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PROJECTION METHODS FOR RELAXED COCOERCIVE VARIATION INEQUALITIES IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce and consider a new system of relaxed cocoercive variational inequalities involving three different operators and the concept of projective nonexpansive mapping. Base on the projection technique, we suggest two kinds of new iterative methods for the approximate solvability of this system. The results presented in this paper extend and improve the main results of [S.S. Chang, H.W.J. Lee, C.K. Chan, Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces, Appl. Math. Lett. 20 (2007) 329-334] and [Z. Huang, M. Aslam Noor, An explicit projection method for a system of nonlinear variational inequalities with different (γ, r)-cocoercive mappings, Appl. Math. Comput. (2007), doi:10.1016/j.amc.2007.01.032].

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1. Introduction

Variational inequality theory, which was introduced by Stampacchia [1] in 1964, has a wide range of applications in the fields of industry, finance, economics, social, ecology, regional, pure and applied sciences. The ideas and techniques of the variational inequalities are being proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems. Projection method and its variant forms, the origin of which can be traced back to Lions and Stampacchia [2], represents an important tool for finding the approximate solutions of variational inequalities. The main and basic idea of this tool is to establish the equivalence between the variational inequalities and the fixed point problems.

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In 1983, Gabay [3] has shown that the convergence of a projection method can be proved for cocoercive operators. In recent years, the researches which based on the convergence of projection methods, on the approximate solvability of a system for relaxed cocoercive nonlinear variational inequality in Hilbert spaces have been studied by many authors (see [4-9]).

Inspired and motivated by research going in this area, we introduce and consider a new system of variational inequalities involving three different nonlinear operators and the concept of projective nonexpansive mapping. This class of system includes the system of variational inequalities which involving one operator and the classical variational inequalities. Using the projection technique, we suggest and analyze two kinds of three-step iterative algorithms for solving this system. We also prove the convergence of the proposed iterative methods under mild conditions. Our results represent in this paper extend and improve the recent results announced by Chang et al [8], Huang and Noor et al [9] and others.

In this paper, we present firstly the concept of projective mappings.

Definition 1.1. Let K be a nonempty closed convex subset of a Hilbert space H. A mapping $S_K : H \to H$ is said to be *projection nonexpansive* if

$$\|P_K S_K x - P_K S_K y\| \leq \|x - y\|, \quad \forall\, x, y \in H.$$

Where P_K is the projection of H onto K.

This class of projection nonexpansive mappings is more general than the class of nonexpansive mappings. $P_K S_K$ is a nonexpansive mapping, so that S_K is a projection nonexpansive mapping which may be not nonexpansive mapping.

Example. Let Z be the complex plane and let $K = \{z \in Z, \|z\| \leq 1\}$ be the unit bull. Let $S : Z \to Z$ be a mapping which defined by

$$S(re^{i\theta}) = \begin{cases} 2re^{i\theta} & if \;\; r > 1, \\ re^{i\theta} & if \;\; r \leq 1. \end{cases} \quad \forall\, re^{i\theta} \in Z.$$

It is easy to see that, S is a expansive mapping(no nonexpansive mapping). In addition, $P_K S$ is a nonexpansive mapping, so that S is a projection nonexpansive mapping which is not nonexpansive mapping.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a closed and convex set in H. Let $T_1, T_2, T_3 : K \times K \times K \to H$ be three nonlinear different operators and S_K is a projection nonexpansive mapping.

We consider the problem of finding $x^*, y^*, z^* \in K$ such that

$$\langle x^* - S_K(y^* - \rho T_1(y^*, z^*, x^*)), x - x^* \rangle \geq 0, \quad \forall\, x \in K,\ \forall\, \rho > 0, \tag{1.1}$$

$$\langle y^* - S_K(z^* - \eta T_2(z^*, x^*, y^*)), x - y^* \rangle \geq 0, \quad \forall\, x \in K,\ \forall\, \eta > 0, \tag{1.2}$$

$$\langle z^* - S_K(x^* - \lambda T_3(x^*, y^*, z^*)), x - z^* \rangle \geq 0, \quad \forall\, x \in K,\ \forall\, \lambda > 0, \tag{1.3}$$

which is system of nonlinear variational inequalities involving three different nonlinear operators.

