n + r$ with a Finsler metric $\overline{F}$. For the sake of notational simplicity, we use bar(or overline) to denote the corresponding quantities for $\overline{M}$. For example, $(\bar{x}, \bar{y})$ is a coordinate on $\overline{M}$ and $\bar{g}_{\alpha\beta} = \frac{1}{2}\partial\overline{F}^2/\partial\bar{y}^\alpha\partial\bar{y}^\beta$. And we use the following conventions on the indices: the Roman indices $i, j, k, \cdots$ vary from 1 to n and the Greek indices $\alpha, \beta, \gamma, \cdots$ vary from 1 to $n+r$ and $\mu, \nu, \cdots$ vary from $n+1$ to $n+r$.

A differentiable immersion $f : (M, F) \to (\overline{M}, \overline{F})$ is isometric if

$$F(y) = \overline{F}(f_* y) \text{ for all } y \in TM . \tag{3.1}$$

Note that (3.1) is equivalent to

$$g_{ij}(y) = \bar{g}_{\alpha\beta}(f_* y)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} .$$

Therefore, we have

Proposition 3.1. *$f_* : p^*TM \to p^*T\overline{M}$ preserves the fundamental tensors and the Cartan tensors:*

$$g = f^*\bar{g} , \quad C = f^*\overline{C} .$$

Now fix orthonormal frame $\{e_i\}$ for p^*TM and let $\bar{e}_i = f_*(e_i)$. Then by Proposition 3.1, we have $\mathfrak{g}_{ij} = \bar{\mathfrak{g}}_{ij}$ and $\mathfrak{C}_{ijk} = \overline{\mathfrak{C}}_{ijk}$. Thus $\{\bar{e}_i\}$ is orthonormal in $p^*T\overline{M}$. And we can find an orthonormal basis

$$\{\bar{e}_1, \cdots, \bar{e}_n, \bar{e}_{n+1} \cdots, \bar{e}_{n+r}\}$$

for $p^*T\overline{M}$.

We now have a coframe $\{\omega^i, \omega^{n+i}\}$ of $\widetilde{TM}$ dual to $\{\mathfrak{h}(e_i), \mathfrak{v}(e_i)\}$ and a coframe $\{\overline{\omega}^\alpha, \overline{\omega}^{m+\alpha}\}$ of $\widetilde{T\overline{M}}$ dual to $\{\bar{\mathfrak{h}}(\bar{e}_\alpha), \bar{\mathfrak{v}}(\bar{e}_\alpha)\}$. As in §2.2, we have connection forms $\{\omega_i^j\}$ of $\widetilde{TM}$ and $\{\overline{\omega}_\alpha^\beta\}$ of $\widetilde{T\overline{M}}$, with respect to the coframes $\{\omega^i, \omega^{n+i}\}$ and $\{\overline{\omega}^\alpha, \overline{\omega}^{m+\alpha}\}$, respectively.

Next we consider the pull-backs of $\overline{\omega}^\alpha$, $\overline{\omega}^{m+\alpha}$, $\overline{\omega}_\beta^\alpha$ by $f_* : \widetilde{TM} \to \widetilde{T\overline{M}}$ and denote them by the same symbols.

Proposition 3.2. *Comparing these pull-backs with the ones we previously defined on $\widetilde{TM}$, we have*

(P1) $$\overline{\omega}^i = \omega^i \,,$$

(P2) $$\overline{\omega}^{m+i} = \omega^{n+i} + V^i_j \omega^j \,,$$

(P3) $$\overline{\omega}^i_j - \omega^i_j = a^i_{jk}\omega^k \,, \quad a^i_{jk} = a^i_{kj} \,,$$

(P4) $$\overline{\omega}^\mu_j = h^\mu_{jk}\omega^k \,, \quad h^\mu_{jk} = h^\mu_{kj} \,,$$

where $e_i = A^j_i \frac{\partial}{\partial x^j}$ *and* $\bar{e}_\alpha = B^\beta_\alpha \frac{\partial}{\partial \bar{x}^\beta}$ *and*

$$V^i_j = (B^{-1})^i_\alpha \frac{\partial^2 f^\alpha}{\partial y^k \partial y^l} y^k A^l_j + (B^{-1})^i_\alpha \overline{N}^\alpha_\beta C^\beta_j - (B^{-1})^i_\alpha \frac{\partial f^\alpha}{\partial x^k} N^k_l A^l_j \,.$$

Proof. By direct calculation with the coordinates, we obtain (P1) and (P2).

