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COMPLETE CONVERGENCE OF MOVING AVERAGE PROCESSES WITH ρ^*-MIXING SEQUENCES

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ABSTRACT. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed and ρ^*-mixing random variables and $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we prove the complete convergence of $\left\{\sum_{k=1}^{n}\sum_{n=-\infty}^{\infty} a_{i+k}\, Y_i/n^{1/t};\ n \geq 1\right\}$ under suitable conditions.

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1. Introduction

We Assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_k = \sum_{i=-\infty}^{\infty} a_{i+k} Y_i,\ k \geq 1. \tag{1.1}$$

Under independence assumptions, i.e., $\{Y_i, -\infty < i < \infty\}$ is a sequence of independent random variables, many limiting results have been obtained for the moving average process $\{X_k;\ k \geq 1\}$. Foe example, Ibragimov(1962) has established the central limit theorem for $\{X_k;\ k \geq 1\}$. Burton and Dehling(1990) have obtained a large deviation principle for $\{X_k;\ k \geq 1\}$ with $E\exp(tY_1) < \infty$ for all t and Liet al.(1992) have obtained the following result on complete convergence.

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Theorem A. *Suppose $\{Y_i, -\infty < i < \infty\}$ is a sequence of independent and identically distributed random variables. Let $\{X_k;\ k \geq 1\}$ be defined as (1.1) and $1 \leq t < 2$. Then $EY_1 = 0$ and $E|Y_1|^{2t} < \infty$ imply*

$$\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{n} X_k\right| \geq n^{\frac{1}{t}}\epsilon\right\} < \infty \text{ for all } \epsilon > 0.$$

Suppose that $\{X_k;\ k \geq 1\}$ is a sequence of random variables and put $F_S = \sigma\{X_k,\ k \in S\}$ where S is a subset of natural number set N.

Define

$$\begin{aligned}\rho_n^* &= \sup\{Corr(f,g) : \text{ For all } S \times T \subset N \times N,\ dist(S,T) \geq n \\ &\quad \forall f \in L^2(F_S),\ g \in L^2(F_T)\},\end{aligned}$$

where

$$\frac{Corr(f,g) = Cov\Big\{f(X_i,\ i \in S), g(X_j,\ j \in T)\ \Big\}}{\Big[Var\{f(X_i,\ i \in S)\}Var\{g(X_j,\ j \in T)\}\Big]^{1/2}}.$$

We call $\{X_k;\ k \geq 1\}$ is a ρ^*-mixing sequence if

$$\lim_{n\to\infty} \rho_n^* < 1. \tag{1.2}$$

Let us note that, since $0 \leq \cdots \leq \rho_n^* \leq \rho_{n-1}^* \leq \cdots \leq \rho_1^* \leq 1$, (1.2) is equivalent to

$$\rho_N^* < 1 \text{ for some } N > 1. \tag{1.3}$$

Bryc and Smolenski(1993) and Peigrad(1998) pointed out the importance of the condition (1.2) in estimating the moments of partial sums or of minimum of partial sums. Various limit properties under the condition $\rho_n^* \to 0$ were studied by Bradly(1992) and Miller(1994). Peligrad and Gut(1999) estimated higher moments of partial sums and of maximum of partial sums.

In this paper, we shall extend Theorem A to the case of ρ^*-mixing dependence. We suppose that $\{Y_i, -\infty < i < \infty\}$ is a sequence of identically distributed and ρ^*-mixing random variables.

Theorem 1.1. *Suppose that $\Big\{Y_i, -\infty < i < \infty\Big\}$ is a sequence of identically distributed and ρ^*-mixing random variables with $\lim_{n\to\infty} \rho^*(n) < 1$ and that $\{X_k;\ k \geq 1\}$ is defined as in (1.1). Let $h(x) > 0 (x > 0)$ be a slowly varying function and $1 \leq t < 2$, $r \geq 1$. Then EY_1 and $E\Big(|Y_1|^{rt}h(|Y_1|^t)\Big) < \infty$ imply*

$$\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\left|\sum_{k=1}^{n} X_k\right| \geq n^{1/t}\epsilon\right\} < \infty \text{ for all } \epsilon > 0.$$

Throughout the sequel, C will represent a positive constant although its value may change from one appearance to the next.

Observe that

$$\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} \sum_{j=1}^{n} a_{j+i} Y_i.$$

Set $a_{ni} = \sum_{j=1}^{n} a_{j+n}$. Then

$$\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} a_{ni} Y_i.$$

The following lemma comes from Burton and Dehling(1990).

