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APPLICATION OF LINKING FOR AN ELLIPTIC SYSTEM

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Abstract. In this article we consider nontrivial solutions of an elliptic system in the bounded smooth domain with homogeneous Dirichlet data. We apply the linking theorem for showing the existence results that is obtained by Massa.

1. Introduction

In this paper, we are interested in the existence of nontrivial solutions of the elliptic system with homogeneous Dirichlet data

$$\tag{1} \begin{cases} -\Delta u = au + bv + C_1(v^+)^p + f_1 + t\phi_1 & \text{in } \Omega \\ -\Delta v = bu + av + C_2(u^+)^q + f_2 + r\phi_1 & \text{in } \Omega \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $u^+(x) = \max\{0, u(x)\}$ and C_1, C_2 are two positive constants. Here Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and ϕ_1 is the first eigenfunction of the Laplacian with Dirichlet boundary conditions. And the nonlinearities will be assumed both superlinear and subcritical, that is, $1 < p, q < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$.

Recently Massa [3] considered the existence of two solutions of problem (1) in a smooth bounded domain $\Omega \subseteq \mathbb{R}^N (N \geq 2)$. He proved the following two theorems.

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THEOREM 1.1. *(Massa [3]) The problem* (1) *is rewritten as a vectorial form*

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} (v^+)^p \\ (u^+)^q \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} t \\ r \end{bmatrix} \phi_1 \text{ in } \Omega \\ \qquad\qquad u = v = 0 \qquad\qquad\qquad \text{in } \partial\Omega \end{cases}$$

where $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

If A has real eigenvalues $a \pm b \notin \sigma(-\Delta)$ and $f_{1,2} \in L^n(\Omega)$ with $n > N \geq 2$, then there exits $(t_0, r_0) \in \mathbb{R}^2$ such that if $(t,r)^T = (t_0, r_0)^T + (\lambda_1 I - A)(\tau, \rho)^T$ with $\tau, \rho < 0$ then a negative solution (u_{neg}, v_{neg}) of (1) *exists.*

THEOREM 1.2. *(Massa [3]) Under the same hypotheses as in Theorem 1.1, then for the same vectors $(t,r) \in \mathbb{R}^2$, a second solution exists.*

Massa is motivated by the results in [1] and [4]. The proof of Theorem 1.1 is relatively simple, since for negative solutions system (1) turns out to be a linear problem. Theorem 1.2 is proved by finding a critical point of a functional defined by

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{2} B(\mathbf{u}, \mathbf{u}) - H(\mathbf{u}) \\ &= \int_\Omega \nabla u \nabla v - \frac{1}{2} \int_\Omega \big(b(u^2 + v^2) + 2auv\big) \\ &\quad -C_1 \int_\Omega \frac{[(v + v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u + u_{neg})^+]^{q+1}}{q+1}, \end{aligned}$$

where

$$B((u,v),(w,z)) = \int_\Omega \nabla u \nabla z + \nabla v \nabla w - a(uz + vw) - b(uw + vz).$$

We know that the functional F is $C^1(E; \mathbb{R})$ and its critical points (u,v) are such that $(u + u_{neg}, v + v_{neg})$ are solutions of (1).

We apply the linking theorem to find the solutions of the system (1). We have the following theorem.

THEOREM 1.3. *If A has real eigenvalues $a \pm b \notin \sigma(-\Delta)$ and $f_{1,2} \in L^n(\Omega)$ with $n > N \geq 2$, then the system* (1) *has at least two nontrivial solutions.*

2. The variational structure

We consider the space $E = H^1_0 \times H^1_0$ equipped with the scalar product

$$\langle (u,v),(w,z)\rangle_E = \int_\Omega \nabla u \nabla w + \nabla v \nabla z,$$

the related norm $\|(u,v)\|_E$. In order to find a orthogonal base for E which diagonalizes B, we consider the eigenvalue problem

$$(u,v) \in E: \quad B((u,v),(\phi,\psi)) = \mu \langle (u,v),(\phi,\psi)\rangle_E \quad \forall (\phi,\psi) \in E:$$

this gives the eigenvalues

$$\mu_{\pm i} = \frac{-b \pm (\lambda_i - a)}{\lambda_i} \quad (i \in \mathbb{N});$$

and the related eigenvectors

$$\Psi_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2\lambda_i}} \quad (i \in \mathbb{N}).$$