By differentiating $\overline{\omega}^i - \omega^i = 0$ and $\omega^\mu = 0$ and using the structure equation (TF) and the Cartan lemma, we have (P3) and (P4). □

3.2. Induced Connection. Via the Chern connection forms ω^i_j, we define a linear connection

$$\nabla : T(\widetilde{TM}) \times \Gamma(p^*TM) \to \Gamma(p^*TM)$$

by $\nabla_X U = \{dU^i(X) + U^j\omega^i_j(X)\} \otimes e_i$. Here $U = U^i e_i \in \Gamma(p^*TM)$ and $X \in T(\widetilde{TM})$.

We define the curvature of ∇ by

$$\Omega(X,Y)U = \nabla_X\nabla_Y U - \nabla_Y\nabla_X - \nabla_{[X,Y]}U$$

for a Finsler vector field U and tangent vector fields X, Y on $\widetilde{TM}$. If we let $\Omega(X,Y)e_j = \Omega^i_j(X,Y)e_i$, then we have $\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k$.

By differentiating the structure equation (TF), we get $\omega^j \wedge \Omega^i_j = 0$ and so

$$\Omega^i_j = \frac{1}{2} R^i_{jkl}\omega^k \wedge \omega^l + P^i_{jkl}\omega^k \wedge \omega^{n+l}.$$

The Riemann curvature tensor R and the Chern curvature tensor P are defined by:

$$R(X,Y)U = R^i_{jkl} X^k X^l U^j e_i,$$
$$P(X,Y)U = P^i_{jkl} X^k X^l U^j e_i$$

for a Finsler vector field U and tangent vector fields X, Y on $\widetilde{TM}$.

Similarly, $\overline{\nabla} : T(\widetilde{T\overline{M}}) \times \Gamma(p^*T\overline{M}) \to \Gamma(p^*T\overline{M})$ is defined in terms of $\overline{\omega}^\alpha_\beta$.

We can decompose $p^*T\overline{M}|_{f_*(\widetilde{TM})}$ into the tangential subbundle $f_*(p^*TM)$ and the normal subbundle $(p^*TM)^\perp$. The tangential subbundle $f_*(p^*TM)$ is identified with p^*TM.

Fix $\xi, \eta \in \Gamma(p^*TM)$ and let $\bar{\xi}, \bar{\eta}$ be local extensions of $f_*(\xi), f_*(\eta)$ to $p^*T\overline{M}$. Now decompose $\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{\eta}$ into the tangential component $(\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{\eta})^T$ and the normal component $B(\xi, \eta)$. Then we have an induced connection ∇' on p^*TM defined by

$$\nabla'_{\mathfrak{h}(\xi)}\eta = (\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{\eta})^T$$

with the connection forms $\overline{\omega}^i_j$. The induced connection ∇' does not coincide with the Chern connection ∇ except the case that $a^i_{jk} = 0$.

Proposition 3.3. *The induced connection ∇' is torsion free and satisfies*

$$\begin{aligned} \nabla'\mathfrak{g}_{ij} &= 2\mathfrak{C}_{ijk}\omega^{n+k} - (\mathfrak{g}_{il}a^l_{jk} + \mathfrak{g}_{jl}a^l_{ik})\omega^k \\ &= 2\mathfrak{C}_{ijk}\omega^{n+k} + 2\mathfrak{C}_{ijl}V^l_k\omega^k. \end{aligned}$$

In particular, $\mathfrak{g}_{il}a^l_{jk} + \mathfrak{g}_{jl}a^l_{ik} = -2\mathfrak{C}_{ijl}V^l_k$ *for all* i, j, k .