Next, we consider some special cases of the problem (1.1), (1.2) and (1.3) as follows:

(I) If $S_K = I$, $T_i : K \times K \to H$ is a bivariate mapping (i=1,2,3) and $\lambda = 0$, then the problem (1.1), (1.2) and (1.3) reduces to the following nonlinear variational inequality problem: to find $x^*, y^* \in K$ such that

$$\langle \rho T_1(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall\, x \in K, \ \forall\, \rho > 0, \tag{1.4}$$

$$\langle \eta T_2(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall\, x \in K, \ \forall\, \eta > 0 \tag{1.5}$$

which is studied by Z. Huang and M. Noor in [9].

(II)If $S_K = I$, $T_1 = T_2 = T : K \times K \to H$ is a bivariate mapping and $\lambda = 0$, then the problem (1.4) and (1.5) reduces to the following nonlinear variational inequality problem: to find $x^*, y^* \in K$ such that

$$\langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall\, x \in K, \ \forall\, \rho > 0, \tag{1.6}$$

$$\langle \eta T(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall\, x \in K, \ \forall\, \eta > 0 \tag{1.7}$$

which is studied by S. S. Chang, H. W. Joseph and C. K. Chan in [8].

Definition 1.2. A mapping $T : K \to H$ is called *r-strongly monotonic* if for all $x, y \in K$, there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2.$$

Definition 1.3. A mapping $T : K \to H$ is called *γ-cocoercive* if for all $x, y \in K$, there exists a constant $\gamma > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma\|Tx - Ty\|^2.$$

Definition 1.4. A mapping $T : K \to H$ is called *relaxed (γ, r)-cocoercive* if for all $x, y \in K$, there exists constants $\gamma > 0$, $r > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2.$$

For $\gamma = 0$, T is r-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. It is easy to see that we have the following implication:

r -strongly monotonicity $\Rightarrow$ relaxed (γ, r)-cocoercivity.

Definition 1.5. A mapping $T : K \to H$ is called *μ-Lipschitzian* if for all $x, y \in K$, there exists a constant $\mu > 0$, such that

$$\|Tx - Ty\| \leq \mu\|x - y\|.$$

In order to prove our results we need the following lemmas:

Lemma 1.1[9]. *For a given element $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0 \ \textit{for all} \ v \in K,$$

if and only if $u \in K$ *satisfies the relation*

$$u = P_K z,$$

where P_K *is a projection from* K *to* H.

Using Lemma 1.1, we can easily show that finding the solution of (1.1), (1.2) and (1.3) is equivalent to finding $x^*, y^*, z^* \in K$ such that

$$x^* = P_K S_K[y^* - \rho T_1(y^*, z^*, x^*)], \tag{1.8}$$

$$y^* = P_K S_K[z^* - \eta T_2(z^*, x^*, y^*)], \tag{1.9}$$

$$z^* = P_K S_K[x^* - \lambda T_3(x^*, y^*, z^*)]. \tag{1.10}$$

Lemma 1.2[8]. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \forall n \geq n_0$$

where n_0 *is some nonnegative integer,* $\lambda_n \in (0,1)$ *with* $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = 0(\lambda_n)$ *and* $\sum_{n=0}^{\infty} c_n < \infty$, *then* $a_n \to 0$ *(as* $n \to \infty$*).*

2. Projection algorithms

In this section, we suggest two kinds of projection algorithms for solving the system of variational inequalities (1.1), (1.2) and (1.3). One is explicit iterative scheme and the other is not explicit iterative scheme. By the explicit projection algorithm, our result will extend the main results of Huang and Noor et al[9]. Using the other algorithm, we improve the results of Chang et al [8].

Algorithm 2.1. For arbitrarily chosen initial points $x_0, y_0, z_0 \in K$ compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K S_K[y_n - \rho T_1(y_n, z_n, x_n)] \\ y_n = (1 - \beta_n)x_n + \beta_n P_K S_K[z_n - \eta T_2(z_n, x_n, y_n)] \\ z_n = (1 - \tau_n)x_n + \tau_n P_K S_K[x_n - \lambda T_3(x_n, y_n, z_n)] \end{cases} \tag{2.1}$$

where P_K is the projection of H onto K, $\rho, \eta, \lambda > 0$ are constants and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\tau_n\}$ are sequences in $[0, 1]$.

Algorithm 2.2. For arbitrarily chosen initial points $x_0, y_0, z_0 \in K$ compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K S_K[y_n - \rho T_1(y_n, z_n, x_n)] \\ y_{n+1} = (1 - \beta_n)x_n + \beta_n P_K S_K[z_n - \eta T_2(z_n, x_n, y_n)] \\ z_{n+1} = P_K S_K[x_{n+1} - \lambda T_3(x_{n+1}, y_n, z_n)] \end{cases} \tag{2.2}$$

where P_K is the projection of H onto K, $\rho, \eta, \lambda > 0$ are constants and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$.

