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MULTIPLE SOLUTIONS FOR A SUSPENDING BEAM EQUATION AND THE GEOMETRY OF THE MAPPING

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Abstract. We investigate the multiple solutions for a suspending beam equation with jumping nonlinearity crossing three eigenvalues, with Dirichlet boundary condition and periodic condition. We show the existence of at least six nontrivial periodic solutions for the equation by using the finite dimensional reduction method and the geometry of the mapping.

1. Introduction

In this paper we investigate the multiplicity of the solutions of the nonlinear suspending beam equation with Dirichlet boundary condition and periodic condition

$$u_{tt} + u_{xxxx} + bu^+ - au^- = f(x,t) \qquad \text{in } [-\frac{\pi}{2}, \frac{\pi}{2}] \times R, \tag{1.1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{1.2}$$

$$u \text{ is } \pi - \text{periodic in } t \text{ and even in } x \text{ and } t, \tag{1.3}$$

where $u^+ = \max\{0, u\}$ and $u^- = -\min\{0, u\}$.

Micheletti and Saccon showed in [11] that there exists $\delta_k > 0$ such that if $f(x,t) = c > 0$ and $\Lambda_k^- - \delta_k < -b < \Lambda_k^-$, $k > 1$, then

$$u_{tt} + u_{xxxx} + bu^+ = c \qquad \text{in } [-\frac{\pi}{2}, \frac{\pi}{2}] \times R, \tag{1.4}$$

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$$u(\pm\frac{\pi}{2},t) = u_{xx}(\pm\frac{\pi}{2},t) = 0,$$
$$u \text{ is } \pi - \text{periodic in } t \text{ and even in } x \text{ and } t$$

has at least four nontrivial solutions, where $c > 0$ and Λ_k^- is a negative eigenvalue of eigenvalue problem

$$u_{tt} + u_{xxxx} = \lambda_{mn}u \tag{1.5}$$

with Dirichlet boundary condition (1.2) and periodic condition (1.3). They proved this result by the abstract result of the critical point theory.

In this paper we assume that

$$f(x,t) = s\phi_{00}.$$

That is, we investigate the multiplicity of the solutions of the equation

$$u_{tt} + u_{xxxx} + bu^+ - au^- = s\phi_{00} \qquad \text{in } [-\frac{\pi}{2},\frac{\pi}{2}] \times R, \tag{1.6}$$
$$u(\pm\frac{\pi}{2},t) = u_{xx}(\pm\frac{\pi}{2},t) = 0,$$
$$u \text{ is } \pi - \text{periodic in } t \text{ and even in } x \text{ and } t,$$

where ϕ_{00} is the eigenfunction corresponding to the eigenvalue $\lambda_{00} = 1$ of the ei genvalue problem (1.5) with (1.2) and (1.3).

The purpose of this paper is to find the number of the weak solutions of (1.6).

Choi and Jung proved in [2] that for $3 < b < 15$, there exist cones R_1, R_2', R_3, R_4' such that (i) if $s\phi_{00} \in \text{Int}R_1$, then (1.4) with $f(x,t) = s\phi_{00}$ has a positive solution and at least two sign-changing solutions, (ii) if $s\phi_{00} \in \partial R_1$, then (1.4) with $f(x,t) = s\phi_{00}$ has a positive solution and at least one sign-changing solution, (iii) if $s\phi_{00} \in \text{Int}R_i'$ $(i = 2,4)$, then (1.4) with $f(x,t) = s\phi_{00}$ has at least one sign-changing solution, (iv) if $s\phi_{00} \in \text{Int}R_3$. then (1.4) with $f(x,t) = s\phi_{00}$ has only the negative solution, (v) if $s\phi_{00} \in \partial R_3$, then (1.4) with $f(x,t) = s\phi_{00}$ has a negative solution. The authors obtain these results by the critical point theory and the finite dimensional reduction method.

