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ON THE RECURSIVE SEQUENCE $x_{n+1} = \frac{a+bx_{n-1}}{A+Bx_n^k}$

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ABSTRACT. In this paper, we investigate the global behavior of the difference equation

$$x_{n+1} = \frac{a+bx_{n-1}}{A+Bx_n^k}, \quad n = 0, 1, ...,$$

where $a, b, B \in [0, \infty)$ and $A, k \in (0, \infty)$ with non-negative initial conditions.

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1. Introduction and preliminaries

Recently there has been a great interest in studying the behavior of rational and non-rational nonlinear difference equations. A. M. Amleh et al [1] studied the dynamics of

$$x_{n+1} = \frac{a+bx_{n-1}}{A+Bx_{n-2}}, \quad n = 0, 1, ...,$$

Gibbons et al [9]investigated the global behavior of the recursive sequence

$$y_{n+1} = \frac{\alpha+\beta y_{n-1}}{\gamma+y_n}, \quad n = 0, 1, ...,$$

where α, β, and γ are nonnegative real numbers. El-Owaidy et al [7] studied the dynamics of

$$x_{n+1} = \frac{bx_{n-1}}{A+Bx_{n-2}^p}, \quad n = 0, 1, ...,$$

where b, A, B and p are nonnegative real parameters. For related results see ([2]-[5],[10]-[20]). Some of these results can be applied to biological and population

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models. In this paper we generalize the results obtained in [9] to the second-order rational difference equation

$$x_{n+1}=\frac{a+bx_{n-1}}{A+Bx_n^k}\ ,\quad n=0,1,..., \tag{1}$$

where $a,b,B\in[0,\infty)$ and $A,k\in(0,\infty)$ with non-negative initial conditions such that $A+Bx_n^k>0 \quad \forall n\geq 0$. The change of variables of Eq.(1) by $x_n=\sqrt[k]{\frac{b}{B}}y_n$, yields

$$y_{n+1}=\frac{s+y_{n-1}}{r+y_n^k}\ ,\quad n=0,1,..., \tag{2}$$

where $s=\frac{a}{b}\sqrt[k]{\frac{B}{b}}$ and $r=\frac{A}{b}$. For definitions and notations, we refer the reader to ([6],[11],[12] and[13]).

Let I be an interval of real numbers and let

$$f:I\times I\to I$$

be a continuously differentiable function.

For every set of initial conditions $\{x_0,x_{-1}\}\subset I$, the difference equation

$$x_{n+1}=f(x_n,x_{n-1})\ ,\quad n=0,1,..., \tag{3}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$. We denote by

$$p=\frac{\partial f}{\partial u}(\bar{x},\bar{x})\quad\text{and}\quad q=\frac{\partial f}{\partial v}(\bar{x},\bar{x})$$

the partial derivatives of $f(u,v)$ evaluated at the equilibrium $\bar{x}$ of Eq.(3). The equation

$$y_{n+1}=py_n+qy_{n-1}\ ,\quad n=0,1,..., \tag{4}$$

is called the linearized equation associated with Eq.(3) about the equilibrium point $\bar{x}$. Then its characteristic equation is

$$\lambda^2-p\lambda-q=0. \tag{5}$$

2. The recursive sequence $y_{n+1}=y_{n-1}/\left(r+y_n^k\right)$

In this section we study the global behavior of Eq.(2), when $s=0$, that is Eq.(2) yields

$$y_{n+1}=\frac{y_{n-1}}{r+y_n^k}\ ,\quad n=0,1,..., \tag{6}$$

where y_{-1} and y_0 are non-negative real numbers, k and r are positive real numbers.

Theorem 1. *The following statements are true*

(1) *If $r > 1$, then Eq.(6) has a unique equilibrium point $\bar{y}_1 = 0$ which is locally asymptotically stable.*

(2) *If $r < 1$, then Eq.(6) has two equilibrium points $\bar{y}_2 = 0$ which is a repeller and $\bar{y}_3 = \sqrt[k]{1-r}$ which is unstable, in fact saddle.*

(3) *If $r = 1$, then Eq.(6) has a unique equilibrium point $\bar{y}_4 = 0$ which is non-hyperbolic point.*

Proof. The characteristic equation of the associated linearized equation about $\bar{y}_i$, $i = 1, 2, 4$ is $\lambda^2 - \frac{1}{r} = 0$, and about $\bar{y}_3$ is $\lambda^2 + k(1-r)\lambda - 1 = 0$, then the results follow directly by applying the Linearized stability Theorem [13]. □

In the following Theorem we prove the global asymptotic stability of the zero equilibrium point $\bar{y} = 0$ when $r > 1$.

Theorem 2. *Assume that $r > 1$. The equilibrium point $\bar{y} = 0$ is globally asymptotically stable.*

Proof. Let $r > 1$, by Theorem (1), $\bar{y} = 0$ is locally asymptotically stable. Now we show that $\bar{y} = 0$ is a global attractor. We have

$$y_{n+1} = \frac{y_{n-1}}{r + y_n^k} < \frac{1}{r} y_{n-1} , \quad n = 0, 1, 2, ...,$$

then by induction, we get

$$y_{2n} < (\frac{1}{r})^n y_0 \quad \text{and} \quad y_{2n-1} < (\frac{1}{r})^n y_{-1} \quad , n = 1, 2, ...$$

Therefore $\lim_{n \to \infty} y_n = 0$. □

Theorem 3. *A necessary and sufficient condition for Eq.(6) to have a prime period two solution is $r = 1$. In this case the prime period two solution is of the form*

$$..., \phi, 0, \phi, 0, ...$$

Furthermore; if $r = 1$, then every solution of Eq.(6) converges to a period (not necessary prime) two solution

$$..., \phi, 0, \phi, 0, ...$$

with $\phi \geq 0$.