If $S_K = I$, $T_1 = T_2 = T_3 = T : K \times K \to H$ is a bivariate mapping, $\lambda = 0$ and $\tau_n = 1$, then the Algorithm 2.1 can be reduced to the following.

Algorithm 2.3. For arbitrarily chosen initial points $x_0, y_0 \in K$ compute the sequence $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[y_n - \rho T(y_n, x_n)] \\ y_n = (1-\beta_n)x_n + \beta_n P_K[x_n - \eta T(x_n, y_n)] \end{cases} \tag{2.3}$$

If $S_K = I$, $T_i : K \times K \to H$ (i=1,2,3) is a bivariate mapping, $\lambda = 0$ and $\beta_n = 1$, then the Algorithm 2.2 can be reduced to the following.

Algorithm 2.4. For arbitrarily chosen initial points $x_0, y_0 \in K$ compute the sequence $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[y_n - \rho T_1(y_n, x_n)] \\ y_{n+1} = P_K[x_n - \eta T_2(x_n, y_n)] \end{cases} \tag{2.3}$$

3. Main results

Based on Algorithms 2.1 and 2.2, we present the approximation solvability of the problem (1.1),(1.2) and (1.3) involving the mappings $T_i : K \times K \times K \to H$ which is relaxed (γ_i, r_i) cocoercive and μ_i-Lipschitz continuous in the first variable(i=1,2,3). For the purpose we first give the following definitions:

Definition 3.1. A three-variable mapping $T : K \times K \times K \to H$ is called *relaxed (γ, r)-cocoercive* if for all $x, y \in K$, there exists constants $\gamma > 0$, $r > 0$ such that

$$\langle T(x,u,p) - T(y,v,q), x - y\rangle \geq -\gamma\|T(x,u,p) - T(y,v,q)\|^2 + r\|x-y\|^2,$$

$\forall u, v, p, q \in K$.

Definition 3.2. A mapping $T : K \times K \times K \to H$ is called *μ-Lipschitzian* if for all $x, y \in K$, there exists a constant $\mu > 0$, such that

$$\|T(x,u,p) - T(y,v,q)\| \leq \mu\|x-y\|, \forall u, v, p, q \in K.$$

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H and $T_i : K \times K \times K \to H$ be three-varible relaxed (γ_i, r_i)-cocoercive and μ_i-Lipschitzian in the first variable, respectively(i=1,2,3). Suppose that $x^*, y^*, z^* \in K \times K \times K$ is a solution to the problem (1.1), (1.2) and (1.3) and that $\{x_n\}, \{y_n\}, \{z_n\}$ are the sequences generated by Algorithm 2.1. If $\{\alpha_n\}, \{\beta_n\}$ and $\{\tau_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty}(1-\beta_n) < \infty, \sum_{n=0}^{\infty}(1-\tau_n) < \infty$;
(iii) $0 < \rho, \eta, \lambda < min\{2(r_1 - \gamma_1\mu_1^2)/\mu_1^2,\ 2(r_2 - \gamma_2\mu_2^2)/u_2^2,\ 2(r_3 - \gamma_3\mu_3^2)/\mu_3^2\}$;
(iv) $r_1 > \gamma_1\mu_1^2,\ r_2 > \gamma_2\mu_2^2,\ r_3 > \gamma_3\mu_3^2$,

then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to x^*, y^*, z^* respectively.

Proof. To get the result, we need first calculate $\|x_{n+1} - x^*\|$ for all $n \geq 0$. From (2.1), (1.8) and the nonexpansive property of $P_K S_K$, we have

$$\begin{aligned} \|x_{n+1} - x^*\| =& \|(1-\alpha_n)x_n + \alpha_n P_K S_K[y_n - \rho T_1(y_n, z_n, x_n)] \\ & - (1-\alpha_n)x^* - \alpha_n P_K S_K[y^* - \rho T_1(y^*, z^*, x^*)]\| \\ \leq & (1-\alpha_n)\|x_n - x^*\| \\ & + \alpha_n \|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|. \end{aligned} \tag{3.1}$$

Using the relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition in the first variable on T_1, we obtain

$$\begin{aligned} & \|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|^2 \\ = & \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*), y_n - y^*\rangle \\ & + \rho^2 \|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\ \leq & \|y_n - y^*\|^2 + 2\rho\gamma_1 \|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\ & - 2\rho r_1 \|y_n - y^*\|^2 + \rho^2 \mu_1^2 \|y_n - y^*\|^2 \\ = & (1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2)\|y_n - y^*\|^2. \end{aligned} \tag{3.2}$$

