J. Appl. Math. & Informatics Vol. **27***(2009), No. 1 - 2, pp. 427 - 430*
Website: http://www.kcam.biz

ON THE INTERMEDIATE DIFFERENTIABILITY OF LIPSCHITZ MAPS BETWEEN BANACH SPACES

CHOON HO LEE

ABSTRACT. In this paper we introduce the intermediate differential of a Lipschitz map from a Banach space to another Banach space and prove that every locally Lipschitz function f defined on an open subset Ω of a superreflexive real Banach space X to a finite dimensional Banach space Y is uniformly intermediate differentiable at every point $\Omega\backslash A$, where A is a σ-lower porous set.

AMS Mathematics Subject Classification : Primary 26A36; Secondary 26A24, 28E99
Key words and phrases : The intermediate defferentiability, Banach spaces, Lipschitz functions, porous sets.

1. Introduction

In this note we will prove the intermediate differentiability of Lipschitz maps between Banach spaces. Several authors([2], [3], [5] and therein) investigate the intermediate differentiability of Lipschitz maps defined on Banach space. Let X be a super-reflexive real Banach space, Y a finite dimensional real Banach space. The open ball with center x and radius r is denote by $B(x,r)$. If f is a Lipschitz function, then the Lipschitz constant of f is denote by $\mathrm{Lip}(f)$.

If f be a function on X into Y and x, $v \in X$, then we introduce the *upper* and *lower(one-side) directional derivatives*

$$\langle \bar{d}f(x,v), y^* \rangle = \limsup_{t\to 0+} \left\langle \frac{f(x+tv)-f(x)}{t}, y^* \right\rangle,$$
$$\langle \underline{d}f(x,v), y^* \rangle = \liminf_{t\to 0+} \left\langle \frac{f(x+tv)-f(x)}{t}, y^* \right\rangle$$

Received May 4, 2008. Accepted September 20, 2008.
Supported by Hoseo University grant 2007.

for each $y^* \in Y^*$. We denote $L(X,Y)$ by the space of all linear maps from X to Y. We say that $m \in L(X,Y)$ is an *intermediate derivative* of a function $f : X \to Y$ at a point $x \in X$ if for each $y^* \in Y^*$ the inequality

$$\langle \underline{d}f(x,v), y^* \rangle \le \langle m(v), y^* \rangle \le \langle \overline{d}f(x,v), y^* \rangle$$

holds for every $v \in X$. Clearly, if f has at x the Gâteaux derivative, then it has also the (unique) intermediate derivative. Therefore Aronszajn's differentiability theorem implies that every (locally) Lipschitz function on a separable Banach space has an intermediate derivative at all points except a set E which is null in Aronszajn's sense.

In case Y is a real scalar, Fabian and Preiss[2], and Zajíček[5] have obtained the results on the intermediate differentiability of Lipschitz maps on the super-reflexive banach space. We extend their result to a finite dimensional Banach space Y. Similarly, we intrduce a uniform intermediate derivative of a Lipschitz map $f : X \to Y$.

Definition 1. A function f defined on an open subset Ω of a Banach space X to a Banach space Y is said to be *uniformly intermediate differentiable* at $x \in X$ if there exists a linear function $m : X \to Y$ and a sequence $t_n \searrow 0$ such that for each $y^* \in Y^*$

$$\lim_{n\to\infty} \left\langle \frac{f(x+t_n v) - f(x)}{t_n} - m(v), y^* \right\rangle = 0$$

for each direction $v \in X$ with $\|v\| = 1$.

2. Main results

The following theorem is due to [1]. We modify their theorem as follows:

Theorem 1. *Let X be nonzero super-reflexive Banach space and Y a finite-dimensional Banach space. Then for each $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that for every ball $B(x,\rho)$ in X and every Lipschitz function $f : B(x,\rho) \subset X \to Y$ there exists a ball $B(y,r) \subset B(x,\rho)$ and an affine function $a : X \to Y$ such that $r \ge c\rho$ and*

$$\|f(z) - a(z)\| \le \varepsilon r \mathrm{Lip}(f) \text{ for each } z \in B(y,r).$$

Let X ba a normed linear space and $A \subset X$. We introduce some terminology via Zajíček[6]: A is *σ-lower porous* if $A = \bigcup_{n=1}^{\infty} P_n$, where each set $P := P_n$ has the following property:

$$\exists \alpha > 0\, \forall x \in X\, \forall r > 0\, \exists y \in X \text{ such that } B(y, \alpha r) \subset B(x,r)\backslash P. \qquad (1)$$

Theorem 2. *Let X be a super-reflexive Banach space. Then every locally Lipschitz function f defined on an open subset Ω of X to a finite dimensional Banach space Y is uniformly intermediate differentiable at every point $\Omega\backslash A$, where A is a σ-lower porous set.*

3. Proof of Theorem 2

Let G_n be the union of all balls $B(c,r) \subset \Omega$ such that $r < 1/n$ and there exists an affine function a on X for which $\|f(z) - a(z)\| \le r/n$ where $z \in B(c, 2r)$. Put $P_n = \Omega \backslash G_n$ and $A = \bigcup_{n=1}^{\infty} P_n$. It is sufficient to prove that

(i) each P_n is σ-lower porous and

(ii) f has a uniform intermediate derivative at each point of $\Omega \backslash A = \bigcap_{n=1}^{\infty} G_n$.