Proof. (*Sufficiency*): Let $r = 1$, for every $\phi > 0$, we have

$$..., \phi, 0, \phi, 0, ...$$

is a prime period two solution for Eq.(6).

(*Necessity*): Assume that Eq.(6) has a prime period two solution

$$,..., \phi, \psi, \phi, \psi, ...$$

Then $\phi = \frac{\phi}{r+\psi^k}$ and $\psi = \frac{\psi}{r+\phi^k}$. Hence,

$$\phi\left[(r-1)+\psi^k\right]=0, \tag{7}$$

and

$$\psi\left[(r-1)+\phi^k\right]=0\ . \tag{8}$$

Assume for the sake of contradiction that $r \neq 1$. From (7), we have either $\phi = 0$ or $\psi = \sqrt[k]{1-r}$.

(1) If $\phi = 0$, by (8), we get $\psi = \phi = 0$, which is a contradiction.

(2) If $\psi = \sqrt[k]{1-r}$, by (8), we get $\phi = \sqrt[k]{1-r}$, which is a contradiction. Therefore $r = 1$. Now assume that $r = 1$ and $\{y_n\}_{n=-1}^{\infty}$ is a solution of Eq.(6). Then

$$y_{n+1} - y_{n-1} = \frac{-y_n y_{n-1}^k}{1+y_n^k} \le 0,$$

hence $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are monotonically decreasing to two limits say ϕ, and ψ. So, we obtain the equations

$$\phi = \frac{\phi}{1+\psi^k} \quad \text{and} \quad \psi = \frac{\psi}{1+\phi^k},$$

which imply that $\phi\psi = 0$. □

Theorem 4. *Let $\bar{y}$ be an equilibrium point of Eq.(6). Then except possibly for the first semicycle, every solution of Eq.(6) has semicycles of length one.*

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6) with at least two semicycles, then there exists $N \ge 0$ such that either

$$y_{N-1} < \bar{y} \le y_N \quad \text{or} \quad y_{N-1} \ge \bar{y} > y_N\ .$$

We assume the first case (the second case is similar and will be omitted). So we get

$$y_{N+1} = \frac{y_{N-1}}{r+y_N^k} < \frac{\bar{y}}{r+\bar{y}^k} = \bar{y} \text{ and } y_{N+2} = \frac{y_N}{r+y_{N+1}^k} > \frac{\bar{y}}{r+\bar{y}^k} = \bar{y}.$$

Thus $y_{N+1} < \bar{y} < y_{N+2}$, and the result follows by induction. □

Theorem 5. *Assume that $r < 1$, and $\bar{y} = \sqrt[k]{1-r}$. The following statements are true:*

(1) *If $y_0 < \bar{y}$, and $y_{-1} > \bar{y}$, then $\{y_{2n-1}\}$ is monotonically increasing to ∞, and $\{y_{2n}\}$ is monotonically decreasing to zero.*

(2) *If $y_0 > \bar{y}$, and $y_{-1} < \bar{y}$, then $\{y_{2n}\}$ is monotonically increasing to ∞, and $\{y_{2n+1}\}$ is monotonically decreasing to zero.*

Proof. Let $r < 1$. (1) Assume that $y_0 < \bar{y} = \sqrt[k]{1-r}$, and $y_{-1} > \bar{y} = \sqrt[k]{1-r}$. Then we have

$$y_1 = \frac{y_{-1}}{r + y_0^k} > \frac{y_{-1}}{r+1-r} = y_{-1} > \bar{y},$$
$$y_2 = \frac{y_0}{r + y_1^k} < \frac{y_0}{r+1-r} = y_0 < \bar{y}.$$

By induction, the odd sequence monotonically increases to a limit (say) $L > 0$, while the even sequence monotonically decreases to a limit (say) $M < \infty$. Hence,

$$M = \frac{M}{r + L^k} \quad \text{and} \quad L = \frac{L}{r + M^k},$$

which implies that $M = 0$, and $L = \infty$.

(2) is similar to (1), and will be omitted. □

3. The recursive sequence $y_{n+1} = (s + y_{n-1}) / (r + y_n^k)$

In this section we investigate the global behavior of

$$y_{n+1} = \frac{s + y_{n-1}}{r + y_n^k}, \qquad n = 0, 1, ..., \tag{9}$$

where s, r, and k are positive real numbers.