Set $\theta_1 = [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]^{1/2}$. It is clear from condition (iii) that $0 \leq \theta_1 < 1$. Hence from (3.2), it follows that

$$\|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| \leq \theta_1 \|y_n - y^*\|. \tag{3.3}$$

Similarly, from the relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian definition in the first variable on T_2, we have

$$\begin{aligned} & \|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\|^2 \\ = & \|z_n - z^*\|^2 - 2\eta\langle T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*), z_n - z^*\rangle \\ & + \eta^2 \|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\ \leq & \|z_n - z^*\|^2 + 2\eta\gamma_2 \|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\ & - 2\eta r_2 \|z_n - z^*\|^2 + \eta^2 \mu_2^2 \|z_n - z^*\|^2 \\ = & (1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2)\|z_n - z^*\|^2. \end{aligned} \tag{3.4}$$

Set $\theta_2 = [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]^{1/2}$. It is clear from condition (iii) that $0 \leq \theta_2 < 1$. Hence from (3.4), it follows that

$$\|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\| \leq \theta_2 \|z_n - z^*\|. \tag{3.5}$$

From relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitzian definition in the first variable on T_3, we get

$$\begin{aligned} & \|x_n - x^* - \lambda[T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)]\|^2 \\ = & \|x_n - x^*\|^2 - 2\lambda\langle T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*), x_n - x^*\rangle \\ & + \lambda^2 \|T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)\|^2 \\ \leq & \|x_n - x^*\|^2 + 2\lambda\gamma_3 \|T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)\|^2 \\ & - 2\lambda r_3 \|x_n - x^*\|^2 + \lambda^2 \mu_3^2 \|x_n - x^*\|^2 \\ = & (1 + 2\lambda\gamma_3\mu_3^2 - 2\lambda r_3 + \lambda^2\mu_3^2)\|x_n - x^*\|^2. \end{aligned} \tag{3.6}$$

Set $\theta_3 = [1 + 2\lambda\gamma_3\mu_3^2 - 2\lambda r_3 + \lambda^2\mu_3^2]^{1/2}$. It is clear from condition (iii) that $0 \leq \theta_3 < 1$. Hence from (3.6), it follows that

$$\|x_n - x^* - \lambda[T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)]\| \leq \theta_3\|x_n - x^*\|. \qquad (3.7)$$

Hence from (3), (2.1) and (3.7),we can make an estimate for $\|z_n - z^*\|$.

$$\begin{aligned}
&\|z_n - z^*\| \\
= \ & \|(1-\tau_n)x_n + \tau_n P_K S_K[x_n - \lambda T_3(x_n, y_n, z_n)] \\
& -(1-\tau_n)z^* - \tau_n P_K S_K[x^* - \lambda T_3(x^*, y^*, z^*)]\| \\
\leq \ & (1-\tau_n)\|x_n - z^*\| + \tau_n\|x_n - x^* - \lambda[T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)]\| \\
\leq \ & (1-\tau_n)\|x_n - x^*\| + (1-\tau_n)\|x^* - z^*\| + \tau_n\theta_3\|x_n - x^*\| \\
\leq \ & \|x_n - x^*\| + (1-\tau_n)\|x^* - z^*\|.
\end{aligned} \qquad (3.8)$$

Hence from (2), (2.1), (3.5) and (3.8),we can make an estimate for $\|y_n - y^*\|$.

$$\begin{aligned}
&\|y_n - y^*\| \\
= \ & \|(1-\beta_n)x_n + \beta_n P_K S_K[z_n - \eta T_2(z_n, x_n, y_n)] \\
& -(1-\beta_n)y^* - \beta_n P_K S_K[z^* - \eta T_2(z^*, x^*, y^*)]\| \\
\leq \ & (1-\beta_n)\|x_n - y^*\| + \beta_n\|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\| \\
\leq \ & (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n\theta_2\|z_n - z^*\| \\
\leq \ & (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n\theta_2[\|x_n - x^*\| \\
& +(1-\tau_n)\|x^* - z^*\|] \\
\leq \ & \|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| + \theta_2(1-\tau_n)\|x^* - z^*\|.
\end{aligned} \qquad (3.9)$$