The eigenvalue problem (1.5) with (1.2) and (1.3) has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m,n = 0,1,2,\ldots) \tag{1.7}$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m,n \geq 0)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi}\cos(2n+1)x \qquad \text{for } \ n \geq 0, \tag{1.8}$$

$$\phi_{mn} = \frac{2}{\pi}\cos 2mt\cos(2n+1)x \qquad \text{for} \;\; m > 0, n \geq 0. \tag{1.9}$$

The main results are the following:

Theorem 1.1. *Assume that* $-17 = \lambda_{41} < a < -1 = -\lambda_{00} < 3 = \lambda_{10} < 15 = -\lambda_{20} < b < -\lambda_{30} = 35$ *and* $s > 0$. *Then (1.6) with (1.2) and (1.3) has at least six nontrivial periodic solutions.*

Generally we have:

Theorem 1.2. *Assume that* $-17 = \lambda_{41} < a < -1 = \lambda_{00} < -\lambda_{k\,0} < b < -\lambda_{k+1\,0}$, $k > 1$, *then (1.6) with (1.2) and (1.3) has at least six nontrivial solutions.*

For the proof of Theorem 1.1 we use the finite dimensional reduction method and the geometry of the mapping defined on the finite dimensional subspace spanned by eigenfunction ϕ_{00}, ϕ_{10}, ϕ_{20}. In section 2 we introduce the Hilbert space defined by the eigenfunction expansion on the restricted region $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and some properties of the operator L, $Lu = u_{tt} + u_{xxxx}$. In section 3 we prove Theorem 1.1 and Theorem 1.2 by using the finite dimensional reduction method and the geometry of the mapping defined on the finite dimensional subspace.

2. Some results on the operator L, $LU = U_{tt} + U_{xxxx}$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) |\; u \text{ is even in } \; x \text{ and } \; t\}. \tag{2.1}$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal base in H_0. We define a subspace H of H_0 as follows

$$H = \{u \in H_0 |\; u = \sum h_{mn}\phi_{mn}, \;\; \sum |\lambda_{mn}| h_{mn}^2 < \infty\} \tag{2.2}$$

with a norm

$$\|u\| = [\sum |\lambda_{mn}| h_{mn}^2]^{\frac{1}{2}}. \tag{2.3}$$

Then this normed space H is complete.

We have a proposition which is proved in [5].

Proposition 2.1. *(i)* $u_{tt} + u_{xxxx} \in H$ *implies* $u \in H$.
(ii) $\|u\| \geq \|u\|_{L^2}$, *where* $\|u\|_{L^2}$ *denotes the* L^2 *norm of* u.
(iii) $\|u\| = 0$ *iff* $\|u\|_{L^2} = 0$.

Proposition 2.2. *Let $w(x,t) \in H_0$. Let a and b be not eigenvalues of (1.4) with (1.2) and (1.3). Then all solution in H_0 of*

$$u_{tt} + u_{xxxx} + bu^+ - au^- = w(x,t) \text{ in } H_0 \tag{2.4}$$

belong to H.

The proof of Proposition 2.2 can be proved by the same method as that of Lemma 1.2 in [5] which is the case that $u_{tt}+u_{xxxx}+\delta u^+ = w(x,t)$, δ is not an eigenvalue of L.

With the aid of Proposition 2.2 it is enough to investigate the existence of solutions of (1.6) in the subspace H of $L_2(Q)$, namely

$$u_{tt} + u_{xxxx} + bu^+ - au^- = s\phi_{00} \qquad \text{in } H, \tag{2.4}$$

3. Proof of Theorem 1.1 and Theorem 1.2

Let L be the operator,

$$Lu = u_{tt} + u_{xxxx}.$$

Assume that $-17 = \lambda_{41} < a < -1 = -\lambda_{00} < 3 = \lambda_{10} < 15 = -\lambda_{20} < b < -\lambda_{30} = 35$ and $s > 0$. We shall use the contraction mapping theorem to reduce the problem from an infinite dimensional one in $L_2(Q)$ to a finite dimensional one.