Proof. Because the connection ∇ satisfies

$$d\mathfrak{g}_{ij} = \langle \nabla e_i, e_j\rangle + \langle e_i, \nabla e_j\rangle + 2\mathfrak{C}_{ijk}\omega^{n+k},$$

we have

$$\begin{aligned} \nabla'\mathfrak{g}_{ij} &= d\mathfrak{g}_{ij} - \langle \nabla' e_i, e_j\rangle + \langle e_i, \nabla' e_j\rangle \\ &= 2\mathfrak{C}_{ijk}\omega^{n+k} - \langle \nabla' e_i - \nabla e_i, e_j\rangle - \langle e_i, \nabla' e_j - \nabla e_j\rangle \\ &= 2\mathfrak{C}_{ijk}\omega^{n+k} - \langle a^l_{ik}\omega^k e_l, e_j\rangle - \langle e_i, a^l_{jk}\omega^k e_l\rangle \\ &= 2\mathfrak{C}_{ijk}\omega^{n+k} - (\mathfrak{g}_{lj}a^l_{ik} + \mathfrak{g}_{il}a^l_{jk})\omega^k. \end{aligned}$$

On the other hand, we can derive $\nabla'\mathfrak{g}_{ij} = 2\overline{\mathfrak{C}}_{ijk}\bar{\omega}^{n+k}$ from $\overline{\nabla}\bar{\mathfrak{g}}_{ij}$. Thus

$$2\mathfrak{C}_{ijk}\omega^{n+k} - (\mathfrak{g}_{ik}a^k_{jl} + \mathfrak{g}_{jk}a^k_{il})\omega^l = 2\overline{\mathfrak{C}}_{ijk}\bar{\omega}^{n+k} = 2\overline{\mathfrak{C}}_{ijk}\omega^{n+k} + 2\overline{\mathfrak{C}}_{ijk}V^k_l\omega^l.$$

Since $\{\omega^k, \omega^{n+k}\}$ is linearly independent, $\mathfrak{C}_{ijk} = \overline{\mathfrak{C}}_{ijk}$ and $\mathfrak{g}_{ik}a^k_{jl} + \mathfrak{g}_{jk}a^k_{il} = -2\overline{\mathfrak{C}}_{ijk}V^k_l$.

□

Now we have

THEOREM 3.4. *The followings are equivalent:*

1. $\nabla' = \nabla$.
2. $\mathfrak{g}_{il}a^l_{jk} + \mathfrak{g}_{jl}a^l_{ik} = 0$ *for all* $i, j, k = 1, \cdots, n$.
3. $\mathfrak{C}_{ijl}V^l_k = 0$ *for all* $i, j, k = 1, \cdots, n$.

Proof. If a connection satisfies (TF) and (AMC), then the connection must coincide with the Chern connection by the uniqueness of the Chern connection in §2.2. And so if the induced connection satisfies $\nabla' \mathfrak{g}_{ij} = 2\mathfrak{C}_{ijk}\omega^{n+k}$, then $\nabla' = \nabla$. The converse is also true. Thus Proposition 3.3 completes the proof. □

3.3. Second Fundamental Form. Note that the tangential component $(\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{\eta})^T$ of $\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{\eta}$ is $\nabla'_{\mathfrak{h}(\xi)}\eta$. The normal component

$$B(\xi, \eta) = \overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{\eta} - \nabla'_{\mathfrak{h}(\xi)}\eta \tag{3.2}$$

is called the second fundamental form of the immersion $f : M \to \overline{M}$.

Recall that $B(e_i, e_j) = \bar{\omega}^\mu_j(\bar{\mathfrak{h}}(\bar{e}_i))\bar{e}_\mu = h^\mu_{ji}\bar{e}_\mu$ and that $h^\mu_{ij} = h^\mu_{ji}$. Thus we readily have

PROPOSITION 3.5. *The mapping* $B : \Gamma(p^*TM) \times \Gamma(p^*TM) \to \Gamma((p^*TM)^\perp)$ *defined by* (3.2) *is bilinear and symmetric.*

Observe that a self-adjoint operator $S_\eta : p^*TM \to p^*TM$ is associated to the symmetric bilinear mapping B by

$$\langle S_\eta(\xi_1), \xi_2\rangle = \langle B(\xi_1, \xi_2), \eta\rangle .$$

Even though $\overline{\nabla}$ is not metric compatible, we have

$$\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{g}_{ij} = 2\overline{\mathfrak{C}}_{ij\alpha}\bar{\omega}^{m+\alpha}(\bar{\mathfrak{h}}(\bar{\xi})) = 0 . \tag{3.3}$$

This is crucial in applying the techniques in Riemannian geometry to the Finsler setting. Because of (3.3), we have the following analogue of the shape operator in Riemannian geometry.