Lemma 1. *The following statements are true:*

(1) *Assume that $r > 1$. Then Eq.(9) has a unique equilibrium point in $(0, \frac{s}{r-1})$.*

(2) *Assume that $r < 1$.*

(*a*) *If $r \geq s$, then Eq.(9) has a unique equilibrium point in $(s, 1]$.*

(*b*) *If $r < s$, then Eq.(9) has a unique equilibrium point in $(1, \frac{s}{r})$.*

(3) *If $r = 1$, then Eq.(9) has a unique equilibrium point $\bar{y} = \sqrt[k+1]{s}$.*

Proof. Clearly, $\bar{y}$ is an equilibrium point of Eq.(9) if and only if $\bar{y}$ is a root of the function

$$f(x) = x^{k+1} + (r-1)x - s, \tag{10}$$

(1) Let $r > 1$, since $f(0) = -s < 0$, $f(\frac{s}{r-1}) = \left(\frac{s}{r-1}\right)^{k+1} > 0$, and $f(x)$ is increasing, then $f(x)$ has a unique root $\bar{y} \in (0, \frac{s}{r-1})$. Equivalently Eq.(9) has a unique equilibrium point $\bar{y} \in (0, \frac{s}{r-1})$.

(2) Let $r < 1$. The function $f(x)$ is decreasing on $\left[s, \sqrt[k]{\frac{1-r}{k+1}}\right]$ and increasing on $[\sqrt[k]{\frac{1-r}{k+1}}, \infty)$.

(a) Assume that $r \geq s$, then $f(1) = r - s \geq 0$. In view of $f(s) = s^{k+1} + rs - 2s < 0$, since $s \leq r < 1$, Eq.(9) has a unique equilibrium point $\bar{y} \in (s, 1]$.

(b) Assume that $r < s$. Then $f(1) = r - s < 0$ and $f(\frac{s}{r}) > 0$, consequently Eq.(9) has a unique equilibrium point $\bar{y} \in (1, \frac{s}{r})$.

(3) By (10), $\bar{y} = \sqrt[k+1]{s}$. □

We apply the linearized stability Theorem to get a sufficient condition for the equilibrium point $\bar{y}$ to be locally asymptotically stable.

Theorem 6. *Assume $r > 0$, and $\bar{y}$ is the equilibrium point of Eq.(9). The following statements are true:*
(1) *If $\bar{y}^{k+1} < \frac{s}{k}$, then $\bar{y}$ is locally asymptotically stable.*
(2) *If $\bar{y}^{k+1} > \frac{s}{k}$, then $\bar{y}$ is unstable, in fact saddle.*
(3) *If $\bar{y}^{k+1} = \frac{s}{k}$, then $\bar{y}$ is non-hyperbolic point.*

Proof. The characteristic equation of the associated linearized equation to Eq.(9) is $\lambda^2 = p\lambda + q$, where

$$p = \frac{-k\bar{y}^k}{r+\bar{y}^k}, \text{ and } \quad q = \frac{1}{r+\bar{y}^k}.$$

By the linearized stability Theorem [13], we have:
(1) If $\bar{y}^{k+1} < \frac{s}{k}$, then $|p| < 1-q < 2$, consequently $\bar{y}$ is locally asymptotically stable.
(2) If $\bar{y}^{k+1} > \frac{s}{k}$, then $p^2 + 4q > 0$ and $|p| > 1-q$, consequently $\bar{y}$ is unstable, in fact saddle.
(3) If $\bar{y}^{k+1} = \frac{s}{k}$, then $|p| = |1-q|$, consequently $\bar{y}$ is non-hyperbolic point. □

Definition ([6]). An Interval I of real numbers is said to be invariant under a real function $G(x, y)$ if $G(x, y) \in I \ \ \forall x, y \in I$.

We need the following Lemmas to prove the main result of this section.

Theorem 7. *Assume that $G(x, y)$ is a continuous function which is non - decreasing (non-increasing) in x for each y and non-increasing (non-decreasing) in y for each x. Assume that every solution of the equation*

$$y_{n+1} = G(y_n, y_{n-k}), \quad n = 0, 1, \ldots \ . \tag{11}$$

has an inferior limit λ and superior limit Λ such that λ and Λ belong to an invariant interval $I = [a, b]$ under G. Let $\bar{y}$ be a unique equilibrium point in I. If the system

$$x = G(x, y) \quad \text{and} \quad y = G(y, x) \tag{12}$$

$$(x = G(y, x) \quad \text{and} \quad y = G(x, y)) \tag{13}$$

has exactly one solution in I^2, then $\bar{y}$ is a global attractor .

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (11), $\lambda = \lim_{n\to\infty} \inf y_n$ and $\Lambda = \lim_{n\to\infty} \sup y_n$. Assume that $G(x, y)$ is non-decreasing (non-increasing) in x for each y and non-increasing (non-decreasing) in y for each x. Take $U_1 = G(\Lambda, \lambda)$ $(U_1 = G(\lambda, \Lambda))$ and $L_1 = G(\lambda, \Lambda)$ $(L_1 = G(\Lambda, \lambda))$. For every $\epsilon \in (0, \lambda - a)$ $\exists n_0 \in \mathbb{N}$ such that

$$\lambda - \epsilon < y_n < \Lambda + \in \quad \forall n \geq n_0.$$

Then $L_1 \leq \lambda \leq \Lambda \leq U_1$. Set $U_{n+1} = G(U_n, L_n)$ $(U_{n+1} = G(L_n, U_n))$ and $L_{n+1} = G(L_n, U_n)$ $(L_{n+1} = G(U_n, L_n))$, $n = 1, 2, ...$ One can see that

$$a \leq ... \leq L_2 \leq L_1 \leq \lambda \leq \Lambda \leq U_1 \leq U_2 \leq ... \leq b.$$