In view of this structure we may define

$$\begin{aligned} E^+ &= \overline{\mathrm{span}\{\Psi_i : \mu_i > 0, i \in \mathbb{Z}^0\}}, \\ E^- &= \overline{\mathrm{span}\{\Psi_i : \mu_i < 0, i \in \mathbb{Z}^0\}}, \\ E^0 &= \mathrm{span}\{\Psi_i : \mu_i = 0, i \in \mathbb{Z}^0\}, \end{aligned}$$

and we have

LEMMA 2.1. *There exists $\xi^* > 0$ such that*

$$B(\mathbf{u},\mathbf{u}) \geq 2\xi^* \|\mathbf{u}\|^2_E \quad \textit{for} \quad \mathbf{u} \in E^+ \tag{2}$$

$$B(\mathbf{u},\mathbf{u}) \leq -2\xi^* \|\mathbf{u}\|^2_E \quad \textit{for} \quad \mathbf{u} \in E^-. \tag{3}$$

Moreover, if $a \pm b \notin \sigma(-\Delta)$, then $E^0 = \{0\}$.

We also define $\widetilde{n}$ such that for $i \geq \widetilde{n}$ we have $\lambda_i - a > |b|$ and

$$E_h = \overline{\mathrm{span}\{\Psi_i : |i| \geq \widetilde{n}, i \in \mathbb{Z}^0\}}, \quad E_l = \mathrm{span}\{\Psi_i : |i| < \widetilde{n}, i \in \mathbb{Z}^0\}.$$

And we have the following

LEMMA 2.2. *$(u,v) \in E^+ \cap E_h$ implies $u = v$ and $(u,v) \in E^- \cap E_h$ implies $u = -v$.*

For the application of the linking inequality, we define

$$H_0 = E_l \cap E^+, \quad H_1 = E_h \cap E^+, \quad H_2 = E_l \cap E^-, \quad H_3 = E_h \cap E^-,$$

then $E = H_0 \oplus H_1 \oplus H_2 \oplus H_3$ and we have

LEMMA 2.3. *There exists* $\mathbf{g} = (g, g) \in H_1$ *with* $\|\mathbf{g}\|_E = 1$ *and* $\|\mathbf{g}\|_{L^\infty} = \infty$.

Lemma 2.1 $\sim$ 2.3 are proved in [3].

3. Proof of Theorem 1.3

In this section we will prove the PS condition and the estimates, which were required for the application of the linking Theorem.

LEMMA 3.1. *(PS Condition). The functional* F *satisfies the PS condition, that is, let* ε_n *be a sequence of positive reals converging to zero and* $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset E$ *be such that*

$$|F(\mathbf{u}_n)| \leq T \tag{4}$$

$$|F'(\mathbf{u}_n)[\phi, \psi]| \leq \varepsilon_n \|(\phi, \psi)\|_E \forall (\phi, \psi) \in E : \tag{5}$$

then $\{\mathbf{u}_n\}$ *admits a convergent subsequence.*

Proof. First, we want to prove that $\|\mathbf{u}_n\|_E$ is bounded. For the sake of contradiction, we consider a sequence $\{\mathbf{u}_n\}$ such that $\|\mathbf{u}_n\|_E \to \infty$. And we define $(U_n, V_n) = (u_n, v_n)/\|\mathbf{u}_n\|_E$, so that $(U_n, V_n) \to (U, V)$ weakly in E and strongly in $[L^r]^2$ for $r < 2^*$.

We observe that

$$\int_\Omega [(v_n + v_{neg})^+]^p v_n = \int_\Omega [(v_n + v_{neg})^+]^{p+1} + [(v_n + v_{neg})^+]^p(-v_{neq})$$

Then, by considering $F(\mathbf{u}_n) - \frac{1}{2}F'(\mathbf{u}_n)[\mathbf{u}_n]$, we get

$$\begin{aligned} &C_1\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega [(v_n + v_{neg})^+]^{p+1} \\ &+C_2\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_\Omega [(u_n + u_{neg})^+]^{q+1} + \frac{C_1}{2}\int_\Omega [(v_n + v_{neg})^+]^p(-v_{neq}) \\ &+\frac{C_2}{2}\int_\Omega [(u_n + u_{neg})^+]^q(-u_{neq}) \leq T + \varepsilon_n\|\mathbf{u}_n\|_E; \end{aligned}$$

by observing that each term in the expression above is nonnegative, we conclude that the estimate from above holds for each of them, and then

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(v_n + v_{neg})^+]^{p+1} \to 0, \quad \frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(u_n + u_{neg})^+]^{q+1} \to 0. \tag{6}$$