PROPOSITION 3.6. *Let* $(x, y) \in \widetilde{TM}, \xi \in p^*TM|_y$ *and* $\eta \in (p^*TM)^\perp|_{f_*y}$. *Choose a local extension* N *of* η *which is normal to* $f_*(p^*TM)$. *Then*

$$S_\eta(\xi) = -(\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}N)^T.$$

4. The equations

Here we will consider the horizontal covariant derivative of normal Finsler vector fields. For a Finsler vector field ξ on p^*TM and a normal Finsler vector field $\bar{\eta}$ on $f_*(\widetilde{TM})$, we define the normal connection $\nabla^\perp$ of the immersion $f : M \to \overline{M}$ by

$$\nabla^\perp_\xi \bar{\eta} = (\overline{\nabla}_{\bar{\mathfrak{h}}(\bar{\xi})}\bar{\eta})^\perp.$$

And the normal curvature $\Omega^\perp$ of the immersion is defined by

$$\Omega^\perp(\xi_1, \xi_2)\bar{\eta} = \nabla^\perp_{\xi_1}\nabla^\perp_{\xi_2}\bar{\eta} - \nabla^\perp_{\xi_2}\nabla^\perp_{\xi_1}\bar{\eta} - \nabla^\perp_{[\xi_1,\xi_2]}\bar{\eta}\,.$$

Now we are ready to generalize the equations of Gauss, Ricci and Codazzi in Riemannian geometry to Finsler manifolds.

4.1. Equations of Gauss and Ricci.

THEOREM 4.1 (Equations of Gauss). *Let ∇ be the Chern connection on M and ∇' the induced connection on M and $\overline{\nabla}$ the Chern connection on $\overline{M}$. We have following relations between the Riemann curvature tensors and the Chern curvature tensors of ∇, ∇' and $\overline{\nabla}$:*

$$\text{(G1)} \qquad \overline{R}^{\,i}_{jkl} = R'^{i}_{jkl} + h^\mu_{jk}h^i_{\mu l} - h^\mu_{jl}h^i_{\mu k},$$

$$\text{(G2)} \qquad \begin{aligned} R'^{i}_{jkl} = {} & R^i_{jkl} + a^i_{jp}\Gamma^p_{kl} + a^i_{pk}\Gamma^p_{jl} - a^p_{jk}\Gamma^i_{pl} - a^i_{jk,l} - a^p_{jk}a^i_{pl} \\ & - (a^i_{jp}\Gamma^p_{lk} + a^i_{pl}\Gamma^p_{jk} - a^p_{jl}\Gamma^i_{pk} - a^i_{jl,k} - a^p_{jl}a^i_{pk}), \end{aligned}$$

$$\text{(G3)} \qquad \overline{P}^{\,i}_{jkl} = P'^{i}_{jkl} - h^\mu_{jk}\overline{\mathfrak{C}}^{\,i}_{\mu l},$$

$$\text{(G4)} \qquad P'^{i}_{jkl} = P^i_{jkl} + a^i_{jp}C^p_{kl} + a^i_{pk}C^p_{jl} - a^p_{jk}C^i_{pl} - a^i_{jk|l},$$

where $\omega^i_j = \Gamma^i_{jk}\omega^k + C^i_{jk}\omega^{n+k}$, $da^i_{jk} = a^i_{jk,l}\omega^l + a^i_{jk|l}\omega^{n+l}$ *and* $h^i_{\mu j} = h^\mu_{ij}$.

Proof. Note that

$$\text{(4.1)} \qquad \overline{\Omega}^{\,i}_j = d\bar{\omega}^i_j - \bar{\omega}^k_j \wedge \bar{\omega}^i_k - \bar{\omega}^\mu_j \wedge \bar{\omega}^i_\mu = \Omega'^i_j - \bar{\omega}^\mu_j \wedge \bar{\omega}^i_\mu.$$

Here $\Omega'^i_j = d\bar{\omega}^i_j - \bar{\omega}^k_j \wedge \bar{\omega}^i_k$ is the curvature of the induced connection ∇'.