Hence $\{U_n\}$ is monotonically increasing to a number say $U \in I$ and $\{L_n\}$ is monotonically decreasing to a number say $L \in I$. This implies that $(U, L) \in I^2$ is a solution of system (12) ((13)). Therefore, $U = L = \bar{y} = \lambda = \Lambda$. □

Corollary 1. *Assume that $G(x, y)$ is a continuous function which is non- decreasing (non-increasing) in x for each y and non-increasing (non-decreasing) in y for each x. Let $I = [a, b]$ be an invariant interval under $G(x, y)$. Assume that $\bar{y} \in I$ is a unique equilibrium point of Eq.(11). Assume that J is a closed interval such that $G(x, y) \in I$ $\forall x, y \in J$. If the system*

$$x = G(x, y) \quad \text{and} \quad y = G(y, x) \qquad (x = G(y, x) \quad \text{and} \quad y = G(x, y))$$

has exactly one solution in I^2, then $\bar{y}$ is a global attractor with basin I^{k+1}.

Theorem 8. *Assume that $r > 0$. Then Eq.(9) has a period (not necessary prime) two solution $\{y_n\}_{n=-1}^{\infty}$ if and only if (y_{-1}, y_0) is a solution of the system*

$$x = \frac{s + x}{r + y^k} \quad \text{and} \quad y = \frac{s + y}{r + x^k}. \tag{14}$$

Furthermore, $y_{-1} \neq y_0$ if and only if $\{y_n\}_{n=-1}^{\infty}$ is a prime period two solution of Eq.(9).

Proof. Assume that (y_{-1}, y_0) is a solution of system (14). Then

$$y_1 = \frac{s + y_{-1}}{r + y_0^k} = y_{-1} \quad \text{and} \quad y_2 = \frac{s + y_0}{r + y_1^k} = \frac{s + y_0}{r + y_{-1}^k} = y_0.$$

By induction $y_{n+2} = y_n$, $n \geq -1$. For the other direction let $\{y_n\}_{n=-1}^{\infty}$ be a period two solution. Then

$$y_1 = \frac{s + y_{-1}}{r + y_0^k} = y_{-1} \quad \text{and} \quad y_2 = \frac{s + y_0}{r + y_1^k} = \frac{s + y_0}{r + y_{-1}^k} = y_0.$$

Clearly, $y_{-1} \neq y_0$ if and only if $\{y_n\}_{n=-1}^{\infty}$ is a prime period two solution. □

Theorem 9. *Let $\bar{y}$ be the positive equilibrium point of Eq.(9). Then except possibly for the first semicycle, every solution of Eq.(9) has semicycles of length one.*

Proof. Let $y_{n-1} < \bar{y} \leq y_n$ or $y_{n-1} \geq \bar{y} > y_n$. Consider $y_{n-1} < \bar{y} \leq y_n$ (the other case is similar and will be omitted), then we get

$$y_{n+1} = \frac{s + y_{n-1}}{r + y_n^k} < \frac{s + \bar{y}}{r + \bar{y}^k} = \bar{y} \quad \text{and} \quad y_{n+2} = \frac{s + y_n}{r + y_{n+1}^k} > \frac{s + \bar{y}}{r + \bar{y}^k} = \bar{y}.$$

By induction we get the result. □

In the following Theorem we determine more precisely necessary conditions (on parameters) for $\bar{y}$ to be locally asymptotically stable and for $\bar{y}$ to be unstable.

Theorem 10. *Assume that $r > 1$, and $\bar{y}$ is the equilibrium point of Eq.(9). The following statements are true*

(1) *If $k \leq 1$, then $\bar{y}$ is locally asymptotically stable.*

(2) *Assume that $k > 1$. We have:*

(*i*) *If $s < k\left(\frac{r-1}{k-1}\right)^{\frac{k+1}{k}}$, then $\bar{y}$ is locally asymptotically stable.*

(*ii*) *If $s > k\left(\frac{r-1}{k-1}\right)^{\frac{k+1}{k}}$, then $\bar{y}$ is unstable, in fact saddle.*

(*iii*) *If $s = k\left(\frac{r-1}{k-1}\right)^{\frac{k+1}{k}}$, then $\bar{y}$ is non-hyperbolic.*

Proof. It is easy to show that

$$\bar{y}^{k+1} < \frac{s}{k} \iff \bar{y} > \frac{s(k-1)}{k(r-1)},$$

$$\bar{y}^{k+1} > \frac{s}{k} \iff \bar{y} < \frac{s(k-1)}{k(r-1)}, \quad \bar{y}^{k+1} = \frac{s}{k} \iff \bar{y} = \frac{s(k-1)}{k(r-1)}.$$

(1) Let $k \leq 1$, then $\bar{y} > \frac{s(k-1)}{k(r-1)}$, whence $\bar{y}^{k+1} < \frac{s}{k}$. Hence, $\bar{y}$ is locally asymptotically stable by Theorem (6).

(2) Let $k > 1$, We have

$$f(\frac{s(k-1)}{k(r-1)}) = \frac{s}{k}\left[\left(\frac{s}{k}\right)^k \left(\frac{k-1}{r-1}\right)^{k+1} - 1\right]$$

where $f(x)$ is defined in (10).

