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INTUITIONISTIC FUZZY AUTOMATA AND INTUITIONISTIC FUZZY REGULAR EXPRESSIONS

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ABSTRACT. A definition of finite automaton (DFA and NDFA) with intuitionistic fuzzy (final) states is proposed. Acceptance of intuitionistic fuzzy regular language by the finite automaton (DFA and NDFA) with intuitionistic fuzzy (final) states are examined. It is found that the finite automaton (DFA and NDFA) with intuitionistic fuzzy (final) states is more suitable for recognizing intuitionistic fuzzy regular language than earlier model. The paper also gives an idea of intuitionistic fuzzy regular expressions through possible definitions.

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1. Introduction

The evolution of fuzzy sets[8] is a milestone in the theory of formal languages. Fuzziness reduces the gap between formal language and natural language in terms of precision. This leads to what is described as fuzzy languages[3]. Fuzzy regular language[6] can be obtained from fuzzy languages if its strings are finite and regular with finite membership value between [0, 1]. Using the notion of intuitionistic fuzzy sets [1, 2] it is possible to obtain intuitionistic fuzzy language[7] by introducing the nonmembership value to the strings of fuzzy language. This is a natural generalization of a fuzzy language as it is characterized by two functions expressing the degree of belongingness and the degree of nonbelongingness. Intuitionistic fuzzy language further reduces the gap between formal language and natural language. An intuitionistic fuzzy language is called intuitionistic fuzzy regular language if its strings are regular having the finite membership and nonmembership values between [0, 1][7].

In the following, the definition of intuitionistic fuzzy set, intuitionistic fuzzy language, intuitionistic fuzzy regular language with some operations are given

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briefly. The finite automaton (NDFA and DFA) with intuitionistic fuzzy transitions is stated. The definition of finite automaton (NDFA and DFA) with intuitionistic fuzzy (final) states is proposed. Two theorems related to intuitionistic fuzzy regular language have been proved. Intuitionistic fuzzy regular expressions have been discussed in the final phase.

2. Preliminaries

Definition 2.1. Let a set 'E' be fixed. An intuitionistic fuzzy set 'A' in 'E' is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in E\}$ where, the functions $\mu_A(x) : E \to [0,1]$ and $\gamma_A(x) : E \to [0,1]$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to the set 'A', the subset of 'E' respectively, and for every $x \in E$; $0 \le \mu_A(x) + \gamma_A(x) \le 1$[1].

Definition 2.2. Let Σ be a set of finite alphabet and $f : \Sigma^* \to M$, $g : \Sigma^* \to N$ functions, where M and N are finite set of real numbers in [0,1]. Then we call the set, $L = \{(w, f(w), g(w)) \mid w \in \Sigma^*\}$ an intuitionistic fuzzy language (in short IFL) over Σ [7]. Here $f(w)$ and $g(w)$ represents the membership and the nonmembership functions of 'L' respectively, such that $0 \le f(w) + g(w) \le 1$.

Let L_1 and L_2 be two IFLs over the finite alphabet set Σ. Let f_1, g_1 and f_2, g_2 represents the membership and the nonmembership functions of L_1 and L_2 respectively. Then the following operations will hold.

Union: $L = L_1 \cup L_2 = \{(w, \max\{f_1(w), g_1(w)\}, \min\{f_2(w), g_2(w)\}) \mid w \in \Sigma^*\}$.

Intersection: $L = L_1 \cap L_2 = \{(w, \min\{f_1(w), g_1(w)\}, \max\{f_2(w), g_2(w)\}) \mid w \in \Sigma^*\}$.

Complement: $\tilde{L}_1 = \{(w, 1 - f_1(w), 1 - f_2(w)) \mid w \in \Sigma^*\}$.

Concatenation:

$$\begin{aligned} L &= L_1 \cdot L_2 \\ &= \{(w, \max\{\min(f_1(x), g_1(y))\}, \min\{\max(f_2(x), g_2(y))\}) \mid w = xy\}. \end{aligned}$$

Star:

$L_1^* = \{(w, \max\{\min(f_{L_1}(x_1), ..., f_{L_1}(x_n))\}, \min\{\max(g_{L_1}(x_1), ..., g_{L_1}(x_n))\}) \mid w = x_1...x_n,\ x_1, ..., x_n \in \Sigma^*, n \ge 0, w \in \Sigma^*\}$, assuming that, min $\Phi = 1$ and max $\Phi = 0$.