Let V be the three dimensional subspace of H spanned by ϕ_{00}, ϕ_{10} and ϕ_{20} and W the orthogonal complement of V in H. Let P be an orthogonal projection from H onto V and $I-P$ the one from H onto W. Then for all $u \in H$, $u = v + w$, where $v = Pu$, $w = (I-P)u$. Therefore (2.4) is equivalent to

(i) $\quad w = L^{-1}(I-P)(-b(v+w)^+ + a(v+w)^-)$,

(ii) $\quad Lv = P(-b(v+w)^+ + a(v+w)^- + s\phi_{00}). \qquad (3.1)$

Let us show that for fixed v, (3.1(i)) has a unique solution $w = \theta(v)$ and that $\theta(v)$ is Lipschitz continuous in terms of v. Let $\alpha = \frac{1}{2}(\lambda_{30} + \lambda_{41}) = \frac{-35+17}{2} = -9$ (3.1.(i)) can be rewritten as

$$w = (L-\alpha)^{-1}(I-P)g_v(w) \tag{3.2}$$

where

$$g_v(w) = -b(v+w)^+ + a(v+w)^- - \alpha(v+w).$$

Since

$$|g_v(w_1) - g_v(w_2)| \le \max\{|-b-\alpha|, |a+\alpha|\}|w_2 - w_1|,$$

$\|g_v(w_1) - g_v(w_2)\| \le \max\{|-b-\alpha|, |a+\alpha|\}\|w_2 - w_1\| < 26\|w_2 - w_1\|$, where $\|\cdot\|$ is the norm in H. Since the operator $(L-\alpha)^{-1}(I-P)$ is a self-adjoint, compact linear map from $(I-P)H$ onto itself, it follows that

$$\|(L-\alpha)^{-1}(I-P)\| \le \frac{1}{26}.$$

Therefore, for fixed $v \in V$, the right hand side of (3.2) defines a Lipschitz mapping from $(I-P)H$ into itself with Lipschitz constant $\gamma < 1$. Therefore by the contraction mapping principle, for given $v \in V$, there exists a unique $w = \theta(v) \in W$ which satisfies (3.2). It follows that, by the standard argument principle, $\theta(v)$ is Lipschitz continuous in terms of v.

Thus equation (2.4) can be reduced to the equivalent equation

$$v_{tt} + v_{xxxx} = P(-b(v+\theta(v))^+ + a(v+\theta(v))^- + s\phi_1) \qquad (3.3)$$

defined on the three dimensional subspace V spanned by $\{\phi_{00}, \phi_{10}, \phi_{20}\}$.

Let

$$\begin{aligned} C_1 = \{v = c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20} \ge 0|\ & c_0 \ge 0, |c_1| \le \epsilon_1 c_0 \text{ and} \\ & |c_2| \le \epsilon_2 c_0 \text{ for some } \ \epsilon_1 > 0, \epsilon_2 > 0 \text{ such that } \ v \ge 0\}, \\ C_2 = \{v = c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20} \le 0|\ & c_0 \le 0,\ |c_1| \le \epsilon_1 |c_0| \text{and} \\ & |c_2| \le \epsilon_2 |c_0| \text{ for some } \ \epsilon_1 > 0, \epsilon_2 > 0 \text{ such that } \ v \le 0\}, \end{aligned}$$

We note that if $v \ge 0$ or $v \le 0$, then $\theta(v) = 0$. If $v \ge 0$ ($v \le 0$) and $\theta(v) = 0$ in (3.1(i)), equation (3.1(i)) is satisfied. We note that $w = \theta(v) = 0$ for $v \in C_1 \cup C_2$, but we do not know $\theta(v)$ for all $v \in V$. We consider the map

$$v \mapsto G(v) = v_{tt} + v_{xxxx} + P(b(v+\theta(v))^+ - a(v+\theta(v))^-).$$

We will consider the images of C_1, C_2 and the complements of $C_1 \cup C_2$ under the map G. First we consider the image of the cone C_1. If $v = c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20} \in C_1$, then we have

$$\begin{aligned} G(v) &= c_0\phi_{00} - 3c_1\phi_{10} - 15c_2\phi_{20} + b(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}) \\ &= (b+1)c_0\phi_{00} + (b-3)c_1\phi_{10} + (b-15)c_2\phi_{20}. \end{aligned}$$