(i) Assume that $s < k\left(\frac{r-1}{k-1}\right)^{\frac{k+1}{k}}$, then $f(\frac{s(k-1)}{k(r-1)}) < 0$. Hence $\bar{y} > \frac{s(k-1)}{k(r-1)}$ and $\bar{y}^{k+1} < \frac{s}{k}$. Therefore $\bar{y}$ is locally asymptotically stable.

(ii) Assume that $s > k\left(\frac{r-1}{k-1}\right)^{\frac{k+1}{k}}$, then $f(\frac{s(k-1)}{k(r-1)}) > 0$. Hence $\bar{y} < \frac{s(k-1)}{k(r-1)}$ and $\bar{y}^{k+1} > \frac{s}{k}$. Therefore $\bar{y}$ is unstable (saddle).

(iii) Assume that $s = k\left(\frac{r-1}{k-1}\right)^{\frac{k+1}{k}}$, then $f(\frac{s(k-1)}{k(r-1)}) = 0$. Then $\bar{y} = \frac{s(k-1)}{k(r-1)}$ and $\bar{y}^{k+1} = \frac{s}{k}$. Therefore $\bar{y}$ is non-hyperbolic. □

Lemma 2. *If $\{y_n\}_{n=-1}^{\infty}$ is a solution of Eq.(9), then*

$$\frac{\alpha}{1+\beta\left[\frac{\alpha(1-\beta^n)}{1-\beta} + y_{-1}\beta^n\right]^k} \leq y_{2n} \leq \frac{\alpha(1-\beta^n)}{1-\beta} + y_0\beta^n, \quad n \geq 1 \tag{15}$$

and

$$\frac{\alpha}{1+\beta\left[\frac{\alpha(1-\beta^n)}{1-\beta} + y_0\beta^n\right]^k} \leq y_{2n-1} \leq \frac{\alpha(1-\beta^n)}{1-\beta} + y_{-1}\beta^n, \quad n \geq 1 \tag{16}$$

where $\alpha = s/r$ *and* $\beta = 1/r$.

Proof. We have $\frac{s}{r+y_n^k} \leq y_{n+1} \leq \frac{s}{r} + \frac{y_{n-1}}{r}$. Hence

$$\frac{\alpha}{1+\beta y_n^k} \leq y_{n+1} \leq \alpha + \beta y_{n-1}, \quad n \geq 0. \tag{17}$$

Inequalities (15) and (16) are obtained by (17), using induction on n. □

Corollary 2. *Assume that* $r > 1$. *Let* $\{y_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(9) and* $\Lambda = \lim_{n\to\infty} \sup y_n$ *and* $\lambda = \lim_{n\to\infty} \inf y_n$, *then* Λ *and* λ *satisfy the two inequalities,*

$$\frac{s}{r + (\frac{s}{r-1})^k} \leq \lambda \leq \Lambda \leq \frac{s}{r-1}, \tag{18}$$

and

$$\frac{s+\lambda}{r+\Lambda^k} \leq \lambda \leq \Lambda \leq \frac{s+\Lambda}{r+\lambda^k}, \tag{19}$$

Proof. Inequality (18) is a direct consequence of (15), and (16). For every $\epsilon \in (0, \lambda)$ $\exists n_0 \in \mathbb{N}$ such that $\lambda - \epsilon \leq y_n \leq \Lambda + \epsilon$ $\forall n \geq n_0$. Then

$$\frac{s+\lambda-\in}{r+(\Lambda+\in)^k} \leq y_n \leq \frac{s+\Lambda+\in}{r+(\lambda-\in)^k} \quad \forall n \geq n_0 + 1.$$

Therefore, $\frac{s+\lambda}{r+\Lambda^k} \leq \lambda \leq \Lambda \leq \frac{s+\Lambda}{r+\lambda^k}$. □

In the following we denote by $I_0 = \left[0, \frac{s}{r-1}\right]$.

Lemma 3. *Assume that* $r > 1$. *The interval* I_0 *is invariant under the function*

$$G(x,y) = \frac{s+x}{r+y^k}. \tag{20}$$

Proof. Let $x, y \in I_0$. Then

$$0 < \frac{s}{r + \left(\frac{s}{r-1}\right)^k} < G(x,y) < \frac{s+x}{r} < \frac{s}{r-1}$$

□

Theorem 11. *Assume that* $r > 1$. *If the system*

$$y = \frac{s+y}{r+x^k} \quad \text{and} \quad x = \frac{s+x}{r+y^k}, \tag{21}$$

has exactly one solution in I_0^2, *then the equilibrium point* $\bar{y}$ *is a global attractor.*

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq.(9), $\lambda = \lim_{n\to\infty} \inf y_n$ and $\Lambda = \lim_{n\to\infty} \sup y_n$. By corollary (2), $\lambda, \Lambda \in I_0$, which is invariant under $G(x,y)$ (defined in (20)). By Theorem (7), $\bar{y}$ is a global attractor. □

Theorem 12. *If* $r \geq 1 + \left(ks^k\right)^{\frac{1}{k+1}}$, *then system (21) has exactly one solution in* I_0^2.