Lemma 1.2. *Let* $\sum_{i=-\infty}^{\infty} a_i$ *be an absolutely convergent series of real numbers with* $a = \sum_{i=-\infty}^{\infty} a_i$ *and* $k \geq 1$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \left| \sum_{j=i+1}^{i+n} a_j \right|^k = |a|^k.$$

The following lemma will be useful. A proof appears in Peligrad and Gut(1999).

Lemma 1.3. Suppose $\{Y_i,\ i \geq 1\}$ be a sequence of ρ^*-mixing random variables with $EY_i = 0$ and $E|Y_i|^q < \infty$ for some $q \geq 2$. Assume that $\lim_{n\to\infty} \rho_n^* < 1$. Then there exists a constant $D(q, N, \rho_N^*)$, depending on q, N, and ρ_N^* with N and ρ_N^* defined via (1.3) such that

$$E|S_n|^q \leq D(q, N, \rho_N^*) \left(\sum_{i=1}^{n} E|Y_i|^q + (\sum_{i=1}^{n} EY_i^2)^{\frac{q}{2}} \right). \tag{1.4}$$

2. Proof of Theorem 1.1

Recall that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_{i+k} Y_i = \sum_{i=-\infty}^{\infty} a_{ni} Y_i$$

From Lemma 1.2, we can assume, without loss of generality, that

$$\sum_{i=-\infty}^{\infty} |a_{ni}| \leq n,\ n \geq 1 \text{ and } \tilde{a} = \sum_{i=-\infty}^{\infty} |a_i| \leq 1.$$

Let $S_n = \sum_{i=-\infty}^{\infty} a_{ni}Y_i \; I\{|a_{ni}Y_i| \le n^{\frac{1}{t}}\}$. Then

$$\begin{aligned}
n^{-\frac{1}{t}}E|S_n| &= n^{-\frac{1}{t}}\left|\sum_{i=-\infty}^{\infty} a_{ni}EY_i \; I\{|a_{ni}Y_i| > n^{\frac{1}{t}}\}\right| \\
&\le n^{-\frac{1}{t}} \sum_{i=-\infty}^{\infty} |a_{ni}| \; E|Y_1| \; I\{|a_{ni}Y_1| > n^{\frac{1}{t}}\} \\
&\le n^{-\frac{1}{t}}n \; E|Y_1| \; I\Big\{\tilde{a}|Y_1| > n^{\frac{1}{t}}\Big\} \\
&\le n^{-\frac{1}{t}}n \; E|Y_1| \; I\Big\{|Y_1| > n^{\frac{1}{t}}\Big\} \\
&\le E|Y_1|^t \; I\Big\{|Y_1| > n^{\frac{1}{t}}\Big\} \to 0 \; as \; n \to \infty.
\end{aligned}$$

So, for large enough n we have $n^{-1/t}E|S_n| < \epsilon/2$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}h(n) \; P\left\{\left|\sum_{k=1}^{n} X_k\right| \ge n^{\frac{1}{t}}\epsilon\right\} \\
&\le C\Big[\sum_{n=1}^{\infty} n^{r-2}h(n) \; P\{\max_i |a_{ni}Y_i| > n^{\frac{1}{t}}\} \\
&\qquad + \sum_{n=1}^{\infty} n^{r-2}h(n) \; P\left\{|S_n - ES_n| \ge n^{\frac{1}{t}}\frac{\epsilon}{2}\right\}\Big] \\
&= I_1 + I_2(say).
\end{aligned}$$

Set $I_{nj} = \left\{i \in \mathcal{Z}; (j+1)^{-\frac{1}{t}} < |a_{ni}| \le j^{-\frac{1}{t}}\right\}$, $j = 1, 2, \cdots$. Then $\bigcup_{j\ge1} I_{nj} = \mathcal{Z}$.

Note that(cf. Li et al. 1992)

$$\sum_{j=1}^{k} (\sharp I_{nj}) \le n(k+1)^{\frac{1}{t}}.$$

For I_1, we have

$$\begin{aligned}
I_1 &\le \sum_{n=1}^{\infty} n^{r-2}h(n) \sum_{i=-\infty}^{\infty} P\{|a_{ni}Y_i| > n^{\frac{1}{t}}\} \\
&\le \sum_{n=1}^{\infty} n^{r-2}h(n) \sum_{j=1}^{\infty}\sum_{i\in I_{nj}} P\{|Y_1| > j^{\frac{1}{t}}n^{\frac{1}{t}}\} \\
&\le \sum_{n=1}^{\infty} n^{r-2}h(n) \sum_{j=1}^{\infty}(\sharp I_{nj}) \sum_{k\ge jn} P\{k \le |Y_1|^t < k+1\}
\end{aligned}$$