Thus the images of the ray $c_0\phi_{00} \pm \epsilon_1 c_0\phi_{10} \pm \epsilon_2 c_0\phi_{20}$ are

$$= (b+1)c_0\phi_{00} \pm (b-3)\epsilon_1 c_0\phi_{10} \pm (b-15)\epsilon_2 c_0\phi_{20}$$

or the rays

$$d_0\phi_{00} \pm \epsilon_1(\frac{b-3}{b+1})d_0\phi_{10} \pm \epsilon_2\frac{b-15}{b+1}d_0\phi_{20}.$$

Thus G maps C_1 into the region

$$D_1 = \{d_0\phi_{00}+d_1\phi_{10}+d_2\phi_{20} | d_0 \geq 0,\ |d_1| \leq \epsilon_1(\frac{b-3}{b+1})d_0,\ |d_2| \leq \epsilon_2\frac{b-15}{b+1}d_0\}.$$

Similarly we consider the image of C_2 by the map G. If $c_0 \leq 0$,

$$G(c_0\phi_{00} \pm \epsilon_1 c_0\phi_{10} \pm \epsilon_2 c_0\phi_{20})$$

$$= (a+1)c_0\phi_{00} \pm (a-3)c_0\epsilon_1\phi_{10} \pm (a-15)c_0\epsilon_2\phi_{20}.$$

Since $(b+1)c_0 > 0$ for $c_0 > 0$ and $(a+1)c_0 > 0$ for $c_0 < 0$, $G(v) = s\phi_{00}$ $(s > 0)$ has a positive solution $\frac{s\phi_{00}}{b+1}$ in C_1 and a negative solution $\frac{s\phi_{00}}{a+1}$ in C_2.

Now we shall find the other solutions in the complements of $C_1 \cup C_2$ of the map $G(v) = s\phi_{00}$, for $s > 0$. We have the following lemma:

LEMMA 3.1. *There exist* $p_1 > 0$ *and* $p_2 > 0$ *such that*
(i) $(G(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}), \phi_{00}) \geq p_1|c_1|$ *and*
(ii) $G(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}), \phi_{00}) \geq p_2|c_2|$.

Proof. (i) $G(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}) = L(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}) + P(b(c_0\phi_{00}+c_1\phi_{10}+c_2\phi_{20}+\theta(c_0\phi_{00}+c_1\phi_{10}+c_2\phi_{20}))^+ - a(c_0\phi_{00}+c_1\phi_{10}+c_2\phi_{20}+\theta(c_0\phi_{00}+c_1\phi_{10}+c_2\phi_{20}))^-)$. If $u = c_0\phi_{00}+c_1\phi_{10}+c_2\phi_{20})+\theta(c_0\phi_{00}+c_1\phi_{10}+c_2\phi_{20})$, then

$$(G(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}), \phi_{00}) = ((L-\lambda_{00})(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}), \phi_{00})$$

$$+P(bu^+ - au^- + \lambda_{00}u, \phi_{00}).$$

Since $(L-\lambda_{00})\phi_{00} = 0$ and L is self adjoint, $((L-\lambda_{00})(c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}), \phi_{00}) = 0$. Moreover $bu^+ - au^- + \lambda_{00}u = (b+1)u^+ - (a+1)u^- \geq \tau|u|$, where $\tau = \min\{b+1, -a-1\} > 0$. Thus $(bu^+ - au^- + \lambda_{00}u, \phi_{00}) \geq \tau\int|u|\phi_{00}$. Then there exists $p_1 > 0$ such that $\tau\phi_{00} \geq p_1|\phi_{10}|$, so that

$$\tau\int|u|\phi_{00} \geq p_1\int|u||\phi_{10}| \geq p_1|\int u\phi_{10}| = p_1|(u,\phi_{10})| = p_1|c_1|.$$

(ii) We also note that there exists $p_2 > 0$ such that $\tau\phi_{00} \geq p_2|\phi_{20}|$, so that

$$\tau\int|u|\phi_{00} \geq p_2\int|u||\phi_{20}| \geq p_2|\int u\phi_{20}| = p_2|(u,\phi_{20})| = p_2|c_2|.$$