Proof. Assume that (x,y) is a solution of system (21) in I_0^2. Then we have

$$r+x^k=\frac{s}{y}+1 \quad \text{and} \quad r+y^k=\frac{s}{x}+1.$$

Hence, $x^k-y^k=\frac{s}{y}-\frac{s}{x}=\frac{s(x-y)}{xy}$. Assume towards a contradiction that $x\neq y$ say $(y<x)$. Hence

$$\frac{x^k-y^k}{x-y}=\frac{s}{xy}.$$

By the Mean Value Theorem, there exists $c\in(y,x)$ such that $\frac{s}{xy}=kc^{k-1}$. When $k\geq 1$, we have $\frac{s}{xy}<kx^{k-1}$ which implies that $r-1<k\left(\frac{s}{r-1}\right)^k$, which is a contradiction. When $k<1$, we have $\frac{s}{xy}<ky^{k-1}$ which implies that $r-1<k\left(\frac{s}{r-1}\right)^k$, which is a contradiction. Then system (21) has exactly one solution $(x,y)=(\bar{y},\bar{y})$. □

In the following Theorem we establish a sufficient condition for the global asymptotic stability of Eq.(9).

Theorem 13. *If* $r\geq 1+\left(ks^k\right)^{\frac{1}{k+1}}$, *then the equilibrium point* $\bar{y}$ *of Eq.(9) is globally asymptotically stable.*

Proof. In the case where $k\leq 1$, $\bar{y}$ is locally asymptotically stable by Theorem (10), and in the case where $k>1$, the condition $r\geq 1+\left(ks^k\right)^{\frac{1}{k+1}}$ implies that $\left(\frac{k-1}{r-1}\right)^{k+1}<\left(\frac{k}{s}\right)^k$, whence $s<k\left(\frac{r-1}{k-1}\right)^{\frac{k+1}{k}}$ which implies that $\bar{y}$ is locally asymptotically stable. By combining Theorems (11) and (12), we see that $\bar{y}$ is globally asymptotically. □

In the following, we show that the equilibrium point $\bar{y}$ of the equation

$$y_{n+1}=\frac{s+y_{n-1}}{r+y_n^k}\quad,\quad n=0,1,..., \tag{22}$$

where $r<1$, is a global attractor with some basin that depends on the coefficients. Let $\bar{y}$ be the equilibrium point of Eq.(22). We can see that

$$\begin{aligned}
\bar{y}^{k+1} &< \frac{s}{k} \quad \text{if and only if} \quad \bar{y}<\frac{s(1-k)}{k(1-r)},\\
\bar{y}^{k+1} &> \frac{s}{k} \quad \text{if and only if} \quad \bar{y}>\frac{s(1-k)}{k(1-r)},\\
\bar{y}^{k+1} &= \frac{s}{k} \quad \text{if and only if} \quad \bar{y}=\frac{s(1-k)}{k(1-r)}.
\end{aligned}$$

In the following Theorem a sufficient condition for $\bar{y}$ to be locally asymptotically stable will be given.

Theorem 14. *Assume that $r < 1$. The following statements are true*

(1) *If $k \geq 1$, then $\bar{y}$ is unstable.*

(2) *Assume that $k < 1$. We have:*

(*i*) *If $s > k\left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$, then $\bar{y}$ is locally asymptotically stable.*

(*ii*) *If $s < k\left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$, then $\bar{y}$ is unstable, in fact saddle.*

(*iii*) *If $s = k\left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$, then $\bar{y}$ is non-hyperbolic.*

Proof. (1) Clearly, if $k \geq 1$, then $\bar{y} > \frac{s(1-k)}{k(1-r)}$, whence, $\bar{y}^{k+1} > \frac{s}{k}$. Hence by Theorem (6), $\bar{y}$ is unstable.

(2) Assume that $k < 1$. Consider the function $f(x) = x^{k+1} + (r-1)x - s$. We have

$$f(\frac{s(1-k)}{k(1-r)}) = \frac{s}{k}\left[\left(\frac{s}{k}\right)^k\left(\frac{1-k}{1-r}\right)^{k+1} - 1\right].$$

(i) Assume that $s > k\left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$, then $f(\frac{s(1-k)}{k(1-r)}) > 0$. Hence $\bar{y} < \frac{s(1-k)}{k(1-r)}$ and $\bar{y}^{k+1} < \frac{s}{k}$. Hence $\bar{y}$ is locally asymptotically stable.

(ii) Assume that $s < k\left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$, then $f(\frac{s(1-k)}{k(1-r)}) < 0$. Hence $\bar{y} > \frac{s(1-k)}{k(1-r)}$ and $\bar{y}^{k+1} > \frac{s}{k}$. Therefore $\bar{y}$ is unstable (saddle).

(iii) Assume that $s = k\left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$, then $f(\frac{s(1-k)}{k(1-r)}) = 0$. Then $\bar{y} = \frac{s(1-k)}{k(1-r)}$ and $\bar{y}^{k+1} = \frac{s}{k}$. Therefore $\bar{y}$ is non-hyperbolic. □

In the following we denote by $G(x,y) = \frac{s+x}{r+y^k}$.