$$\le \sum_{n=1}^{\infty} n^{r-2}h(n)\sum_{k=n}^{\infty}\sum_{j=1}^{[k/n]}(\sharp I_{nj})P\{k\le |Y_1|^t<k+1\}$$
$$\le \sum_{n=1}^{\infty} n^{r-2}h(n)\sum_{k=1}^{n}(\frac{k}{n}+1)^{\frac{1}{t}}\ n\ P\{k\le |Y_1|^t<k+1\}$$
$$\le C\sum_{n=1}^{\infty} n^{r-1}h(n)n^{-\frac{1}{t}}\sum_{k=n}^{\infty}k^{\frac{1}{t}}P\{k\le |Y_1|^t<k+1\}$$
$$< C\sum_{k=1}^{\infty}\sum_{n=1}^{k} n^{r-1}h(n)n^{-\frac{1}{t}}k^{\frac{1}{t}}P\{k\le |Y_1|^t<k+1\}$$
$$< C\sum_{k=1}^{\infty} k^{r-\frac{1}{t}}h(k)k^{\frac{1}{t}}P\{k\le |Y_1|^t<k+1\}$$
$$= \sum_{k=1}^{\infty} k^{r}h(k)P\{k\le |Y_1|^t<k+1\}$$
$$\le C\ E|Y_1|^n\ h(|Y_1|^t)<\infty.$$

For I_2, we have for $q\ge 2$, by Lemma 1.3

$$P\left\{|S_n-ES_n|\ge \frac{\epsilon}{2}n^{\frac{1}{t}}\right\}\le Cn^{-\frac{q}{t}}E|S_n-ES_n|^q$$
$$\le C\ n^{-\frac{q}{t}}\left\{\left(\sum_{i=-\infty}^{\infty}a_{ni}^2\ EY_1^2\ I\{|a_{ni}Y_1|\le n^{\frac{1}{t}}\}\right)^{\frac{q}{2}}+\sum_{i=-\infty}^{\infty}E|a_{ni}Y_1|^q\ I\{|a_{ni}Y_1|<n^{\frac{1}{t}}\}\right\}.$$

Then

$$I_2\le C\sum_{n=1}^{\infty}n^{r-2}h(n)n^{-\frac{q}{t}}\left(\sum_{i=-\infty}^{\infty}a_{ni}^2\ EY_1^2\ I\{|a_{ni}Y_1|\le n^{\frac{1}{t}}\}\right)^{\frac{q}{2}}$$
$$+C\sum_{n=1}^{\infty}n^{r-2}h(n)n^{-\frac{q}{t}}\sum_{i=-\infty}^{\infty}E|a_{ni}Y_1|^q\ I\{|a_{ni}Y_1|\le n^{\frac{1}{t}}\}.$$
$$=I_3+I_4(say).$$

If $r\ge 2$, we choose q large enough such that $q\Big(\dfrac{1}{t}-\dfrac{1}{2}\Big)>r-2$, then

$$I_3\le \sum_{n=1}^{\infty}n^{r-2}h(n)n^{-\frac{q}{t}}\left(\sum_{i=-\infty}^{\infty}a_{ni}^2EY_1^2\right)^{\frac{q}{2}}$$
$$\le \sum_{n=1}^{\infty}n^{r-2}h(n)n^{-q(\frac{1}{t}-\frac{1}{2})}<\infty,$$

and

$$
\begin{aligned}
I_4 &\le \sum_{n=1}^{\infty} n^{r-2}h(n)n^{-\frac{q}{t}} \sum_{j=1}^{\infty}\sum_{i\in I_{nj}} E|a_{ni}Y_1|^q I\{|a_{ni}Y_1| \le n^{\frac{1}{t}}\} \\
&\le \sum_{n=1}^{\infty} n^{r-2}h(n)n^{-\frac{q}{t}} \sum_{j=1}^{\infty}(\sharp I_{nj})j^{-\frac{q}{t}} E|Y_1|^q I\{|Y_1|^t \le n(j+1)\} \\
&\le \sum_{n=1}^{\infty} n^{r-2}h(n)n^{-\frac{q}{t}} \sum_{j=1}^{\infty}(\sharp I_{nj})j^{-\frac{q}{t}} \sum_{0\le k\le (j+1)n} E|Y_1|^q I\{k \le |Y_1|^t < k+1\} \\
&\le \sum_{n=1}^{\infty} n^{r-2}h(n)n^{-\frac{q}{t}} \sum_{j=1}^{\infty}(\sharp I_{nj})j^{-\frac{q}{t}} \sum_{k=0}^{2n} E|Y_1|^q I\{k \le |Y_1|^t < k+1\} \\
&\quad + \sum_{n=1}^{\infty} n^{r-2}h(n)n^{-\frac{q}{t}} \sum_{j=1}^{\infty}(\sharp I_{nj})j^{-\frac{q}{t}} \sum_{k=2n+1}^{(j+1)n} E|Y_1|^q I\{k \le |Y_1|^t < k+1\} \\
&= I_5 + I_6 (say).
\end{aligned}
$$

