Korean J. Math. **17** (2009), No. 2, pp. 161–167

ON THE INJECTIVITY OF THE WEAK TOPOS FUZ

Ig Sung Kim

Abstract. Category *Fuz* of fuzzy sets has a similar function to the Category *Set*. We study injective, absolute retract, enough injectives, injective hulls and essential extension in the Category *Fuz* of fuzzy sets.

1.Introduction

Category *Fuz* of fuzzy sets has a similar function to the topos *Set*. *Fuz* has finite products, middle object, equalizers, exponentials and weak subobject classifier. But *Fuz* is not a topos, it forms a weak topos. There are some comparisons between weak topos *Fuz* and topos *Set*. In this paper, first we show that in *Fuz* there exist objects that are not injectives and there exist monomorphisms that are not essential extension. But with some conditions, *Fuz* has injectives and absolute retract. Secondly we show that *Fuz* has enough injectives and every object in *Fuz* has an injective hull.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. *An elementary topos is a category $\mathcal{E}$ that satisfies the following;*
(T1) $\mathcal{E}$ is finitely complete,
(T2) $\mathcal{E}$ has exponentiation,
(T3) $\mathcal{E}$ has a subobject classifier.

Received March 13, 2009. Revised April 8, 2009.
2000 Mathematics Subject Classification: 18B25.
Key words and phrases: category *Fuz*, injective, enough injective, injective hull, essential extension.
This paper was supported by Sangji University Research Fund, 2007.

(T2) means that for every object A in $\mathcal{E}$, the endofunctor $(-)\times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A \circ (g \times i_A) = f$;

$$
\begin{array}{ccc}
Y \times A & \xrightarrow{\ f\ } & B \\
{\scriptstyle g\times i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\
B^A \times A & \xrightarrow[ev_A]{} & B
\end{array}
$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : 1 \to \Omega$ such that for any monomorphism $h : D \to C$, there is a unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ which makes the following diagram a pull-back;

$$
\begin{array}{ccc}
D & \xrightarrow{\ !\ } & 1 \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\
C & \xrightarrow[\chi_h]{} & \Omega
\end{array}
$$

Example 2.2. Category *Set* is a topos. $\{*\}$ is a terminal object. $\Omega = \{0, 1\}$ and $\top : \{*\} \to \Omega$ with $\top(*) = 1$ is a subobject classifier. If we define

$\chi_h(c) = 1$ if $c = h(d)$ for some $d \in D$,

$\chi_h(c) = 0$ otherwise

then χ_h is a characteristic function of D .

Category *Fuz* of fuzzy sets is a category whose object is (A, α_A) where A is an object and $\alpha_A : A \to I$ is a morphism with $I = (0, 1]$ in *Set* and morphism from (A, α_A) to (B, α_B) is a function $f : A \to B$ in *Set* such that $\alpha_A(a) \leqq \alpha_B \circ f(a)$.

Definition 2.3. *A middle object in a category $\mathcal{C}$ is a monomorphism $m : X \to Y$ such that*

1. $Hom(A, Y)$ is partially ordered for all object $A \in \mathcal{C}$.

2. There is a unique smallest morphism a so that the square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{\ a\ } & Y \end{array}$$

is a pull-back

3. For any monomorphism $f : B \to A$, there is a unique

Characteristic morphism $\chi_f : A \to Y$ such that $\chi_f \leqq a$ and the square

$$\begin{array}{ccc} B & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{\ \chi_f\ } & Y \end{array}$$

is a pull-back [7].

Definition 2.4. *A weak topos is a Cartesian closed category with middle object [7].*

Proposition 2.5. *Category Fuz is a weak topos.*

For the proof see Yuan and Lee [7].

