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THE $M/G/1$ FEEDBACK RETRIAL QUEUE WITH BERNOULLI SCHEDULE

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ABSTRACT. We consider an M/G/1 feedback retrial queue with Bernoulli schedule in which after being served each customer either joins the retrial group again or departs the system permanently. Using the supplementary variable method, we obtain the joint generating function of the numbers of customers in two groups.

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1. Introduction

Queueing systems in which arriving customers who find all servers and waiting positions occupied may retry for service after a period of time are called retrial queues or queues with repeated calls. Retrial queues are useful in modelling many problems in telephone switching systems, computer and communication systems. A review of the literature on retrial queues can be found in Falin [6] and Yang and Templeton [7]. Most papers on retrial queues have treated the system with no waiting position (see[2],[3] and [5]). Choi and Park [4] have investigated an M/G/1 retrial queue with infinite waiting space in which arriving customers who find the server busy join either the retrial group or the infinite waiting space. Choi and Kulkarni [2] have investigated M/G/1 feedback retrial queue with no waiting position. The phenomena of feedback in the retrial queueing systems occur in many practical situations; for instance, multiple access telecommunication systems,where messages turned out as errors are sent again, is modeled as retrial queue with feedback.

In this paper, we deal with feedback retrial queue where after being served each customer either joins the retrial group again or departs permanently the

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system. In section 2, we describe the mathematical queueing model in detail. In section 3, we derive the joint generating function of the numbers of customers in the priority group and retrial group by using the supplementary variable method, and mean queue lengths in the two groups. It is shown that our results are consistent with known results when $\beta = 0$.

2. Mathematical model

We consider an M/G/1 feedback retrial queueing system with infinite waiting space, as shown in figure 1.

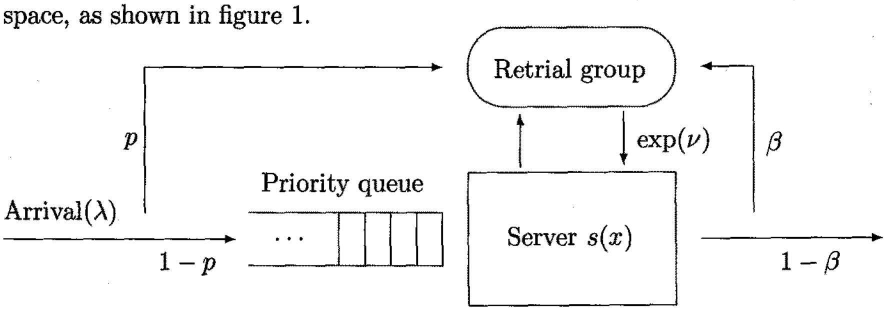

FIGURE 1. M/G/1 feedback retrial queue with Bernoulli schedule

Customers arrive according to a poisson process with rate λ. An arriving customer obtains service immediately if the server is idle and otherwise he joins either the retrial group with probability p in order to seek service again after a random amount of time, or the infinite waiting space (called priority group) with probability $q = 1 - p$, where he waits to be served. Customers who are blocked on their first attempts are allowed to join the priority group, but a customer in the retrial group has no option to join the priority group and always returns to the retrial group when he finds the server busy on his retrial attempt. Customers in the retrial group are accepted on retrial only when the priority group is empty. According to the above rule, customers in the priority group have non-preemtive priority over those in the retrial group. The retrial time (the time interval between two consecutive attempts made by a customer in the retrial group) is exponentially distributed with mean $\frac{1}{\alpha}$ and is independent of all previous retrial times and all other stochastic processes in the system. The service times of customers are independently and identically distributed with a probability density function (p.d.f) $s(x)$ and mean b. Let

$$s^*(\theta) = \int_0^\infty e^{-\theta x} s(x)dx$$

be the Laplace transform of service time S. After the customer is served completely, he will decide either to rejoin the retrial group again for another service with probability β or to leave the system forever with probability $1 - \beta$, where $0 \leq \beta < 1$. When $\beta = 0$, our model coincides with retrial queue with infinite waitng space by Choi and Park [4]. It is easy to show that the system is stable provided that $\rho = \lambda b < 1$. In this paper, we always assume that the system is stable.