Hence, we have $L^* = \bigcup_{i=0}^{\infty} L^i = L^0 \cup L^1 \cup L^2 \cup \ldots$. i.e., Kleene Closure is satisfied.

The language for the operation + on L_1 is L_1^+ which is defined as,

$L_1^+ = \{(w, \max\{\min(f_{L_1}(x_1), ..., f_{L_1}(x_n))\}, \min\{\max(g_{L_1}(x_1), ..., g_{L_1}(x_n))\}) \mid w = x_1...x_n,\ x_1, ..., x_n \in \Sigma^*, n \ge 1, w \in \Sigma^*\}$.

Hence, we have $L^+ = \bigcup_{i=1}^{\infty} L^i$.

The equivalence and inclusion relations between two intuitionistic fuzzy languages are the equivalence and inclusion relations between two intuitionistic fuzzy sets, since intuitionistic fuzzy languages are a special class of intuitionistic fuzzy sets.

Let L_1 and L_2 be two intuitionistic fuzzy languages defined over Σ the finite alphabet set. Then,
$L_1 = L_2$ iff $f_1(w) = g_1(w)\ \forall\ w \in \Sigma^*$ and $f_2(w) = g_2(w)\ \forall\ w \in \Sigma^*$ and
$L_1 \subseteq L_2$ iff $f_1(w) \leq g_1(w)\ \forall\ w \in \Sigma^*$ and $f_2(w) \geq g_2(w)\ \forall\ w \in \Sigma^*$.

Let 'L' be an IFL over Σ the finite alphabet set and $f_L : \Sigma^* \to M$, $g_L : \Sigma^* \to N$ represents the membership and the nonmembership functions of 'L' respectively. Then for each $m \in M$, denote by $S_L(m)$ the set, $S_L(m) = \{w \in \Sigma^* \mid f_L(w) = m\}$ and for each $l \in N$, denote by $S_L(l)$ the set, $S_L(l) = \{w \in \Sigma^* \mid g_L(w) = l\}$. Note that $S_L(m) = S_L(l)$.

Definition 2.3. Let 'L' be an IFL over Σ the finite alphabet set and $f_L : \Sigma^* \to M$, $g_L : \Sigma^* \to N$ represents the membership and the nonmembership functions of 'L' respectively. We call 'L' an intuitionistic fuzzy regular language (in short IFRL) [7] if

i) the sets $\{m \in M \mid S_L(m) \neq \Phi\}$ and $\{l \in N \mid S_L(l) \neq \Phi\}$ are finite, also,

ii) for each $m \in M$ the string $S_L(m)$ and for each $l \in N$ the string $S_L(l)$ are regular.

Theorem 2.1. *Intuitionistic fuzzy regular languages are closed under union, intersection, complement, concatenation and star operations* [7].

Proof. Let L_1 and L_2 be two intuitionistic fuzzy regular languages over Σ the finite alphabet set. Let $f_{L_1} : \Sigma^* \to M_1, g_{L_1} : \Sigma^* \to N_1$ and $f_{L_2} : \Sigma^* \to M_2, g_{L_2} : \Sigma^* \to N_2$ represents the membership and the nonmembership functions of L_1 and L_2 respectively.

Let L be the resulting language after the operation (union,....,star) and $f_L : \Sigma^* \to M, g_L : \Sigma^* \to N$ the membership and the nonmembership functions of L respectively. Then obviously, $M \subseteq M_1 \cup M_2$ and $N \subseteq N_1 \cup N_2$ (in the case of union, intersection, concatenation and star operation) or $M = \{1-m \mid m \in M_1\}$ and $N = \{1 - n \mid n \in N_1\}$(in the case of complement) are finite for which the corresponding strings are regular. Let m and n be finite values in M and N respectively. Then $S_L(m)$ and $S_L(n)$ are defined as follows.