Besides, combining (3.1), (3.2), (3.3) with (3.9), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
\leq \ & (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| \\
\leq \ & (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1[\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| \\
& +\theta_2(1-\tau_n)\|x^* - z^*\|] \\
\leq \ & (1-\alpha_n(1-\theta_1))\|x_n - x^*\| + \theta_1(1-\beta_n)\|x^* - y^*\| + \theta_1\theta_2(1-\tau_n)\|x^* - z^*\|.
\end{aligned} \qquad (3.10)$$

Taking $a_n = \|x_n - x^*\|$, $\lambda_n = \alpha_n(1-\theta_1)$, $b_n = 0$ and $c_n = \theta_1(1-\beta_n)\|x^* - y^*\| + \theta_1\theta_2(1-\tau_n)\|x^* - z^*\|$ in Lemma 1.2, we know that all conditions in Lemma 2.2 are satisfied, so $\|x_n - x^*\| \to 0$ as $(n \to \infty)$. Apply the Lemma 1.2 to (3.9) and (3.8), we can get $\|y_n - y*\| \to 0$ and $\|z_n - z*\| \to 0$ as well as $\|x_n - x^*\| \to 0$(as $n \to \infty$). This completes the proof. □

Remark 3.1. Theorem 3.1 extends and improves the results of Theorem 3.1 in S.S.Chang[8] and others.

The following Corollary can be obtained from Theorem 3.1 immediately.

Corollary 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \times K \to H$ be two-varible relaxed (γ, r)-cocoercive and μ-Lipschitzian in the first variable. Suppose that $x^*, y^* \in K \times K$ is a solution to the SNVI problem (1.6), (1.7) and that $\{x_n\}, \{y_n\}$ are the sequences generated by*

Algorithm 2.3. If $\{\alpha_n\}, \{\beta_n\}$ *are two sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty}(1-\beta_n) < \infty$;
(iii) $0 < \rho, \eta < 2(r - \gamma\mu^2)/\mu^2$;
(iv) $r > \gamma\mu^2$,

then the sequences $\{x_n\}, \{y_n\}$ *converge strongly to* x^*, y^* *respectively.*

Theorem 3.2 *Let* K *be a nonempty closed convex subset of a real Hilbert space* H *and* $T_i : K \times K \times K \to H$ *be three-varible relaxed* (γ_i, r_i)*-cocoercive and* μ_i*-Lipschitzian in the first variable, respectively(i=1,2,3). Suppose that* $x^*, y^*, z^* \in K \times K \times K$ *is a solution to the SNVID1 problem (1.1), (1.2) and (1.3) and that* $\{x_n\}, \{y_n\}, \{z_n\}$ *are the sequences generated by Algorithm 2.2. If* $\{\alpha_n\}, \{\beta_n\}$ *are two sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty}(1-\beta_n) < \infty$;
(iii) $0 < \rho, \eta, \lambda < min\{2(r_1 - \gamma_1\mu_1^2)/\mu_1^2,\ 2(r_2 - \gamma_2\mu_2^2)/\mu_2^2,\ 2(r_3 - \gamma_3\mu_3^2)/\mu_3^2\}$;
(iv) $r_1 > \gamma_1\mu_1^2,\ r_2 > \gamma_2\mu_2^2,\ r_3 > \gamma_3\mu_3^2$,

then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ *converge strongly to* x^*, y^*, z^* *respectively.*

Proof. The beginning several steps of the proof process in this Theorem is similar with the proof of Theorem 3.1. Hence the process will refine for these steps.

To get the result, we need first calculate $\|x_{n+1} - x^*\|$ for all $n \geq 0$. From (2.2),(1) and the nonexpansive property of $P_K S_K$, we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ =&\|(1-\alpha_n)x_n + \alpha_n P_K S_K[y_n - \rho T_1(y_n, z_n, x_n)] \\ &- (1-\alpha_n)x^* - \alpha_n P_K S_K[y^* - \rho T_1(y^*, z^*, x^*)]\| \\ \leq&(1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|. \end{aligned} \tag{3.11}$$

Set $\theta_1 = [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]^{1/2}$, $\theta_2 = [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]^{1/2}$, $\theta_3 = [1 + 2\lambda\gamma_3\mu_3^2 - 2\lambda r_3 + \lambda^2\mu_3^2]^{1/2}$, it is clear from condition (iii) that $0 \leq \theta_i < 1$ (i=1,2,3).