Evaluating (4.1) at $(\bar{\mathfrak{h}}(\bar{e}_k), \bar{\mathfrak{h}}(\bar{e}_l))$, we have

$$\overline{R}^{\,i}_{jkl} = R'^i_{jkl} - \bar{\omega}^\mu_j \wedge \bar{\omega}^i_\mu(\bar{\mathfrak{h}}(\bar{e}_k), \bar{\mathfrak{h}}(\bar{e}_l))$$

and

$$\bar{\omega}^\mu_j \wedge \bar{\omega}^i_\mu(\bar{\mathfrak{h}}(\bar{e}_k), \bar{\mathfrak{h}}(\bar{e}_l)) = h^\mu_{jk}(-h^\mu_{il}) - h^\mu_{jl}(-h^\mu_{ik}) = -h^\mu_{jk}h^i_{\mu l} + h^\mu_{jl}h^i_{\mu k}\,.$$

Plugging $\bar{\omega}^i_j = \omega^i_j + a^i_{jk}\omega^k$ into ${\Omega'}^i_j$, we have

$$(4.2) \qquad {\Omega'}^i_j = \Omega^i_j + \omega^k \wedge (a^i_{jp}\omega^p_k + a^i_{pk}\omega^p_j - a^p_{jk}\omega^i_p - da^i_{jk} - a^p_{jk}a^i_{pq}\omega^q)\,.$$

Evaluating (4.1) at $(\bar{\mathfrak{h}}(\bar{e}_k), \bar{\mathfrak{v}}(\bar{e}_l))$, we have (G3). And evaluating (4.2) at $(\mathfrak{h}(e_k), \mathfrak{v}(e_l))$, we have (G4). □

Theorem 4.2 (Equations of Ricci). *Let ∇ be the Chern connection on M and ∇' the induced connection on M and $\overline{\nabla}$ the Chern connection on $\overline{M}$. We have following relations between the Riemann curvature tensors and the Chern curvature tensors of $\overline{\nabla}$ and $\nabla^{\perp}$:*

$$\text{(R1)} \qquad \overline{R}^{\mu}_{\nu ij} = {R^{\perp}}^{\mu}_{\nu ij} + h^k_{\nu i}h^{\mu}_{kj} - h^k_{\nu j}h^{\mu}_{ki}\,,$$

$$\text{(R2)} \qquad \overline{P}^{\mu}_{\nu ij} = {P^{\perp}}^{\mu}_{\nu ij} + h^{\mu}_{ki}\overline{\mathfrak{C}}^k_{\nu j},$$

where $h^k_{\mu j} = h^{\mu}_{kj}$ and $\overline{\mathfrak{C}}^k_{\mu j} = \overline{\mathfrak{C}}_{k\mu j}$.

Proof. Note that

$$(4.3) \qquad \overline{\Omega}^{\mu}_{\nu} = {\Omega^{\perp}}^{\mu}_{\nu} - \bar{\omega}^k_{\nu} \wedge \bar{\omega}^{\mu}_k\,.$$

Now evaluating (4.3) at $(\bar{\mathfrak{h}}(\bar{e}_i), \bar{\mathfrak{h}}(\bar{e}_j))$, we have (R1). Then evaluating (4.3) at $(\mathfrak{h}(e_i), \mathfrak{v}(e_j))$, we have (R2).

□

4.2. Locally Minkowskian Finsler manifolds.

Definition 4.1. A Finsler manifold (M, F) is called locally Minkowskian if for each $p \in M$, there exists a local coordinate system (x^i, y^i) on TM such that F is a function of (x^i) only.

Because the Riemann curvature tensor and the Chern curvature tensor of locally Minkowskian Finsler manifolds vanish, the equations of Gauss and Ricci of an isometric immersions into locally Minkowskian Finsler manifolds can be simplified. From Theorem 4.1 and 4.2, we have

Theorem 4.3 (Equations of Gauss and Ricci). *Let $f : (M, F) \to (\overline{M}, \overline{F})$ be an isometric immersion of a Finsler manifold (M, F) into a* *locally Minkowskian Finsler manifold* $(\overline{M},\overline{F})$. *Then we have*

$$\begin{aligned} R^i_{jkl} &= -\, h^\mu_{jk}h^i_{\mu l} + h^\mu_{jl}h^i_{\mu k} - (a^i_{jp}\Gamma^p_{kl} + a^i_{pk}\Gamma^p_{jl} - a^p_{jk}\Gamma^i_{pl} - a^i_{jk,l} - a^p_{jk}a^i_{pl}) \\ &\quad + (a^i_{jp}\Gamma^p_{lk} + a^i_{pl}\Gamma^p_{jk} - a^p_{jl}\Gamma^i_{pk} - a^i_{jl,k} - a^p_{jl}a^i_{pk}), \\ P^i_{jkl} &= +\, h^\mu_{jk}\overline{\mathfrak{C}}^i_{\mu l} - (a^i_{jp}C^p_{kl} + a^i_{pk}C^p_{jl} - a^p_{jk}C^i_{pl} - a^i_{jk|l}), \\ R^{\perp\mu}{}_{\nu ij} &= -\, h^k_{\nu i}h^\mu_{kj} + h^k_{\nu j}h^\mu_{ki} \\ P^{\perp\mu}{}_{\nu ij} &= -\, h^\mu_{ki}C^k_{\nu j}\,. \end{aligned}$$