Union:

$$S_L(m) = \begin{cases} S_{L_1}(m) - \cup_{m'>m} S_{L_2}(m') & \text{if m} \in M_1 - M_2 \\ S_{L_2}(m) - \cup_{m'>m} S_{L_1}(m') & \text{if m} \in M_2 - M_1 \\ ((S_{L_1}(m) \cup S_{L_2}(m)) - \cup_{m'>m} S_{L_1}(m')) & \\ - \cup_{m''>m}\, S_{L_2}(m'') & \text{if m} \in M_1 \cap M_2 \end{cases}$$

$$S_L(n) = \begin{cases} S_{L_1}(n) - \cup_{n'<n} S_{L_2}(n') & \text{if n} \in N_1 - N_2 \\ S_{L_2}(n) - \cup_{n'<n} S_{L_1}(n') & \text{if n} \in N_2 - N_1 \\ ((S_{L_1}(n) \cup S_{L_2}(n)) & \\ - \cup_{n'<n}\, S_{L_1}(n')) - \cup_{n''<n} S_{L_2}(n'') & \text{if n} \in N_1 \cap N_2 \end{cases}$$

Intersection:

$$S_L(m) = \begin{cases} S_{L_1}(m) - \cup_{m'<m} S_{L_2}(m') & \text{if m} \in M_1 - M_2 \\ S_{L_2}(m) - \cup_{m'<m} S_{L_1}(m') & \text{if m} \in M_2 - M_1 \\ ((S_{L_1}(m) \cup S_{L_2}(m)) - \cup_{m'<m} S_{L_1}(m')) & \\ - \cup_{m''<m} S_{L_2}(m'') & \text{if m} \in M_1 \cap M_2 \end{cases}$$

$$S_L(n) = \begin{cases} S_{L_1}(n) - \cup_{n'>n} S_{L_2}(n') & \text{if n} \in N_1 - N_2 \\ S_{L_2}(n) - \cup_{n'>n} S_{L_1}(n') & \text{if n} \in N_2 - N_1 \\ ((S_{L_1}(n) \cup S_{L_2}(n)) - \cup_{n'>n} S_{L_1}(n')) & \\ - \cup_{n''>n} S_{L_2}(n'') & \text{if n} \in N_1 \cap N_2 \end{cases}$$

Complement: $M = \{1 - m \mid m \in M_1\}$ *and* $N = \{1 - n \mid n \in N_1\}$, $S_L(m) = S_{L_1}(1 - m)$ *and* $S_L(n) = S_{L_1}(1 - n)$.

Concatenation:

$$S_L(m) = \cup_{\substack{min(m_1,m_2)=m \\ m_1 \in M_1 \\ m_2 \in M_2}} S_{L_1}(m_1) S_{L_2}(m_2) - \cup_{\substack{min(m_1',m_2')>m \\ m_1' \in M_1 \\ m_2' \in M_2}} S_{L_1}(m_1') S_{L_2}(m_2').$$

$$S_L(n) = \cup_{\substack{min(n_1,n_2)=n \\ n_1 \in N_1 \\ n_2 \in N_2}} S_{L_1}(n_1) S_{L_2}(n_2) - \cup_{\substack{min(n_1',n_2')<n \\ n_1' \in N_1 \\ n_2' \in N_2}} S_{L_1}(n_1') S_{L_2}(n_2').$$

Star: Assuming that $M = \{m_1, m_2, ..., m_n\}$ and $1 \geq m_1 > m_2 > ... > m_n \geq 0$,

$N = \{n_1, n_2, ...n_n\}$ and $0 \leq n_1 < n_2 < ... < n_n \leq 1$,

$S_L(m_1) = (S_{L_1}(m_1))^*$ if $m_1 = 1$,

$S_L(1) = \{\lambda\}$ and $S_L(m_1) = (S_{L_1}(m_1))^+ - \{\lambda\}$ if $m_1 \neq 1$,

$S_L(m_i) = (\cup_{j \leq i} S_{L_1}(m_j))^+ - \cup_{k<i} S_L(m_k) - \{\lambda\}, 1 < i \leq n$,

$S_L(n_1) = (S_{L_1}(n_1))^*$ if $n_1 = 0$,

$S_L(0) = \{\lambda\}$ and $S_L(n_1) = (S_{L_1}(n_1))^+ - \{\lambda\}$ if $n_1 \neq 0$,

$S_L(n_i) = (\cup_{j \geq i} S_{L_1}(n_j))^+ - \cup_{k>i} S_L(n_k) - \{\lambda\}, 1 < i \leq n$.