Note that for $q \ge 1$ and $m \ge 1$, we have

$$
\begin{aligned}
n &\ge \sum_{i=-\infty}^{\infty} |a_{ni}| \ge \sum_{j=1}^{\infty}\sum_{i\in I_{nj}} |a_{ni}| \ge \sum_{j=1}^{\infty}(\sharp I_{nj})(j+1)^{-\frac{1}{t}} \\
&\ge \sum_{j=m}^{\infty}(\sharp I_{nj})(j+1)^{-\frac{1}{t}} \ge \sum_{j=m}^{\infty}(\sharp I_{nj})(j+1)^{-\frac{q}{t}}(m+1)^{(\frac{q}{t}-\frac{1}{t})}.
\end{aligned}
$$

So,

$$
\sum_{j=m}^{\infty}(\sharp I_{nj})j^{-\frac{q}{t}} < C\, n\, m^{-\frac{q-1}{t}}.
$$

Then, we have

$$
\begin{aligned}
I_5 &\le C\sum_{n=1}^{\infty} n^{r-2}\, h(n)\, n^{-\frac{q}{t}} \cdot n \sum_{k=0}^{2n} E|Y_1|^q\, I\{k \le |Y_i|^t < k+1\} \\
&\le C\sum_{k=1}^{\infty}\sum_{n=[\frac{k}{2}]}^{\infty} n^{r-1}h(n)n^{-\frac{q}{t}}\, E|Y_1|^q I\{k \le |Y_i|^t < k+1\} \\
&\le C\sum_{n=1}^{\infty} k^{r-\frac{q}{t}}\, h(k)\, E|Y_1|^q\, I\{k \le |Y_i|^t < k+1\} \\
&\le C\, E|Y_1|^{rt}\, h(|Y_1|^t) < \infty,
\end{aligned}
$$

and

$$
I_6 < C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-\frac{q}{t}} \sum_{k=2n+1}^{\infty}\sum_{j\ge\frac{k}{n}-1} (\sharp I_{nj})j^{-\frac{q}{t}} E|Y_1|^q\, I\{k \le |Y_1|^t < k+1\}
$$

$$< C\sum_{n=1}^{\infty} n^{r-2}\, h(n)\, n^{-\frac{q}{t}} \sum_{k=2n+1}^{\infty} n(\frac{k}{n})^{-\frac{q-1}{t}}\, E|Y_1|^q\, I\{k \le |Y_i|^t < k+1\}$$

$$= \sum_{n=1}^{\infty} n^{r-1}\, h(n)\, n^{-\frac{1}{t}} \sum_{k=2n+1}^{\infty} k^{-\frac{q-1}{t}}\, E|Y_1|^q\, I\{k \le |Y_i|^t < k+1\}$$

$$< C\sum_{k=2}^{\infty}\sum_{n=1}^{[k/2]} n^{r-1}\, h(n)\, n^{-\frac{1}{t}}\, k^{-\frac{q-1}{t}}\, E|Y_1|^q\, I\{k \le |Y_i|^t < k+1\}$$

$$\le \sum_{k=2}^{\infty} k^r\, h(k)\, k^{-\frac{1}{t}}\, k^{-\frac{q-1}{t}}\, E|Y_1|^q\, I\{k \le |Y_i|^t < k+1\}$$

$$= \sum_{k=2}^{\infty} k^{r-\frac{q}{t}}\, h(k)\, E|Y_1|^q\, I\{k \le |Y_i|^t < k+1\}$$

$$\le C\, E|Y_1|^{rt}\, h(|Y_i|^t) < \infty.$$

So, $I_4 < \infty$ and then $I_2 < \infty$. If $r < 2$, we choose $q = 2$. Then

$$I_2 = \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-\frac{2}{t}} \sum_{i=-\infty}^{\infty} E|a_{ni}Y_1|^2\, I\{|a_{ni}Y_1| \le n^{\frac{1}{t}}\}.$$

Similary to I_4, we have $I_2 < \infty$.

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