Definition 2.6. *We say that an object A of a category $\mathcal{C}$ is an absolute retract if any monomorphism $f : A \to B$ has a left inverse and an object A of a category $\mathcal{C}$ is an injective if, for any morphism $f : B \to A$ and any monomorphism $h : B \to C$, there exists a morphism $g : C \to A$ such that $f = g \circ h$.*

Definition 2.7. *We say that a monomorphism $m : A \to B$ of a category $\mathcal{C}$ is an essential extension if any morphism $n : B \to C$ is a monomorphism whenever $n \circ m : A \to C$ is a monomorphism. Also we say that an essential extension $m : A \to E$ where E is injective is an injective hull of A.*

3. Main parts

PROPOSITION 3.1. *In Fuz, there exist objects that are not injectives.*

proof. Consider an object $(J = \{x, y\}, \alpha_J)$ satisfying $\alpha_J(x) = 0.4$ and $\alpha_J(y) = 0.5$, and a monomorphism $m : (A = \{a, b\}, \alpha_A) \to (B = \{u, v, w\}, \alpha_B)$ defined by $m(a) = u, m(b) = v$ satisfying $\alpha_A(a) = 0.2, \alpha_A(b) = 0.3$, $\alpha_B(u) = 0.7, \alpha_B(v) = 0.8$ and $\alpha_B(w) = 1$. We assume that the object (J, α_J) is injective. Then for a morphism $s : (A, \alpha_A) \to (J, \alpha_J)$ defined by $s(a) = x$ and $s(b) = y$, there exists a morphism $t : (B, \alpha_B) \to (J, \alpha_J)$ defined by $t(u) = x$, $t(v) = y$ and $t(w) = x$ or y such that $t \circ m = s$. But it does not satisfy that $\alpha_B(w) \leqq \alpha_J \circ t(w)$. So the morphism $t : B \to J$ does not exist in *Fuz*. Hence (J, α_J) is not an injective object in *Fuz*.

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ J & = & J \end{array}$$

□

THEOREM 3.2. *In Fuz, (J, α_J) is injective if J is normal and $max\{\alpha_A(a), \alpha_B(m(a))\} \leqq \alpha_J(f(a))$ for all $a \in A$, where $m : (A, \alpha_A) \to (B, \alpha_B)$ is a monomorphism and $f : (A, \alpha_A) \to (J, \alpha_J)$ is a morphism.*

Proof. Let $m : (A, \alpha_A) \to (B, \alpha_B)$ be a monomorphism and $f : (A, \alpha_A) \to (J, \alpha_J)$ be a morphism. Define a morphism $g : (B, \alpha_B) \to (J, \alpha_J)$ by $g(b) = f(a)$ for all $b = m(a)$ and $g(b) = v$ for all $b \in B - m[A]$ satisfying $\alpha_J(v) = 1$. Then $g : B \to J$ is the morphism in *Fuz* and $g \circ m = f$. So (J, α_J) is injective. □

PROPOSITION 3.3. *In Fuz, there exist objects that are not absolute retracts.*

Proof. Consider an object (A, α_A) and define a monomorphism $m : (A = \{a, b\}, \alpha_A) \to (B = \{u, v, w\}, \alpha_B)$ by $m(a) = u, m(b) = v$

satisfying $\alpha_A(a) = 0.4, \alpha_A(b) = 0.5$, $\alpha_B(u) = 0.8, \alpha_B(v) = 0.9$ and $\alpha_B(w) = 1$.

We assume that there exists a morphism $n : (B, \alpha_B) \to (A, \alpha_A)$ such that $n \circ m = i_A$. Then $n(u) = a, n(v) = b$ and $n(w) = a$ or b. But it does not satisfy that $\alpha_B \leqq \alpha_A \circ n$. So the morphism $n : B \to A$ does not exist in *Fuz*. Hence an object (A, α_A) is not an absolute retract.□

THEOREM 3.4. *In Fuz, an object* (A, α_A) *is an absolute retract if* (A, α_A) *is normal and the square*

$$\begin{array}{ccc} A & \xrightarrow{\;m\;} & B \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ I & \xrightarrow{\;i_I\;} & I \end{array}$$

commutes, for any monomorphism $m : (A, \alpha_A) \to (B, \alpha_B)$.

Proof. For any monomorphism $m : (A, \alpha_A) \to (B, \alpha_B)$, we define a morphism $f : (B, \alpha_B) \to (A, \alpha_A)$ by $f(b) = a$ for all $b = m(a)$ and $f(b) = v$ for all $b \in B - m[A]$ satisfying $\alpha_A(v) = 1$. Then $f : B \to A$ is the morphism in *Fuz* and $f \circ m = i_A$. So (A, α_A) is an absolute retract. □