Hence, Kleene Closure is satisfied. □

Definition 2.4. A nondeterministic finite automaton $\tilde{A}$ with intuitionistic fuzzy transitions (in short NDFA-IFT) is a 6-tuple $\tilde{A} = (Q, \Sigma, \tilde{\delta}, \tilde{\gamma}, s, F)$ where, Q is the set of finite states, Σ is the finite set of input alphabets, $\tilde{\delta}$ and $\tilde{\gamma}$ are the fuzzy subsets of $Q \times \Sigma \times Q$ called the membership and the nonmembership functions of '$\tilde{A}$' respectively. s is the starting state and $F \subseteq Q$ the set of final states [7]. For $x \in \Sigma^*$ and p, $q \in Q$ define,

$$\tilde{\delta}^*(p, x, q) = \begin{cases} 0 \text{ if } x = \lambda \text{ and p} \neq \text{q} \\ 1 \text{ if } x = \lambda \text{ and p=q} \\ \max_{r \in Q}\{\min(\tilde{\delta}^*(p, x', r), \tilde{\delta}(r, a, q)) \mid x = x'a \ \forall \ x' \in \Sigma^*, a \in \Sigma\}, \\ \text{otherwise.} \end{cases}$$

$$\tilde{\gamma}^*(p,x,q) = \begin{cases} 1 \text{ if } x = \lambda \text{ and p}\neq\text{q} \\ 0 \text{ if } x = \lambda \text{ and p=q} \\ \min_{r\in Q}\{\max(\tilde{\gamma}^*(p,x',r),\tilde{\gamma}(r,a,q)) \mid x = x'a\ \forall\ x' \in \Sigma^*, a \in \Sigma\}, \\ \text{otherwise.} \end{cases}$$

Then we say that $x \in \Sigma^*$ is accepted by '$\tilde{A}$' with a degree $d_{\tilde{A}}(x)$ and a nondegree $n_{\tilde{A}}(x)$ such that $0 \leq d_{\tilde{A}}(x) + n_{\tilde{A}}(x) \leq 1$. Where, $d_{\tilde{A}}(x) = \max\{\tilde{\delta}^*(s,x,q) \mid q \in F\}$ and $n_{\tilde{A}}(x) = \min\{\tilde{\gamma}^*(s,x,q) \mid q \in F\}$. We denote it by $L(\tilde{A})$ and is given by $L(\tilde{A}) = \{(x, d_{\tilde{A}}(x), n_{\tilde{A}}(x) \mid x \in \Sigma^*\}$.

Definition 2.5. A deterministic finite automaton with intuitionistic fuzzy transitions (in short DFA-IFT) is $\tilde{A} = (Q,\Sigma,\tilde{\delta},\tilde{\gamma},s,F)$. It is a NDFA-IFT with the condition that for each p, q $\in$ Q and a $\in \Sigma$, if $\tilde{\delta}(p,a,q) > 0, \tilde{\gamma}(p,a,q) < 1$ and $\tilde{\delta}(p,a,q') > 0, \tilde{\gamma}(p,a,q') < 1$ then $q = q'$ [7].

Theorem 2.2. *'L' is an intuitionistic fuzzy regular language iff it is accepted by a NDFA-IFT with the exception of* λ [7].

Theorem 2.3. *Let 'L' be an IFRL then, 'L' is accepted by a DFA-IFT iff it satisfies the following conditions: For* $x, y \in \Sigma^+$, $u \in \Sigma^*, x = yu$ *and* $\tilde{\delta}(y) > 0$, $\tilde{\gamma}(y) < 1$ *implies that,* $\tilde{\delta}(x) \leq \tilde{\delta}(y)$ *and* $\tilde{\gamma}(x) \geq \tilde{\gamma}(y)$ [7].