3. The joint distributions of queue sizes

We will investigate the joint distribution of the numbers of customers in the priority queue and the retrial group at an arbitrary time by using the supplementary variable method. Here we take supplementary variable as the remaining service time.

At an arbitrary time, the steady state of the system can be characterized by the following variables;

N_1 = the number of customers in the priority group (excluding the customer in the service),

N_2 = the number of customers in the retrial group,

$\tilde{S}$ = the remaining service time of the customer in the service, and

$$\xi = \begin{cases} 0, & \text{when server is idle,} \\ 1, & \text{when server is busy.} \end{cases}$$

Define the probabilities;

$$q_j = P\Big\{\xi = 0, N_2 = j\Big\}, \quad j = 0, 1, 2, \dots;$$

$$\begin{aligned} p_{ij}(x)dx \quad &= \quad P\Big\{\xi = 1, N_1 = i, N_2 = j, \tilde{S} \in (x, x + dx]\Big\}, \\ &\qquad i, j = 0, 1, 2, \dots, \text{ and } x \geq 0; \end{aligned}$$

and their Laplace transforms

$$p^*_{ij}(\theta) = \int_0^\infty e^{-\theta x} p_{ij}(x) d\, x, \quad i, j = 0, 1, 2, \dots.$$

The usual arguments lead to the following differential difference equations;

$$(\lambda + j\alpha) q_j \;= (1 - \beta) p_{0j}(0) + \beta p_{0j-1}(0), j \geq 0, \tag{1a}$$

$$\begin{aligned} -p'_{0j}(x) = &- \lambda p_{0j}(x) + p\lambda p_{0j-1}(x) + (1 - \beta) s(x) p_{1j}(0) \\ &+ \beta s(x) p_{1j-1}(0) + (j + 1)\alpha s(x) q_{j+1} + \lambda s(x) q_j, j \geq 0, \end{aligned} \tag{1b}$$

$$\begin{aligned} -p'_{ij}(x) = &- \lambda p_{ij}(x) + p\lambda p_{ij-1}(x) + q\lambda p_{i-1j}(x) \\ &+ (1 - \beta) s(x) p_{i+1j}(0) + \beta s(x) p_{i+1j-1}(0), i \geq 1, j \geq 0, \end{aligned} \tag{1c}$$

where $p_{i,-1}(x) = 0$ for any x.

By taking the Laplace transform of equations (1b) and (1c), we obtain

$$\begin{aligned}(\theta-\lambda)p_{0j}^*(\theta)&+p\lambda p_{0j-1}^*(\theta)\\&=p_{0j}(0)-(1-\beta)s^*(\theta)p_{1j}(0)-\beta s^*(\theta)p_{1j-1}(0)\\&-(j+1)\alpha s^*(\theta)q_{j+1}-\lambda s^*(\theta)q_j,\end{aligned} \tag{2b}$$

$$\begin{aligned}(\theta-\lambda)p_{ij}^*(\theta)&+p\lambda p_{ij-1}^*(\theta)+q\lambda p_{i-1j}^*(\theta)\\&=p_{ij}(0)-(1-\beta)s^*(\theta)p_{i+1j}(0)-\beta s^*(\theta)p_{i+1j-1}(0).\end{aligned} \tag{2c}$$