For (i), it is sufficient to prove that each P_n is lower porous at each point $x \in \Omega$. To prove this choose n, x and find $\delta > 0$, $K > 0$ such that $B(x,\delta) \subset \Omega$, $\delta < 1/n$ and f is Lipschitz with constant K on $B(x,\delta)$. Now find $c = c(\frac{1}{2nK})$ by Theorem 1 and consider an arbitrary $0 < \rho < \delta$. By the choice of c there exists a ball $B(y,r) \subset B(x,\rho)$ and an affine function a on X such that $r \ge c\rho$ and

$$\|f(z) - a(z)\| \le \frac{1}{2nK} rK = \frac{r}{2n} \quad \text{for each } z \in B(y,r).$$

Therefore $B(y, r/2) \subset G_n$ and we see that P_n is lower porous at x (with $\eta = c/2$).

To prove (ii), suppose that $z \in \bigcap_{n=1}^{\infty} G_n$ is given. Then there exist sequences $(B(c_n, r_n))$ of balls and (a_n) of affine functions on X such that $0 < r_n < 1/n$, $z \in B(c_n, r_n)$ and

$$\|f(y) - a_n(y)\| < r_n/n \quad \text{for each } y \in B(c_n, 2r_n). \tag{2}$$

Let $a_n(t) = q_n + m_n(t)$, where $q_n \in Y$ and m_n be a linear function of X to Y. If $v \in X$, $\|v\| = 1$, then (1) implies that for any $\|y^*\| \le 1$

$$\begin{aligned} &\left|\left\langle \frac{f(z + r_n v) - f(z)}{r_n} - m_n(v), y^* \right\rangle\right| \\ &= \left|\left\langle \frac{f(z + r_n v) - f(z)}{r_n} - \frac{a_n(z + r_n v) - a_n(z)}{r_n}, y^* \right\rangle\right| \\ &\le \left|\left\langle \frac{f(z + r_n v) - a(z + r_n v)}{r_n}, y^* \right\rangle\right| + \left|\left\langle \frac{f(z) - a_n(z)}{r_n}, y^* \right\rangle\right| \\ &< \frac{2}{n} \end{aligned} \tag{3}$$

Since f is locally Lipschitz, there exist $L > 0$ and $n_0 \in \mathbb{N}$ such that $|\langle m_n(v), y^* \rangle| < L + \frac{2}{n}$ whenever $n \ge n_0$ and $\|v\| = 1$. Therefore $\{m_n\}_{n=n_0}^{\infty}$ is an uniformly bounded sequence in the space $L(X,Y)$ with the uniform operator topology. Because of the uniform boundedness of $\{m_n\}$ in $L(X,Y)$ we can choose a subsequence $\{m_{n_k}\}_{k=1}^{\infty}$ and $x \in L(X,Y)$ such that for all $v \in X$, $m_{n_k}(v)$ coverges to $m(v)$ weakly in Y. Put $t_k := r_{n_k}$. Then (3) and the weak convergence of $\{m_{n_k}(v)\}$ imply that

$$\frac{f(z + t_k v) - f(z)}{t_k} \to m(v) \text{ weakly in } Y$$

for each $v \in X$, $\|v\| = 1$, which completes the proof.

REFERENCES

1. S. Bates, W. B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Affine Approximation of Lipschitz Functions and Nonlinear Quotients*, GAFA, Geom. Funct. Anal. 9(1999), 1092-1127
2. M. Fabian and D. Preiss, *On Intermediate Differentiability of Lipschitz functions on Certain Banach spaces*, Proc. Amer. Math. Soc. 113(1991), 733-740
3. J. Giles and S. Schiffer, *Generalizing generic differentiability properties from convex to locally Lipschitz functions*, J. Math. Anal. Appl. 188(1994), 833–854
4. J. Lindenstrauss and D. Priess, *On Fréchet Differentiability of Lipschitz maps betweeen Banach spaces*, Ann. of Math. 157(2003), 257–288
5. L. Zajíček, *A note on intermediate differentiability of Lipschitz functions*, Comment. Math. Univ. Carolinae 40(1999)795–799
6. L. Zajíček, *On σ-porous sets in Abstract spaces*, Abst. Appl. Analysis 2005(2005), 509–534

Choon Ho Lee received his BS from Sogang University and Ph.D at Seoul national University under the direction of Jongsik Kim. His research interests focus on the Partial Differential Equations, Geometric Measure Theory and Mathematics Education.

Department of Mathematics and Institute of Basic Sicences, Hoseo University, ChoongNam 336-795, Korea
e-mail: `chlee@hoseo.edu`