Lemma 4. *Assume that $r < 1$ and $k < 1$.*

(1) *If $r \geq s^{\frac{k-1}{k}}$, then $I = [1, \frac{s}{r}]$ is invariant under G and I contains $\bar{y}$.*

(2) *If $r \geq 1 + s - s^k$, then $I = [s, 1]$ is invariant under G and I contains $\bar{y}$.*

Proof. (1) The condition $r \geq s^{\frac{k-1}{k}}$ implies that $s > 1$. Hence by Lemma (1), $\bar{y} \in \left(1, \frac{s}{r}\right)$. Let $x, y \in \left[1, \frac{s}{r}\right]$, then we have

$$1 = \frac{s+1}{1+\left(\frac{s}{r}\right)^k} \leq G(x,y) = \frac{s+x}{r+y^k} \leq \frac{s+\frac{s}{r}}{r+1} \leq \frac{s}{r}.$$

(2) The condition $r \geq 1 + s - s^k$, implies that $s < 1$. Also we can see that $s \leq r$. Hence by Lemma (1) $\bar{y} \in [s, 1]$. Let $x, y \in [s, 1]$, then we have

$$s = \frac{s+s}{1+1} \leq G(x,y) = \frac{s+x}{r+y^k} \leq \frac{s+1}{r+s^k} \leq 1.$$

□

Theorem 15. *Assume that* $r < 1$, $k < 1$.

(*i*) *If* $r < s < \left(\frac{1}{k}r^{k+1}\right)^{\frac{1}{k}}$, *then system (21) has exactly one solution* $(x, y) \in \left[1, \frac{s}{r}\right]^2$.

(*ii*) *If* $k < s \leq r$, *then system (21) has exactly one solution* $(x, y) \in [s, 1]^2$.

Proof. Assume that $(x, y) \in I^2$ is a solution of system (21), and $y < x$, where $I = \left[1, \frac{s}{r}\right]$ in statement (i) and $I = [s, 1]$ in statement (ii). Then as before

$$\frac{x^k - y^k}{x - y} = \frac{s}{xy}.$$

There exists $c \in (y, x)$ such that $\frac{s}{xy} = kc^{k-1} \leq ky^{k-1}$. Hence $ky^k \geq \frac{s}{x}$.
(i) Since $r < s$, then $\left[1, \frac{s}{r}\right]$ contains $\bar{y}$. Since $1 \leq x, y \leq \frac{s}{r}$, then $k\left(\frac{s}{r}\right)^k \geq ky^k \geq \frac{s}{x} \geq \frac{rs}{s} = r$, whence $s^k \geq \frac{r^{k+1}}{k}$, which is a contradiction. Therefore, $x = y = \bar{y}$.
(ii) Since $s \leq r < 1$, then $[s, 1]$ contains $\bar{y}$. Since $s \leq x, y \leq 1$, then $k \geq ky^k \geq \frac{s}{x} \geq s$, which is a contradiction. Therefore, $x = y = \bar{y}$. □

Theorem 16. *Assume that* $r < 1$ *and* $k < 1$.

(*i*) *If* $r^{\frac{k}{k-1}} < s < \left(\frac{1}{k}r^{k+1}\right)^{\frac{1}{k}}$, *then* $\bar{y}$ *is a global attractor with basin* $\left[1, \frac{s}{r}\right]^2$.

(*ii*) *If* $r \geq 1 + s - s^k$ *and* $k < s$, *then* $\bar{y}$ *is a global attractor with basin* $[s, 1]^2$.

Proof. (i) By Lemma (4) the interval $\left[1, \frac{s}{r}\right]$ is invariant under G and contains $\bar{y}$. The condition $r^{\frac{k}{k-1}} < s < \left(\frac{1}{k}r^{k+1}\right)^{\frac{1}{k}}$ implies that system (21) has a unique solution in $\left[1, \frac{s}{r}\right]^2$. By Corollary (1) $\bar{y}$ is a global attractor with basin $\left[1, \frac{s}{r}\right]^2$.

(ii) By the same argument of (i) we can prove (ii). □

Theorem 17. *Assume that* $r < 1$, $k < 1$.

(*i*) *If* $r^{\frac{k}{k-1}} < s < \left(\frac{1}{k}r^{k+1}\right)^{\frac{1}{k}}$, *then* $\bar{y}$ *is globally asymptotically stable with basin* $\left[1, \frac{s}{r}\right]^2$.

(*ii*) *If* $r \geq 1 + s - s^k$ *and* $k < s$, *then* $\bar{y}$ *is globally asymptotically stable with basin* $[s, 1]^2$.

Proof. (i) Since $r < 1$ and $k < 1$, then $r < r^{\frac{k}{k-1}} < s$. The condition $r^{\frac{k}{k-1}} < s < \left(\frac{1}{k}r^{k+1}\right)^{\frac{1}{k}}$ implies that $k < r^{\frac{1}{1-k}} < r$. Hence, $k < r < r^{\frac{k}{k-1}} < s$. Therefore, $\frac{s}{k} > 1 > \left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$, hence $s > k\left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$. In view of Theorem (14), $\bar{y}$ is locally asymptotically stable and by Theorem (16), $\bar{y}$ is globally asymptotically stable with basin $\left[1, \frac{s}{r}\right]^2$.

(ii) Since, $r \geq 1 + s - s^k$, then $s < 1$. The condition $s - s^k + 1 \leq r < 1$ implies that $s < 1$, which in turn that $s \leq s - s^k + 1$. Hence $k < s \leq r$. Therefore $\frac{s}{k} > 1 > \left(\frac{1-r}{1-k}\right)^{\frac{k+1}{k}}$. By Theorem (14), $\bar{y}$ is locally asymptotically stable. Again, by Theorem (16), whence $\bar{y}$ is globally asymptotically stable with basin $[s, 1]^2$. □

Now, we give examples in which conditions of Theorem (17) hold.