We introduce the following generating function for complex z_2 with $|z_2|\le 1$,

$$Q(z_2)=\sum_{j=0}^{\infty}q_jz_2^j$$

$$P_i^*(\theta,z_2)=\sum_{j=0}^{\infty}p_{ij}^*(\theta)z_2^j,$$

$$P_i(0,z_2)=\sum_{j=0}^{\infty}p_{ij}(0)z_2^j.$$

Multiplying equations (1a),(2b) and (2c) by z_2^j and summing over j, we obtain the following basic equations;

$$\lambda Q(z_2)+\alpha z_2Q'(z_2)=(1-\beta+\beta z_2)P_0(0,z_2), \tag{3a}$$

$$\begin{aligned}(\theta-\lambda&+p\lambda z_2)P_0^*(\theta,z_2)\\&=P_0(0,z_2)-(1-\beta+\beta z_2)s^*(\theta)P_1(0,z_2)\\&-\alpha s^*(\theta)Q'(z_2)-\lambda s^*(\theta)Q(z_2),\end{aligned} \tag{3b}$$

$$\begin{aligned}(\theta-\lambda&+p\lambda z_2)P_i^*(\theta,z_2)+q\lambda P_{i-1}^*(\theta,z_2)\\&=P_i(0,z_2)-(1-\beta+\beta z_2)s^*(\theta)P_{i+1}(0,z_2),\end{aligned} \tag{3c}$$

Next we introduce the generating functions of $P_i^*(\theta,z_2)$ and $P_i(0,z_2)$;

$$P^*(\theta,z_1,z_2)=\sum_{i=0}^{\infty}P_i^*(\theta,z_2)z_1^i$$

$$P(0,z_1,z_2)=\sum_{i=0}^{\infty}P_i(0,z_2)z_1^i.$$

Note that $P^*(0,z_1,z_2)=E(z_1^{N_1}z_2^{N_2};\xi=1)$ which is the joint generating function of (N_1,N_2) when the server is busy.From equations (3b) and (3c), we obtain

$$\begin{aligned}(\theta-\lambda+p\lambda z_2+q\lambda z_1)&P^*(\theta,z_1,z_2)\\ &=\left\{1-(1-\beta)s^*(\theta)\frac{1}{z_1}-\beta s^*(\theta)\frac{z_2}{z_1}\right\}P(0,z_1,z_2)\\ &+\left\{(1-\beta)s^*(\theta)\frac{1}{z_1}+\beta s^*(\theta)\frac{z_2}{z_1}\right\}P_0(0,z_2)\\ &-\alpha s^*(\theta)Q'(z_2)-\lambda s^*(\theta)Q(z_2).\end{aligned} \tag{4}$$

Moreover, the left hand side of the equation (4)vanishes at $\theta=\lambda-p\lambda z_2-q\lambda z_1$, thus we have

$$\begin{aligned}\Big\{z_1-(1-\beta+\beta z_2)&s^*(\lambda-p\lambda z_2-q\lambda z_1)\Big\}P(0,z_1,z_2)\\ &=-(1-\beta+\beta z_2)s^*(\lambda-p\lambda z_2-q\lambda z_1)P_0(0,z_2)\\ &+z_1s^*(\lambda-p\lambda z_2-q\lambda z_1)\Big\{\alpha Q'(z_2)+\lambda Q(z_2)\Big\}.\end{aligned} \tag{5}$$

Now we consider the function

$$f(z_1,z_2)=z_1-(1-\beta+\beta z_2)s^*(\lambda-p\lambda z_2-q\lambda z_1). \tag{6}$$

For each fixed z_2 with $|z_2|<1$, regard $f(z_1,z_2)$ is a function of z_1 . On the unit cirle $|z_1|=1$, we see that

$$Re(\lambda-p\lambda z_2-q\lambda z_1)=\lambda(1-pRe(z_2)-qRe(z_1))>0.$$

It is easy to see that $|s^*(\theta)|<1$ for $Re(\theta)>0$. Thus we have $|s^*(\lambda-p\lambda z_2-q\lambda z_1)|<|z_1|$ on $|z_1|=1$. By Rouche's theorem it follows that for each z_2 with $|z_2|<1$, there exists a unique solution $z_1=\phi(z_2)$ of the equation $f(z_1\ z_2)=0$ in the unit circle, i.e.,

$$f(\phi(z_2),z_2)=\phi(z_2)-(1-\beta+\beta z_2)s^*(\lambda-p\lambda z_2-q\lambda\phi(z_2))=0.$$