For any $(\phi, \psi) \in E$ with $\|(\phi, \psi)\|_E = 1$, we consider $\frac{F'(\mathbf{u}_n)[\phi,\psi]}{\|\mathbf{u}_n\|_E}$. We get

$$
\begin{aligned}
(7) \qquad B((U_n, V_n), (\phi, \psi)) &- C_1 \int_\Omega \frac{[(v_n + v_{neg})^+]^p}{\|\mathbf{u}_n\|_E} \psi \\
&- C_2 \int_\Omega \frac{[(u_n + u_{neg})^+]^q}{\|\mathbf{u}_n\|_E} \phi \to 0.
\end{aligned}
$$

By using the weak convergence of (U_n, V_n) and (6), (7) implies that

$$B((U_n, V_n), (\phi, \psi)) \to 0.$$

Now consider $\frac{F'(\mathbf{u}_n)[v_n,u_n]}{\|\mathbf{u}_n\|_E}$;

$$
\begin{aligned}
B((U_n, V_n), (V_n, U_n)) &- C_1 \int_\Omega \frac{[(v_n + v_{neg})^+]^p}{\|\mathbf{u}_n\|_E} U_n \\
&- C_2 \int_\Omega \frac{[(u_n + u_{neg})^+]^q}{\|\mathbf{u}_n\|_E} V_n \to 0,
\end{aligned}
$$

which implies $B((U_n, V_n), (V_n, U_n)) \to 0$ and then $\int_\Omega |\Delta U_n|^2 + |\Delta V_n|^2 \to 0$. But this gives rise to a contradiction since by definition we have $\|(U_n, V_n)\|_E = 1$. We conclude that $\|\mathbf{u}_n\|_E$ is bounded. It is now simple to see that $\mathbf{u}_n$ admits a convergent subsequence. In fact, up to a subsequence, $(u_n, v_n) \to (u, v)$ weakly in E and strongly in $[L^r]^2$ for $r < 2^*$, then we may consider $F'(u_n, v_n)[v_n - v, u_n - u]$ to obtain

$$\int_\Omega \Delta u_n \Delta(u_n - u) + \Delta v_n \Delta(v_n - v) \to 0,$$

which implies that the convergence is in fact strong. □

For any Y subspace of E, consider $B_\rho(Y) := \{u \in Y : \|u\| \le \rho\}$ and denote by $\partial B_\rho(Y)$ the boundary of $B_\rho(Y)$ relative to Y. Furthermore define, for any $e \in E$,

$$Q_R(Y, e) = \{u + ae \in Y \oplus [e] : u \in Y, a \ge 0, \|u + ae\| \le R\}$$

and denote by $\partial Q_R(Y, e)$ its boundary relative to $Y \oplus [e]$.

Lemma 3.2. *There exists $\rho > 0$ such that*

$$\inf_{B_\rho(H_0 \oplus H_1)} F \ge 0 \quad \textit{and} \quad \inf_{\partial B_\rho(H_0 \oplus H_1)} F > 0$$

Proof. Applying the proof of Lemma 3.6 in [3], if $\|\mathbf{u}\|_E \leq \rho$ then we have

$$\begin{aligned} F(\mathbf{u}) &\geq \|u\|^2_{H^1_0}\left(\xi^* - C\|u\|^{q-1}_{H^1_0}\right) + \|v\|^2_{H^1_0}\left(\xi^* - C\|v\|^{p-1}_{H^1_0}\right) \\ &\geq \|u\|^2_{H^1_0}\left(\xi^* - C\rho^{q-1}\right) + \|v\|^2_{H^1_0}\left(\xi^* - C\rho^{p-1}\right). \end{aligned}$$

Since $p, q > 1$, for $\rho > 0$ small enough we obtain that $\xi^* - C\rho^{q-1} > 0$ and $\xi^* - C\rho^{p-1} > 0$. Let $\widetilde{C} = \min\{\xi^* - C\rho^{q-1}, \xi^* - C\rho^{p-1}\}$, then

$$F(\mathbf{u}) \geq \widetilde{C}\left(\|u\|^2_{H^1_0} + \|v\|^2_{H^1_0}\right) = \widetilde{C}\|\mathbf{u}\|^2_E.$$

Hence if $\|\mathbf{u}\|_E \leq \rho$ then $F(\mathbf{u}) \geq \widetilde{C}\|\mathbf{u}\|^2_E \geq 0$ and if $\|\mathbf{u}\|_E = \rho$ $F(\mathbf{u}) \geq \widetilde{C}\|\mathbf{u}\|^2_E = \widetilde{C}\rho^2 > 0$ □

LEMMA 3.3. *Let* $\mathbf{g}$ *as in the Lemma 2.3. We have*

$$\sup_{Q_R(H_2 \oplus H_3, \mathbf{g})} F < +\infty, \qquad \text{for any } R > 0.$$