Using the relaxed (γ_i, r_i)-cocoercive and μ_i-Lipschitzian definition in the first variable on T_i, we obtain

$$\|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| \leq \theta_1\|y_n - y^*\|. \tag{3.12}$$

$$\|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\| \leq \theta_2\|z_n - z^*\|. \tag{3.13}$$

$$\|x_{n+1} - x^* - \lambda[T_3(x_{n+1}, y_n, z_n) - T_3(x^*, y^*, z^*)]\| \leq \theta_3\|x_{n+1} - x^*\|. \tag{3.14}$$

Hence from (3), (2.2) and (3.14), we can make an estimate for $\|z_{n+1} - z^*\|$.

$$\begin{aligned} &\|z_{n+1} - z^*\| \\ =& \|P_K S_K[x_{n+1} - \lambda T_3(x_{n+1}, y_n, z_n)] - P_K S_K[x^* - \lambda T_3(x^*, y^*, z^*)]\| \\ \leq& \|x_{n+1} - x^* - \lambda[T_3(x_{n+1}, y_n, z_n) - T_3(x^*, y^*, z^*)]\| \\ \leq& \theta_3\|x_{n+1} - x^*\|, \end{aligned} \tag{3.15}$$

which implies that for all $n \geq 1$,

$$\|z_n - z^*\| \leq \theta_3 \|x_n - x^*\|. \tag{3.16}$$

From (2), (2.2), (3.13) and (3.16),we can make an estimate for $\|y_{n+1} - y^*\|$.

$$\begin{aligned} & \|y_{n+1} - y^*\| \\ = \;& \|(1-\beta_n)x_n + \beta_n P_K S_K[z_n - \eta T_2(z_n, x_n, y_n)] \\ & -(1-\beta_n)y^* - \beta_n P_K S_K[z^* - \eta T_2(z^*, x^*, y^*)]\| \\ \leq \;& (1-\beta_n)\|x_n - y^*\| + \beta_n \|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\| \\ \leq \;& (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n \theta_2 \|z_n - z^*\| \\ \leq \;& (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n \theta_2 \theta_3 \|x_n - x^*\| \\ \leq \;& \|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\|. \end{aligned} \tag{3.17}$$

Besides, combining (3.11), (3.12) with (3.17), we have

$$\begin{aligned} & \|x_{n+1} - x^*\| \\ \leq \;& (1-\alpha_n)\|x_n - x^*\| + \alpha_n \theta_1 \|y_n - y^*\| \\ \leq \;& (1-\alpha_n)\|x_n - x^*\| + \alpha_n \theta_1 [\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\|] \\ \leq \;& (1-\alpha_n(1-\theta_1))\|x_n - x^*\| + \theta_1(1-\beta_n)\|x^* - y^*\|. \end{aligned} \tag{3.18}$$

Taking $a_n = \|x_n - x^*\|$, $\lambda_n = \alpha_n(1-\theta_1)$, $b_n = 0$ and $c_n = \theta_1(1-\beta_n)\|x^* - y^*\|$ in Lemma 1.2, we know that all conditions in Lemma 2.2 are satisfied, so $\|x_n - x^*\| \to 0$ as $(n \to \infty)$. Apply the Lemma 1.2 to (3.17) and (3.16), we can get $\|y_n - y*\| \to 0$ and $\|z_n - z*\| \to 0$ as well as $\|x_n - x^*\| \to 0$(as $n \to \infty$). This completes the proof.

Remark 3.2. Theorem 3.2 extends the solvability of (1.4) and (1.5) to the more general form. Besides, Theorem 3.2 improves the results of Theorem 3.1 in Huang and Noor et. al[9].

The following Corollary can be obtained from Theorem 3.2 immediately.

Corollary 3.2. *Let K be a nonempty closed convex subset of a real Hilbert space H and $T_i : K \times K \to H$ be two-varible relaxed (γ_i, r_i)-cocoercive and μ_i-Lipschitzian in the first variable, respectively(i=1,2). Suppose that $x^*, y^* \in K \times K$ is a solution to the SNVID2 problem (1.4), (1.5) and that $\{x_n\}, \{y_n\}$ are the sequences generated by Algorithm 2.4. If $\{\alpha_n\}$ are two sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \rho, \eta < min\{2(r_1 - \gamma_1\mu_1^2)/\mu_1^2,\ 2(r_2 - \gamma_2\mu_2^2)/\mu_2^2\}$;

(iii) $r_1 > \gamma_1\mu_1^2,\ r_2 > \gamma_2\mu_2^2$,

then the sequences $\{x_n\}, \{y_n\}$ converge strongly to x^*, y^* respectively.

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