Since

$$\left.\frac{\partial f(z_1,z_2)}{\partial z_1}\right|_{z_1=1=z_2}=1-q\rho>0,$$

we conclude by the implicit function theorem that $z_1=\phi(z_2)$ is analytic on $|z_2|<1$ and is continuous at $z_2=1$ and $\phi(1)=1$. The first and second derivative of $\phi(z_2)$ at $z_2=1$ are derived as follows

$$\begin{cases}\phi'(1)=\dfrac{p\rho+\beta}{1-q\rho}\\[2ex] \phi''(1)=\dfrac{1}{(1-q\rho)^3}\Big\{\lambda^2(p+q\beta)^2E(s^2)+2\beta\rho(p+q\beta)(1-q\rho)\Big\}.\end{cases} \tag{7}$$

By substituting $z_1 = \phi(z_2)$ in equation (5), we obtain

$$P_0(0, z_2) = \frac{\alpha\phi(z_2)}{1-\beta+\beta z_2} Q'(z_2) + \frac{\lambda\phi(z_2)}{1-\beta+\beta z_2} Q(z_2). \tag{8}$$

From equations (3a)and (8),we obtain the differential equation

$$Q'(z_2) = \frac{\lambda}{\alpha} \frac{1-\phi(z_2)}{\phi(z_2)-z_2} Q(z_2), \tag{9}$$

whose solution is

$$Q(z_2) = c \exp\left\{-\frac{\lambda}{\alpha}\int_{z_2}^{1} \frac{1-\phi(x)}{\phi(x)-x} dx\right\}, \tag{10}$$

where c is a constant .
By substituting equation (9) into equation (8),we obtain

$$P_0(0, z_2) = \frac{\lambda\phi(z_2)}{1-\beta+\beta z_2} \frac{1-z_2}{\phi(z_2)-z_2} Q(z_2). \tag{11}$$

Thus $Q(z_2)$ and $P_0(0, z_2)$ are known, and $P(0, z_1, z_2)$ is found from the equation (5). Next we find $P^*(0, z_1, z_2)$. Letting $\theta = 0$ in equation (4) and substituting $Q(z_2)$, $P(0, z_2)$ and $P(0, z_1, z_2)$ into equation (4), we obtain

$$\begin{aligned} P^*(0, z_1, z_2) = c \cdot & \frac{1 - s^*(\lambda - p\lambda z_2 - q\lambda z_1)}{1 - pz_2 - qz_1} \\ \times & \frac{z_1 - \phi(z_2)}{z_1 - (1-\beta+\beta z_2)s^*(\lambda - p\lambda z_2 - q\lambda z_1)} \\ \times & \frac{1-z_2}{\phi(z_2)-z_2} \exp\left\{-\frac{\lambda}{\alpha}\int_{z_2}^{1} \frac{1-\phi(x)}{\phi(x)-x} dx\right\}. \end{aligned} \tag{12}$$

In order to find $P^*(0,1,1)$, we first let $z_2 \to 1$ and then $z_1 \to 1$ in equation (11). Using $\phi(1) = 1$ and (7), we obtain by the L'Hospital's rule that

$$\begin{aligned} P^*(0,1,1) &= \lim_{z_1 \to 1} \frac{c}{q} \cdot \frac{s^*(q\lambda - q\lambda z_1) - 1}{z_1 - s^*(q\lambda - q\lambda z_1)} \cdot \frac{1}{1-\phi'(1)} \\ &= c \cdot \frac{\rho}{1-\rho-\beta} . \end{aligned} \tag{13}$$

Finally, to find c which is the probability that the server is idle , we use the total probability $1 = Q(1) + P^*(0,1,1)$, then we have that $c = \frac{1-\rho-\beta}{1-\beta}$ and $P^*(0,1,1) = \frac{\rho}{1-\beta}$ as the probability that the server is busy. Thus we have the following theorem.