Proof. Since $\mathbf{w} \in E^-$, $B(\mathbf{w}, \mathbf{w}) \leq -2\xi^*\|\mathbf{w}\|^2_E$. And since E^+ and E^- are orthogonal, $\langle \mathbf{w}, \mathbf{g}\rangle_E = 0 = B(\mathbf{w}, \mathbf{g})$ and $B(\mathbf{u}, \mathbf{u}) = B(\mathbf{w}, \mathbf{w}) + s^2 B(\mathbf{g}, \mathbf{g})$. Hence

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - C_1\int_\Omega \frac{[(v + v_{neq})^+]^{p+1}}{p+1} - C_2\int_\Omega \frac{[(u + u_{neq})^+]^{q+1}}{q+1} \\ &\leq -\xi^*\|\mathbf{w}\|^2_E + \frac{1}{2}s^2 B(\mathbf{g}, \mathbf{g}) \\ &\leq \frac{1}{2}s^2 B(\mathbf{g}, \mathbf{g}) \end{aligned}$$

We know that

$$\|\mathbf{w} + s\mathbf{g}\|^2_E = \|\mathbf{w}\|^2_E + s^2\|\mathbf{g}\|^2_E = \|\mathbf{w}\|^2_E + s^2.$$

For any $R > 0$, if $\|\mathbf{w} + s\mathbf{g}\|_E \leq R$ then $s \leq R$ and hence

$$F(\mathbf{u}) \leq \frac{1}{2}s^2 B(\mathbf{g}, \mathbf{g}) \leq \frac{1}{2}R^2 B(\mathbf{g}, \mathbf{g}) < +\infty.$$

□

LEMMA 3.4. *Let* ρ *as in the lemma 3.2. Then there exists* $R > \rho$ *such that* $F(\mathbf{u}) \leq 0$ *for:*

(a) $\mathbf{u} \in E^-$,
(b) $\mathbf{u} = \mathbf{w} + s\mathbf{g}$; $\quad \mathbf{w} \in E^-$, $\quad \|\mathbf{w} + s\mathbf{g}\|_E = R$, $\quad 0 < s \leq R$.

Proof. (a) It is proved in Lemma 3.8 of [3].

(b) Applying the proof of Lemma 3.8 (c) in [3], we have

$$-C_1\int_\Omega \frac{[(v+v_{neq})^+]^{p+1}}{p+1} - C_2\int_\Omega \frac{[(u+u_{neq})^+]^{q+1}}{q+1} \le -\widetilde{C}R^{\min\{p,q\}+1}.$$

We get

$$\begin{aligned} F(\mathbf{u}) &\le \frac{1}{2}B(\mathbf{u},\mathbf{u}) - \widetilde{C}R^{\min\{p,q\}+1} \\ &\le -\xi^*\|\mathbf{w}\|_E^2 + \frac{1}{2}s^2B(\mathbf{g},\mathbf{g}) - \widetilde{C}R^{\min\{p,q\}+1} \\ &\le \frac{1}{2}s^2B(\mathbf{g},\mathbf{g}) - \widetilde{C}R^{\min\{p,q\}+1} \\ &= R^2\left\{\frac{1}{2}\left(\frac{s}{R}\right)^2 B(\mathbf{g},\mathbf{g}) - \widetilde{C}R^{\min\{p,q\}-1}\right\} \\ &\le R^2\left\{\frac{1}{2}B(\mathbf{g},\mathbf{g}) - \widetilde{C}R^{\min\{p,q\}-1}\right\}. \end{aligned}$$

Choose $R > 1$(and also $R > \rho$) large enough to make $\frac{1}{2}B(\mathbf{g},\mathbf{g}) - \widetilde{C}R^{\min\{p,q\}-1} < 0$: which completes the proof. □

Proof of Theorem 1.3. By Lemma 3.2 $\sim$ 3.4, there exists $0 < \rho < R$ such that

$$\inf_{B_\rho(H_0\oplus H_1)} F < \inf_{\partial B_\rho(H_0\oplus H_1)} F.$$

Moreover the functional F satisfies the PS condition. By the linking theorem, F has at least two critical values c_1 and c_2

$$\inf_{B_\rho(H_0\oplus H_1)} F \le c_1 \le \sup_{\partial Q_R(H_2\oplus H_3,\mathbf{g})} F < \inf_{\partial B_\rho(H_0\oplus H_1)} F \le c_2 \le \sup_{Q_R(H_2\oplus H_3,\mathbf{g})} F.$$

Since $\inf_{B_\rho(H_0\oplus H_1)} F \ge 0$ and $\inf_{B_\rho(H_0\oplus H_1)} F$, $c_1 = 0$. We know that (1) has a negative solution. Since $c_2 > 0$, (1) has at least one nontrivial solution and the proof is completed.

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