Finally, we derive an analogue of the equations of Codazzi in Riemannian geometry. If we plug $\bar\omega^\mu_i = h^\mu_{ij}\omega^j$ into

$$\overline{\Omega}^\mu_i = d\bar\omega^\mu_i - \bar\omega^k_i \wedge \bar\omega^\mu_k - \bar\omega^\nu_i \wedge \bar\omega^\mu_\nu\,,$$

we have

$$\begin{aligned} \overline{\Omega}^\mu_i &= dh^\mu_{ij}\bar\omega^j + h^\mu_{ij}\bar\omega^k \wedge \bar\omega^j_k - \bar\omega^k_i \wedge h^\mu_{kj}\bar\omega^j - h^\nu_{ij}\bar\omega^j \wedge \bar\omega^\mu_\nu \\ &= (dh^\mu_{ij} - h^\mu_{ik}\bar\omega^k_j - h^\mu_{kj}\bar\omega^k_i + h^\nu_{ij}\bar\omega^\mu_\nu) \wedge \bar\omega^j\,. \end{aligned}$$

If $\overline{M}$ is a locally Minkowskian manifold, then $\overline{\Omega}^\mu_i = 0$. By the Cartan lemma,

$$dh^\mu_{ij} - h^\mu_{ik}\bar\omega^k_j - h^\mu_{kj}\bar\omega^k_i + h^\nu_{ij}\bar\omega^\mu_\nu = h^\mu_{ijk}\omega^k\,, \quad h^\mu_{ijk} = h^\mu_{ikj}\,.$$

If we put

$$\begin{aligned} dh^\mu_{ij} &= h^\mu_{ij,k}\omega^k + h^\nu_{ij|k}\omega^{n+k}\,, \\ \omega^i_j &= \Gamma^i_{jk}\omega^k + C^i_{jk}\omega^{n+k}\,, \\ \bar\omega^\mu_\nu &= \overline{\Gamma}^\mu_{\nu k}\bar\omega^k + \overline{C}^\mu_{\nu k}\bar\omega^{m+k}\,, \end{aligned}$$

we have

Theorem 4.4 (Equations of Codazzi).

$$\begin{aligned} & h^\mu_{ij,k} - h^\mu_{il}\Gamma^l_{jk} - h^\mu_{lj}\Gamma^l_{ik} + h^\nu_{ij}\overline{\Gamma}^\mu_{\nu k} - h^\mu_{il}a^l_{jk} - h^\mu_{lj}a^l_{ik} = h^\mu_{ijk}\,, \\ & h^\mu_{ij|k} - h^\mu_{il}C^l_{jk} - h^\mu_{lj}C^l_{ik} + h^\nu_{ij}\overline{C}^\mu_{\nu k} = 0\,, \\ & h^\mu_{ij,k} - h^\mu_{ik,j} = h^\mu_{il}\Gamma^l_{jk} + h^\mu_{lj}\Gamma^l_{ik} - h^\nu_{ij}\overline{\Gamma}^\mu_{\nu k} + h^\mu_{lj}a^l_{ik} \\ & \qquad\qquad - (h^\mu_{il}\Gamma^l_{kj} + h^\mu_{lk}\Gamma^l_{ij} - h^\nu_{ik}\overline{\Gamma}^\mu_{\nu j} + h^\mu_{lk}a^l_{ij})\,. \end{aligned}$$

Remark . In Riemannian geometry, the equations of Gauss, Ricci and Codazzi play an analogous role of the compatibility equations in the local theory of surfaces. Thus we may conjecture that these equations are the necessary and sufficient condition for Finsler manifolds to be isometrically immersed into a Minkowskian Finsler manifold. But this seems unlikely in that as we weaken the hypothesis on the metric, we need more equations.

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