3. Finite automaton with intuitionistic fuzzy (final) states

In this section, the definition of nondeterministic and deterministic finite automaton with intuitionistic fuzzy (final) states (in short NDFA-IFS and DFA-IFS) is proposed and two theorems have been discussed for the acceptance of intuitionistic fuzzy regular language through them. Unlike previous models (NDFA-IFT and DFA-IFT, proposed in [7] and given in section 2) in this model, NDFA-IFS and DFA-IFS are equivalent in accepting intuitionistic fuzzy regular language. In this model intuitionistic fuzzy regular language is accepted by NDFA-IFS and DFA-IFS without any restrictions and vice versa. So, this model is more suitable for the study of intuitionistic fuzzy regular languages.

Definition 3.1. A nondeterministic finite automaton with intuitionistic fuzzy (final) states (in short NDFA-IFS) '$\tilde{A}$' is a 7-tuple $\tilde{A} = (Q,\Sigma,\delta,\gamma,s,\tilde{F}_{1\tilde{A}},\tilde{F}_{2\tilde{A}})$ where, Q is the finite set of states, Σ is the finite set of input alphabets, δ and γ are the transition functions $Q \times \Sigma :\longrightarrow 2^Q$, s is the intuitionistic fuzzy starting state and $\tilde{F}_{1\tilde{A}}$, $\tilde{F}_{2\tilde{A}}$: $Q \rightarrow [0,1]$ called respectively the membership and the nonmembership functions of intuitionistic fuzzy (final) state set.

Define,

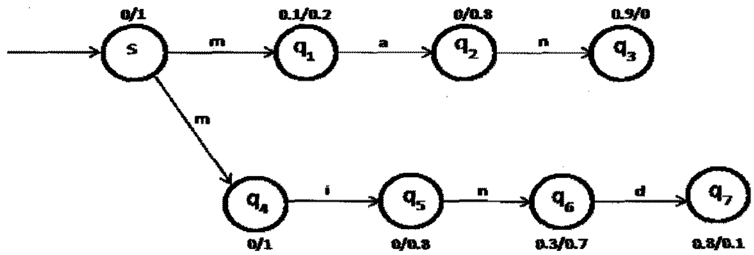
$d_{\tilde{A}}(x) = \max\{\tilde{F}_{1\tilde{A}}(q) \mid (s,x,q) \in \delta^*\}$ and

$n_{\tilde{A}}(x) = \min\{\tilde{F}_{2\tilde{A}}(q) \mid (s,x,q) \in \gamma^*\}$

where, $\delta^* : Q \times \Sigma^* \rightarrow 2^Q$ and $\gamma^* : Q \times \Sigma^* \rightarrow 2^Q$ called the reflexive and transitive closure of δ and γ respectively.

The string 'x' is accepted by '$\tilde{A}$' with degree $d_{\tilde{A}}(x)$ and nondegree $n_{\tilde{A}}(x)$ respectively with the condition $0 \leq d_{\tilde{A}}(x) + n_{\tilde{A}}(x) \leq 1$.

The intuitionistic fuzzy language accepted by '$\tilde{A}$' denoted $L(\tilde{A})$, is the set, $\{(x, d_{\tilde{A}}(x), n_{\tilde{A}}(x)) \mid x \in \Sigma^*\}$.



Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, s, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ be a NDFA-IFS with $\Sigma = \{a, d, i, m, n\}$, $Q = \{s, q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$, s the intuitionistic fuzzy starting state with membership $\tilde{F}_{1\tilde{A}}(s) = 0$ and nonmembership $\tilde{F}_{2\tilde{A}}(s) = 1$, δ and γ the transition functions $Q \times \Sigma :\rightarrow 2^Q$ with, $\delta(s, m) = q_1$, $\gamma(s, m) = q_1$, $\delta(s, m) = q_4$, $\gamma(s, m) = q_4$, $\delta(q_1, a) = q_2$, $\gamma(q_1, a) = q_2$, $\delta(q_2, n) = q_3$, $\gamma(q_2, n) = q_3$, $\delta(q_4, i) = q_5$, $\gamma(q_4, i) = q_5$, $\delta(q_5, n) = q_6$, $\gamma(q_5, n) = q_6$, $\delta(q_6, d) = q_7$, $\gamma(q_6, d) = q_7$, and $\tilde{F}_{1\tilde{A}}$, $\tilde{F}_{2\tilde{A}}$: $Q \rightarrow [0, 1]$ the membership and the nonmembership functions of intuitionistic fuzzy (final) state set respectively with, $\tilde{F}_{1\tilde{A}}(q_1) = 0.1$, $\tilde{F}_{2\tilde{A}}(q_1) = 0.2$, $\tilde{F}_{1\tilde{A}}(q_2) = 0$, $\tilde{F}_{2\tilde{A}}(q_2) = 0.8$, $\tilde{F}_{1\tilde{A}}(q_3) = 0.9$, $\tilde{F}_{2\tilde{A}}(q_3) = 0$, $\tilde{F}_{1\tilde{A}}(q_4) = 0$, $\tilde{F}_{2\tilde{A}}(q_4) = 1$, $\tilde{F}_{1\tilde{A}}(q_5) = 0$, $\tilde{F}_{2\tilde{A}}(q_5) = 0.8$, $\tilde{F}_{1\tilde{A}}(q_6) = 0.3$, $\tilde{F}_{2\tilde{A}}(q_6) = 0.7$, $\tilde{F}_{1\tilde{A}}(q_7) = 0.8$, $\tilde{F}_{2\tilde{A}}(q_7) = 0.1$.

Then $d_{\tilde{A}}(\text{man}) = 0.9$ and $n_{\tilde{A}}(\text{man}) = 0$, $d_{\tilde{A}}(\text{mind}) = 0.8$ and $n_{\tilde{A}}(\text{mind}) = 0.1$ etc.

Definition 3.2. A deterministic finite automaton with intuitionistic fuzzy (final) states (in short DFA-IFS) $\tilde{A} = (Q, \Sigma, \delta, \gamma, s, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ is a NDFA-IFS with δ and γ being functions $Q \times \Sigma \rightarrow Q$ instead of a relation.

For each $x \in \Sigma^*$, $d_{\tilde{A}}(x) = \tilde{F}_{1\tilde{A}}(q)$ where, $q = \delta^*(s, x)$ and $n_{\tilde{A}}(x) = \tilde{F}_{2\tilde{A}}(q)$ where, $q = \gamma^*(s, x)$.

Define, $d_{\tilde{A}}(x) = 0$ and $n_{\tilde{A}}(x) = 1$ if $\delta^*(s, x)$ and $\gamma^*(s, x)$ are not defined.

Theorem 3.1. *Let 'L' be an intuitionistic fuzzy language. Then 'L' is an intuitionistic fuzzy regular language iff it is accepted by a DFA-IFS.*

Proof. Let 'L' be an intuitionistic fuzzy language with $f_L : \Sigma^* \rightarrow M$ and $g_L : \Sigma^* \rightarrow N$ as its membership and nonmembership functions having the finite set of real numbers M and N in [0,1].

Assume that 'L' is an intuitionistic fuzzy regular language. So, for each $m \in M$ and $l \in N$, the strings $S_L(m)$ and $S_L(l)$ are regular.

Assume that, $M = \{m_1, m_2, ..., m_n\}$ and $N = \{l_1, l_2, ..., l_n\}$. For each i, $1 \leq i \leq n$, we construct a DFA-IFS, $\tilde{A}_i = (Q_i, \Sigma, \delta_i, \gamma_i, s_i, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ Such that, $L(\tilde{A}_i) = S_L(m_i) = S_L(l_i)$.