THEOREM 3.5. *Fuz has enough injectives.*

Proof. Let (A, α_A) be an object in *Fuz*. Then there exists a monomorphism $m : (A, \alpha_A) \to (A, \alpha'_A)$ defined by $m(a) = a$ satisfying $\alpha'_A(a) = 1$ for all $a \in A$. We have that $\alpha_A \leqq \alpha'_A \circ m$. We only claim that (A, α'_A) is injective. Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ be a monomorphism and $g : (X, \alpha_X) \to (A, \alpha'_A)$ be a morphism. Then (A, α'_A) is normal and $max\{\alpha_X(a), \alpha_Y(f(a))\} \leqq \alpha'_A(g(a))$ for all $a \in X$. By Theorem 3.2, (A, α'_A) is injective. □

THEOREM 3.6. *In Fuz, every object has an injective hull.*

Proof. For any object (A, α_A) in *Fuz*, by Theorem 3.5, there exists a monomorphism $m : (A, \alpha_A) \to (A, \alpha'_A)$ defined by $m(a) = a$ where (A, α'_A) is injective satisfying $\alpha'_A(a) = 1$ for all $a \in A$. For any morphism $n : (A, \alpha'_A) \to (B, \alpha_B)$ such that $n \circ m$ is a monomorphism, we only claim that n is a monomorphism. If $h, g : (X, \alpha_X) \to (A, \alpha'_A)$ are morphisms in *Fuz*, then we have $\alpha'_A \circ h \geqq \alpha_X$ and $\alpha'_A \circ g \geqq \alpha_X$. Since $\alpha_B \circ n \geqq \alpha'_A$, we get $\alpha_B \circ n \circ h \geqq \alpha_X$. So we only claim that n is injective. Let $n(a) = n(b)$, then $n \circ m(a) = n \circ m(b)$. Since $n \circ m$ is a monomorphism, we get that $a = b$. So the object (A, α_A) has an injective hull. □

$$\begin{array}{ccc} A & \xrightarrow{m} & A \\ {\scriptstyle n\circ m}\downarrow & & \downarrow{\scriptstyle n} \\ B & \xrightarrow{i_B} & B \end{array}$$

Proposition 3.7. *In Fuz, there exist monomorphisms that are not essential extensions.*

Proof. Consider a monomorphism $m : (A = \{a, b\}, \alpha_A) \to (B = \{u, v, w\}, \alpha_B)$ defined by $m(a) = u$ and $m(b) = w$ such that $\alpha_A \leqq \alpha_B \circ m$ satisfying $\alpha_A(a) = 0.1$, $\alpha_A(b) = 0.2$, $\alpha_B(u) = 0.4$, $\alpha_B(v) = 0.6$ and $\alpha_B(w) = 0.6$, and a morphism $n : (B = \{u, v, w\}, \alpha_B) \to (C = \{x, y, z\}, \alpha_C)$ defined by $n(u) = x, n(v) = x$ and $n(w) = z$ satisfying $\alpha_C(x) = \alpha_C(y) = \alpha_C(z) = 0.6$. Then $\alpha_A \leqq \alpha_C \circ n \circ m$. So $n \circ m$ is also a monomorphism in *Fuz*. But n is not a monomorphism in *Fuz* by $\alpha_C \circ n(u) \leqq \alpha_B(u)$. So the monomorphism $m : (A, \alpha_A) \to (B, \alpha_B)$ is not an essential extension. □

References

1. B. Banaschwski, *On the injectivity of Boolean algebra*, Seminar of the Categorical Topology Research Group at the Univ. of Cape Town (1990).
2. B. Banaschwski, M. M. Ebrahimi and M. Mahmoudi, *On the normal completion of a Boolean algebra*, J. of pure and applied algebra **181** (2003).
3. M. M. Ebrahimi, *Internal Completeness and Injectivity of Boolean Algebras in the Topos of M-Sets*, Bull. Austral. Math. Soc. **41** (1990).
4. R. Goldblatt, *Topoi, North-Holland* (1984).
5. P. T. Johnstone, *Topos Theory*, Academic Press, N. Y. (1977).
6. X. Yuan and E. S. Lee, *Category Fuz, Wtopos and Category FuzFuz*, J. of Math. Anal. and Appl. **201** (1996), 489-501.

7. X. Yuan, H. Li and E. S. Lee, *Categories of fuzzy sets and weak topos*, Fuzzy Sets and Systems **127** (2002), 291-297.

Sangji University,
Wonju, 660, Korea
E-mail: iskim@sangji.ac.kr