□

Now we are looking for the preimages of $G(v) = s\phi_{00}$, for $s > 0$, in the complement of $C_1 \cup C_2$. Let us consider the image under G of $c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}$ with $c_1 \geq \epsilon_1|c_0|$, $c_2 \geq \epsilon_2|c_0|$, $c_1 = k$ for some $k > 0$. By Lemma 3.1, the image $G(L)$ of $c_1 = k$, $|c_0| \leq \frac{1}{\epsilon_1}k$ and $c_2 \geq \epsilon_2|c_0|$ must lie to the right of the line $c_0 = p_1k$ and must cross the positive ϕ_{00} axis in the image space. Thus we have shown that if $u = c_0\phi_{00} + k\phi_{10} + c_2\phi_{20} + \theta(c_0\phi_{00} + k\phi_{10} + c_2\phi_{20})$, $k > 0$, $|c_0| \leq \frac{k}{\epsilon_1}$. Then u satisfies, for some c_0, $u_{tt} + u_{xxxx} + bu^+ - au^- = s_1\phi_{00}$ for some $s_1 > p_1k$ and $k > 0$. Letting $\tilde{u} = \frac{s}{s_1}u$, we see that $\tilde{u}$ satisfies

$$(\tilde{u})_{tt} + (\tilde{u})_{xxxx} + b\tilde{u}^+ - a\tilde{u}^- = s\phi_{00}.$$

Similarly we can show the existence of another solution $\breve{u}$ satisfying

$$(\breve{u})_{tt} + (\breve{u})_{xxxx} + b\breve{u}^+ - a\breve{u}^- = s\phi_{00}$$

with $(\breve{u}, \phi_{00}) < 0$.

Now we consider the image under G of $c_0\phi_{00} + c_1\phi_{10} + c_2\phi_{20}$ with $c_1 \geq \epsilon_1|c_0|$, $c_2 \geq \epsilon_2|c_0|$, and $c_2 = l$ for some $l > 0$. By Lemma 3.1, the image $G(M)$ of $c_2 = l$, $|c_0| \leq \frac{1}{\epsilon_2}l$ must lie to the right of the line $c_0 = p_2l$ and must cross the positive ϕ_{00} axis in the image space. Thus we have shown that if $u = c_0\phi_{00} + c_1\phi_{10} + l\phi_{20} + \theta(c_0\phi_{00} + c_1\phi_{10} + l\phi_{20})$, $l > 0$, $c_1 \geq \epsilon_1|c_0|$, $l \geq \epsilon_2|c_0|$, then u satisfies, for some c_0, $u_{tt} + u_{xxxx} + bu^+ - au^- = s_2\phi_{00}$ for some $s_2 > p_2l$ and $l > 0$. Letting $\tilde{\tilde{u}} = \frac{s}{s_2}u$, we see that $\tilde{\tilde{u}}$ satisfies

$$(\tilde{\tilde{u}})_{tt} + (\tilde{\tilde{u}})_{xxxx} + b\tilde{\tilde{u}}^+ - a\tilde{\tilde{u}}^- = s\phi_{00}.$$

Similarly we can show the existence of another solution $\breve{\breve{u}}$ satisfying

$$(\breve{\breve{u}})_{tt} + (\breve{\breve{u}})_{xxxx} + b\breve{\breve{u}}^+ - a\breve{\breve{u}}^- = s\phi_{00}$$

with $(\breve{\breve{u}}, \phi_{00}) < 0$.

Thus $G(v) = s\phi_{00}$, $s > 0$ has six solutions, one in each of the six regions. The six solutions are a positive solution $\frac{s\phi_{00}}{b+1}$ in C_1, a negative solution $\frac{s\phi_{00}}{a+1}$ in C_2, $\tilde{u}$, $\breve{u}$, $\tilde{\tilde{u}}$, and $\breve{\breve{u}}$, which we state in above. We prove Theorem 1.1. For the proof of Theorem 1.2 we set V the $k+1$ dimensional subspace of H spanned by ϕ_{00}, ϕ_{10} and $\phi_{k\ 0}$ and W the orthogonal complement of V in H. The other parts of the proof of Theorem 1.2 are the similar process to those of the proof of Theorem 1.1.

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