Now define a DFA-IFS, $\tilde{A} = (Q, \Sigma, \delta, \gamma, s, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ to be the cross product of $\tilde{A}_i, 1 \leq i \leq n$, with

$$\tilde{F}_{1\tilde{A}}(q^{(1)}, q^{(2)}, ..., q^{(n)}) = \begin{cases} m_i & q^{(i)} \in F_i \text{ for some i,} 1 \leq i \leq n \text{ and } q^{(j)} \notin F_j \forall j \neq i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{F}_{2\tilde{A}}(q^{(1)}, q^{(2)}, ..., q^{(n)}) = \begin{cases} l_i & q^{(i)} \in F_i \text{ for some i,} 1 \leq i \leq n \text{ and } q^{(j)} \notin F_j \forall j \neq i \\ 1 & \text{otherwise} \end{cases}$$

Note that if $(q^{(1)}, q^{(2)}, ..., q^{(n)})$ is reachable from $(s_1, s_2, ..., s_n)$, for each $i, 1 \leq i \leq n$ in $\tilde{A}$, then it is not possible to get $q^{(i)} \in F_i$ and $q^{(j)} \in F_j$ for $i \neq j$ because $L(\tilde{A}_i) \cap L(\tilde{A}_j) = \Phi$ for $i \neq j$, $1 \leq i, j \leq n$. Hence $\tilde{A}$ accepts L.

For the reverse proof, Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, s, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ be a DFA-IFS. Define $M = \{m \mid \tilde{F}_{1\tilde{A}}(q) = m \text{ for some } q \in Q\}$ and $N = \{l \mid \tilde{F}_{2\tilde{A}}(q) = l \text{ for some } q \in Q\}$. Thus M and N are finite sets. For each $m \in M$ and $l \in N$,
define $A_{m,l} = (Q, \Sigma, \delta, \gamma, s, F_m, F_l)$ where $F_m = \{q \mid \tilde{F}_{1\tilde{A}}(q) = m\}$ and $F_l = \{q \mid \tilde{F}_{2\tilde{A}}(q) = l\}$.

Let $L = L(\tilde{A})$ i.e., $f_L = d_{\tilde{A}}$ and $g_L = n_{\tilde{A}}$. Thus for each $m \in M$ the string $S_L(m)$ and for each $n \in N$ the string $S_L(l)$ are regular and hence L is an IFRL. □

Theorem 3.2. *An intuitionistic fuzzy language is accepted by a NDFA-IFS iff it is accepted by a DFA-IFS.*

Proof. Here we have to show if '$\tilde{A}$' is a NDFA-IFS and $L = L(\tilde{A})$ then $L = L(\tilde{A}')$ where, $\tilde{A}'$ is a DFA-IFS. Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, s, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ represents a NDFA-IFS. We can construct a DFA-IFS $\tilde{A}' = (Q', \Sigma, \delta', \gamma', s', \tilde{F}'_{1\tilde{A}}, \tilde{F}'_{2\tilde{A}})$ by using the method of standard subset construction and, for each $P \in Q'(P \subseteq Q)$, define $\tilde{F}'_{1\tilde{A}}(P) = \max\{m \mid \tilde{F}_{1\tilde{A}}(q) = m,\ q \in P\}$ and $\tilde{F}'_{2\tilde{A}}(P) = \min\{l \mid \tilde{F}_{2\tilde{A}}(q) = l,\ q \in P\}$ where, m and l represents the membership and the nonmembership values of the strings in the language. Hence $L = L(\tilde{A}')$. □

4. Intuitionistic fuzzy regular expressions (in short IFREs)

Each intuitionistic fuzzy regular language has finite membership and nonmembership values. The set of finite words associated with these values forms a regular language. Hence, an intuitionistic fuzzy regular language may be represented by a modified regular expression.

Let $\Sigma = \{a, b\}$ be a finite set of alphabet. If $\{\lambda, a, aa, ...\}/0.5/0.4 \cup \{b, ab, ba, aba, ...\}/0.7/0.3$ represents an intuitionistic fuzzy regular language, then we can represent it by intuitionistic fuzzy regular expression as $a^*/0.5/0.4 + a^*ba^*/0.7/0.3$

We give a formal definition of intuitionistic fuzzy regular expression (IFRE) as follows.

Definition 4.1. Let Σ be a finite alphabet set and M, N be finite sets of real numbers in [0, 1].