Example 1. Consider the recursive sequence

$$x_{n+1} = \frac{s + x_{n-1}}{0.3 + x_n^{0.1}}, \quad n = 0, 1, \dots .$$

It is easy to show that for every $s \in (1.15, 17714.7)$, $\bar{y}$ is globally asymptotically stable with basin $\left[1, \frac{s}{0.3}\right]$.

The following Theorem shows that outside the basins indicated in the above Theorem, we can establish unbounded solutions for Eq.(22).

Theorem 18. *Assume that $r < 1$, $y_{-1}^k < (1 - r)$, $y_0^k > 1 - r + \frac{s}{\sqrt[k]{1-r}}$. Then*

$$\lim_{n\to\infty} y_{2n} = \infty \quad \textit{and} \quad \lim_{n\to\infty} y_{2n-1} = 0.$$

Proof. By induction on n, we can see that $y_{2n-1} < \sqrt[k]{1 - r}$, and $y_{2n} > ns + y_0$. Hence $\lim_{n\to\infty} y_{2n} = \infty$, and consequently, $\lim_{n\to\infty} y_{2n-1} = 0$. □

Now, we study the global behavior of Eq.(9) when $r = 1$, that is Eq.(9) yields

$$y_{n+1} = \frac{s + y_{n-1}}{1 + y_n^k}, \qquad n = 0, 1, \dots . \tag{23}$$

In the sequel we assume that $s \neq 0$, otherwise see section (2).

Lemma 5. (*a*) *Assume that $k < 1$. Then the unique equilibrium point $\bar{y} = \sqrt[k+1]{s}$ of Eq.(23) is locally asymptotically stable.*

(*b*) *Assume that $k > 1$. Then the unique equilibrium point $\bar{y} = \sqrt[k+1]{s}$ of Eq.(23) is unstable. In fact, saddle.*

Proof. It is clear that Eq.(23) always has a unique equilibrium point $\bar{y} = \sqrt[k+1]{s}$. We can show that

$$p = \frac{-k\bar{y}^k}{1 + \bar{y}^k} \quad \text{and} \quad q = \frac{1}{1 + \bar{y}^k}.$$

(a) Let $k < 1$, whence, the condition $|p| < 1 - q \iff (k - 1)\bar{y}^k < 0$, and $1 - q < 2 \iff 0 < 2 + \bar{y}^k$ is always true. Therefore, the equilibrium point $\bar{y} = \sqrt[k+1]{s}$ is locally asymptotically stable.

(b) Let $k > 1$, whence, the condition $|p| > |1 - q| \iff (k - 1)\bar{y}^k > 0$, and $p^2 + 4q > 0$ is always true. Therefore, the equilibrium point $\bar{y} = \sqrt[k+1]{s}$ is unstable. In fact, saddle. □

In the sequel we denote $\bar{y}$ the unique equilibrium point of Eq.(23).

Theorem 19. *The system*

$$x = \frac{s + x}{1 + y^k} \quad \textit{and} \quad y = \frac{s + y}{1 + x^k} \tag{24}$$

has exactly one solution $(\bar{y}, \bar{y})$.

Proof. Assume on the contrary that the system (24) has a solution (x, y) such that $x \neq y$, then we get $xy(y^{k-1} - x^{k-1}) = 0$, since, $x \neq y$. Hence $xy = 0$ so either $x = 0$ or $y = 0$ both of them imply that $s = 0$, which is a contradiction. Therefore the system has exactly one solution $(\bar{y}, \bar{y})$. □

N.B: Clearly, system (25) has exactly one solution if and only if Eq.(23) has no prime period two solutions. Consider, the function $G(x, y) = \frac{s+y}{1+x^k}$.

Theorem 20. *Assume that* $k < 1$.

(*a*) *If* $s \leq 1$, *then* $[s, 1]$ *is invariant under* $G(x, y)$ *and* $\bar{y} \in [s, 1]$.

(*b*) *If* $s > 1$, *then* $[1, s]$ *is invariant under* $G(x, y)$ *and* $\bar{y} \in [1, s]$.

Proof. (a) Let $s \leq 1$ and $k < 1$, we have

$$s = \frac{s+s}{1+1} \leq G(x, y) = \frac{s+y}{1+x^k} \leq \frac{s+1}{1+s^k} \leq 1$$

Therefore, $[s, 1]$ is invariant under G. One can see that $\bar{y} \in [s, 1]$.

(b) Let $s > 1$ and $k < 1$, we have

$$1 \leq \frac{s+1}{1+s^k} \leq \frac{s+y}{1+x^k} = G(x, y) \leq \frac{s+s}{1+1} = s$$

Therefore, $[1, s]$ is invariant under G. One can see that $\bar{y} \in [1, s]$. This completes the proof. □

We combine Corollary (1), Lemma (5), Theorems (19) and (20), to obtain the following results.

Theorem 21. *Assume that* $k < 1$. *Then the equilibrium point* $\bar{y} = \sqrt[k+1]{s}$ *of Eq.(23) is globally asymptotically stable with basin* $[s, 1]^2$ *when* $s \leq 1$ *and with basin* $[1, s]^2$ *when* $s > 1$.

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