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A STUDY ON GENERALIZED QUASI-CLASS A OPERATORS

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Abstract. In this paper, we consider the operator T satisfying $T^{*k}(|T^2| - |T|^2)T^k \geq 0$ and prove that if the operator is injective and has the real spectrum, then it is self-adjoint.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on a Hilbert space $\mathscr{H}$. Recall ([2]) that $T \in \mathscr{L}(\mathscr{H})$ is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $p \in (0,1]$, and T is called *paranormal* if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathscr{H}$. Following [3] and [2] we say that $T \in \mathscr{L}(\mathscr{H})$ belongs to *class A* if $|T^2| \geq |T|^2$. We shall denote classes of p-hyponormal operators, paranormal operators, and class A operators by $\mathcal{H}(p)$, $\mathcal{PN}$, and $\mathcal{A}$, respectively. It is well known that

$$\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{PN}. \tag{1.1}$$

In [5] Jeon and Kim considered an extension of the notion of class A operators; we say that $T \in \mathscr{L}(\mathscr{H})$ is *quasi-class A* if

$$T^*|T^2|T \geq T^*|T|^2T.$$

We shall denote the set of quasi-class A operators by $\mathcal{QA}$. As shown in [5], the class of quasi-class A operators properly contains classes of class A operators, i.e., the following inclusions holds;

$$\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{QA}. \tag{1.2}$$

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In view of inclusions (1.1) and (1.2), it seems reasonable to expect that the operators in class $\mathcal{QA}$ are paranormal. But there exists an example [5] that one would be wrong in such an expectation.

Now, we consider a following generalization of quasi-class A operators in [10].

Definition 1.1. *We say that $T \in \mathscr{L}(\mathscr{H})$ is of quasi-class (A, k) class if*

$$T^{*k}(|T^2| - |T|^2)T^k \geq 0 \quad for\ k \in \mathbb{N} \tag{1.3}$$

We denote the spectrum and the closure of numerical range of an operator $T \in \mathscr{L}(\mathscr{H})$ by $\sigma(T)$ and $\overline{W(T)}$, respectively.

In 1966, I. H. Sheth [9] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ for any operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint, and then I. H. Kim [7] extended this result of Sheth to the class of p-hyponormal operators. Very recently, Jeon, Kim, Tanahashi and Uchiyama [6] also extended this result to the class of quasi-class A operators as follows.

Proposition 1.2 ([6], Theorem 2.6). *If T is a quasi-class A operator and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

The aim of this paper is to extend this result to more generalized quasi-class A operators(i.e., quasi-class (A, k) operators) as follows.

Theorem 1.3. *Let T be of injective quasi-class (A, k) with the real spectrum. Then T is self-adjoint.*

In [11], J.P. Williams showed that if $T \in \mathscr{L}(\mathscr{H})$ is any operator such that $ST = T^*S$, where $0 \notin \overline{W(S)}$, then the spectrum of T is real. So, for a $T \in \mathscr{L}(\mathscr{H})$, the condition that

$$\text{there exists an operator S such that } ST = T^*S, \text{ where } 0 \notin \overline{\mathrm{W(S)}}$$

is stronger than that the spectrum of T is real, which shows that the above Theorem extends Proposition 1.2. under the injectiveness of T.

2. Proofs

In this section we give a proof of Theorem1.3, modifying arguments used in proofs of [6]. We need some lemmas.

LEMMA 2.1. *Let T be of quasi-class (A, k). Then the following assertions hold:*

(1) *Assume that* ran T^k *is not dense, and decompose*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathscr{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker T^{k*}$$

where $\overline{\operatorname{ran}(T^k)}$ is the closure of ran T^k. *Then T_1 is of class A, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

(2) *The restriction $T|_{\mathcal{M}}$ to an invariant subspace $\mathcal{M}$ of T is also of quasi-class (A, k).*

LEMMA 2.2. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class A operator. Then we have an inequality*

$$\| |T^2| - |T|^2 \| \leq \| |T|U|T| - |T|U^*|T| \| \leq \frac{1}{\pi} \text{meas } \sigma(T), \tag{2.1}$$

where $T = U|T|$ is the polar decomposition of T.

LEMMA 2.3. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class A operator with the real spectrum. Then T is self-adjoint.*

Proof. Since T is of class A and it has the real spectrum, from (2.1), we have $|T^2| = |T|^2$. Now let

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \overline{\operatorname{ran}(T)} \oplus \ker(T^*)$$

be a 2×2 matrix representation of T, and let P be the orthogonal projection onto $\overline{\operatorname{ran}(T)}$. Then since

$$|T^2| - |T|^2 = 0 \;\Rightarrow\; T^*(T^*T - TT^*)T = 0,$$

we have $P(T^*T - TT^*)P = 0$. Therefore, by simple calculation, $A^*A - AA^* = BB^*$ and hence A is hyponormal. Since the spectrum of A is contained in the spectrum of T, it is also real. Thus A is self-adjoint and $B = 0$, which implies that T is self-adjoint. □

Proof of Theorem 1.3. If T is of quasi-class (A, k) and the range of T^k is dense, then T is of class A from Lemma 2.1. Hence Theorem 1.3 is reduced to Lemma 2.3. Assume that the range of T^k is not dense. From Lemma 2.1 we have a decomposition

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathscr{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker T^{k*}.$$

Then T_1 is of class A and $T_3^k = 0$. Since the spectrum of T_1 is contained in the spectrum of T, T_1 is self-adjoint by Lemma 2.3. Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection onto $\overline{\operatorname{ran}(T^k)}$. Then

$$Q|T|^2Q = QT^*TQ = \begin{pmatrix} T_1^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and so we may write

$$|T|^2 = \begin{pmatrix} T_1^2 & C \\ C^* & D \end{pmatrix}.$$

On the other hand, let $|T| = \begin{pmatrix} E & F \\ F^* & G \end{pmatrix}$. Then we have

$$\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = (Q|T|^2Q)^{\frac{1}{2}} \geq Q|T|Q = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q(T^*T)^{\frac{1}{2}}Q \geq Q(T^*QT)^{\frac{1}{2}}Q = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $E = T_1$ and $|T| = \begin{pmatrix} T_1 & F \\ F^* & G \end{pmatrix}$. By straight forward calculation we have

$$\begin{pmatrix} T_1^2 & T_1T_2 \\ T_2^*T_1 & |T_2|^2 + |T_3|^2 \end{pmatrix} = |T|^2 = \begin{pmatrix} T_1^2 + FF^* & T_1F + FG \\ F^*T_1 + GF^* & F^*F + G^2 \end{pmatrix},$$

which implies that $F = 0$ and $T_1T_2 = 0$. Since T_1 is injective, $T_2 = 0$. Thus $\overline{\operatorname{ran}(T^k)}$ and $\ker T^{k*}$ are reducing subspaces. Since T is injective, T_3 is also injective. Therefore we have that $T_3 = 0$. Hence T is self-adjoint. □

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