1) Let 'e' be a regular expression over Σ and $m \in$ M, $l \in$ N. Then, we call $\tilde{e} = e/m/l$ an intuitionistic fuzzy regular expression (in short IFRE) where, m and l represents the membership and the nonmembership value of 'e' respectively such that $0 \leq m + l \leq 1$.

2) Let $\tilde{e}_1$ and $\tilde{e}_2$ be two intuitionistic fuzzy regular expressions then, following will holds.

i) $\Phi \in$ IFRE with membership 1 and nonmembership 0, ii) $\lambda \in$ IFRE with membership 1 and nonmembership 0, iii) $a \in$ IFRE with membership 1 and nonmembership 0, for all $a \in \Sigma$.

iv) For all $\tilde{e}_1$ and $\tilde{e}_2 \in$ IFREs, $(\tilde{e}_1 + \tilde{e}_2) \in$ IFRE, $(\tilde{e}_1 \cdot \tilde{e}_2) \in$ IFRE, also $(\tilde{e}_1)^* \in$ IFRE.

3) An intuitionistic fuzzy regular expression in formed by applying 1) and 2) a finite number of times.

Definition 4.2. Let $\tilde{e}$ be an intuitionistic fuzzy regular expression. Then, the corresponding language i.e., intuitionistic fuzzy regular language (IFRL) $L(\tilde{e})$ is defined by;

$L(\tilde{e}) = \{(x, m, l) \mid x \in L(e)\}$. Here $L(e)$ represents language for regular expression.

If $\tilde{e} = \tilde{e}_1 + \tilde{e}_2$, $\tilde{e} = (\tilde{e}_1) \cdot (\tilde{e}_2)$ or $\tilde{e} = (\tilde{e}_1)^*$. Then,
$L(\tilde{e}) = L(\tilde{e}_1) \cup L(\tilde{e}_2)$, $L(\tilde{e}) = L(\tilde{e}_1) \cdot L(\tilde{e}_2)$ or $L(\tilde{e}) = (L(\tilde{e}_1))^*$ respectively.

Definition 4.3. An intuitionistic fuzzy regular expression over Σ is normalized if it is of the form $e_1/m_1/l_1 + e_2/m_2/l_2 + ... + e_n/m_n/l_n$ where, $e_1, e_2, ..., e_n$ are regular expressions over Σ, $m_1, m_2, ..., m_2$ and, $l_1, l_2, ..., l_n$ are real numbers in $[0, 1]$, $n \geq 1$.

Note that, if $m = 1$ and $l = 0$ then $e/m/l$ can simply be written as 'e'. We assume that '$\cdot$' and '$*$' have higher priorities than '/'. So, certain pairs of parenthesis can be omitted.

Example 4.1. The following are all valid IFREs.

1) $a^*b/0.5/0.3 + a^*ba^*/0.3/0.7$
2) $ab^* + a$
3) $(1+10)^*/0.8/0.1$
4) $(0+\lambda)(1+10)^*/0.6/0.3$
5) $(b^*ab^*/0.7/0.2) \cdot (a^*ba^*/0.5/0.2) + a^* + b$

Above 1) to 4) are normalized and 5) is not normalized. Following is not a valid IFRE.

1) $(a^*/0.4/0.3)/0.5/0.4 + a^*ba^*/0.9/0$

Definition 4.4. An intuitionistic fuzzy regular expression $\tilde{e}$ is called a strictly normalized intuitionistic fuzzy regular expression, if it is normalized.
i.e., $\tilde{e} = e_1/m_1/l_1 + e_2/m_2/l_2 + ... + e_n/m_n/l_n$ and for any $m_i \neq m_j$, $l_i \neq l_j$, $L(e_i) \cap L(e_j) = \Phi$.

Example 4.2. 1) $(0+\lambda)(1+10)^*1/0.6/0.3 + (0+1)^*00(0+1)^*1/0.5/0.4$ shows an IFRE which is strictly normalized.

2) 1*1/0.7/0.2+(0+1)*0(0+1)*1/0.7/0.2 shows an IFRE which is not strictly normalized.

One can see that the families of languages as represented by IFREs, normalized IFREs, and strictly normalized IFREs, respectively, are same as the family of intuitionistic fuzzy regular languages.

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