Theorem. *The stationary distribution of* (ξ, N_1, N_2) *has the following generating functions*

$$Q(z_2) = E(z_2^{N_2}\ ;\ \xi = 0) = \frac{1-\rho-\beta}{1-\beta}\exp\Big\{-\frac{\lambda}{\alpha}\int_{z_2}^{1}\frac{1-\phi(x)}{\phi(x)-x}dx\Big\}. \tag{14}$$

$$\begin{aligned} P^*(0, z_1, z_2) &= E(z_1^{N_1} z_2^{N_2}; \xi = 1) \\ &= \frac{1-\rho-\beta}{1-\beta}\cdot\frac{1-s^*(\lambda - p\lambda z_2 - q\lambda z_1)}{1-pz_2-qz_1} \\ &\times \frac{z_1-\phi(z_2)}{z_1-(1-\beta+\beta z_2)s^*(\lambda-p\lambda z_2-q\lambda z_1)} \\ &\times \frac{1-z_2}{\phi(z_2)-z_2}\exp\Big\{-\frac{\lambda}{\alpha}\int_{z_2}^{1}\frac{1-\phi(x)}{\phi(x)-x}\,dx\Big\}. \end{aligned} \tag{15}$$

From (14), (15) and (8) we obtain the following information about the mean queue lengths ;

$$E(N_1; \xi = 1) = \frac{q\lambda^2 E(s^2)}{2(1-\beta)(1-q\rho)}. \tag{16}$$

$$E(N_2; \xi = 0) = \frac{\lambda}{\alpha}\frac{p\rho+\beta}{1-\beta}. \tag{17}$$

$$\begin{aligned} E(N_2; \xi = 1) &= \frac{\lambda^2 E(s^2)}{2(1-\beta)}\frac{p(1-\beta)+q\beta\rho}{(1-\beta-\rho)(1-q\rho)} \\ &+ \frac{1}{1-\beta}\frac{\lambda\rho(p\rho+\beta)+\alpha\beta\rho^2}{\alpha(1-\beta-\rho)}. \end{aligned} \tag{18}$$

Corollary. *The mean queue lengths in the priority queue and retrial group are given by*

$$E(N_1) = \frac{q\lambda^2 E(s^2)}{2(1-\beta)(1-q\rho)}, \tag{19}$$

$$\begin{aligned} E(N_2) &= \frac{\lambda^2 E(s^2)}{2(1-\beta)}\cdot\frac{p(1-\beta)+q\beta\rho}{(1-\beta-\rho)(1-q\rho)} \\ &+ \frac{\lambda(1-\beta)(p\rho+\beta)+\alpha\beta\rho^2}{\alpha(1-\beta)(1-\beta-\rho)}. \end{aligned} \tag{20}$$

Remark. Theorem and corollary are consistent with known results when $\beta = 0$. When $\beta = 0$, our model becomes the M/G/1 retrial queue with Bernoulli schedule . In this case $\phi(z_2) = s^*(\lambda - p\lambda z_2 - q\lambda\phi(z_2))$. Equations (14) and (15) reduce to

$$\begin{aligned} Q(z_2) &= E(z_2^{N_2}; \xi = 0) \\ &= (1-\rho)\exp\Big\{ -\frac{\lambda}{\alpha}\int_{z_2}^{1} \frac{1-\phi(x)}{\phi(x)-x}dx \Big\}, \end{aligned} \tag{21}$$

$$\begin{aligned} P^*(0, z_1, z_2) &= E(z_1^{N_1} z_2^{N_2}; \xi = 1) \\ &= (1-\rho)\frac{1 - s^*(\lambda - p\lambda z_2 - q\lambda z_1)}{1 - pz_2 - qz_1} \\ &\times \frac{z_1 - \phi(z_2)}{z_1 - s^*(\lambda - p\lambda z_2 - q\lambda z_1)} \\ &\times \frac{1 - z_2}{\phi(z_2) - z_2}\exp\Big\{ -\frac{\lambda}{\alpha}\int_{z_2}^{1} \frac{1-\phi(x)}{\phi(x)-x}dx \Big\}, \end{aligned} \tag{22}$$

which agree with theorem in Choi and Park [4].

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