J. Appl. Math. & Informatics Vol. **27**(2009), No. 1 - 2, pp. 205 - 215
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A NOTE ON THE EXISTENCE OF SOLUTIONS OF HIGHER-ORDER DISCRETE NONLINEAR STURM-LIOUVILLE TYPE BOUNDARY VALUE PROBLEMS

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ABSTRACT. Sufficient conditions for the existence of at least one solution of the boundary value problems for higher order nonlinear difference equations

$$\begin{cases} \Delta^n x(i-1) = f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2}x(i)), \ i \in [1, T+1], \\ \Delta^m x(0) = 0, \ m \in [0, n-3], \\ \Delta^{n-2}x(0) = \phi(\Delta^{n-1}(0)), \\ \Delta^{n-1}x(T+1) = -\psi(\Delta^{n-2}x(T+1)) \end{cases}$$

are established.

AMS Mathematics Subject Classification : 34B10, 34B15.
Key words and phrases : Solution, higher order difference equation, Sturm-Liouville boundary value problems, fixed-point theorem, growth condition.

1. Introduction

Solvability of boundary value problems for finite difference equations were studied in many papers, one may see the text books [1,2] and the papers [3,4].

In [6], the authors studied the solvability of the problem

$$(BVP) \begin{cases} (E)\ \Delta^n x(i-1) + f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2}x(i)) = 0 \text{ for } i \in [1, T+1] \\ \qquad \text{and } n \geq 2, \\ (BC) \begin{cases} \Delta^m x(0) = 0, \ m \in [0, n-3], \\ \alpha\Delta^{n-2}x(0) - \beta\Delta^{n-1}(0) = 0, \\ \gamma\Delta^{n-2}x(T+1) + \delta\Delta^{n-1}x(T+1) = 0. \end{cases} \end{cases} \tag{1}$$

Suppose

(H_1). $\alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0$ with $\gamma\beta + \gamma\alpha(T+1) + \alpha\delta > 0$;

(H_2). $f \in C([1, T+1] \times R^{n-1}, R)$;

Received September 12, 2007. Revised January 28, 2008. Accepted February 25, 2008.
This paper was supported by Natural Science Foundation of Hunan province (06JJ5008).

(H_3). Suppose v, w are lower and upper solutions of BVP(1), respectively, satisfying

$$\Delta^{n-2}v(i) \leq \Delta^{n-2}w(i) \text{ for } i \in [0, T+1].$$

Then

$$\begin{aligned} f(i, v(i), \Delta v(i), \cdots, \Delta^{n-3}v(i), u_{n-1}) &\leq f(i, u_1, \cdots, u_{n-2}, u_{n-1}) \\ &\leq f(i, w(i), \Delta w(i), \cdots, \Delta^{n-3}w(i), u_{n-1}) \end{aligned}$$

for all $i \in [1, , T+1]$, $u_{n-1} \in R$ and

$$(v(i), \Delta v(i), \cdots, \Delta^{n-3}v(i)) \leq (u_1, \cdots, u_{n-2}) \leq (w(i), \Delta w(i), \cdots, \Delta^{n-3}w(i)),$$

where $(x_1, \cdots, x_{n-2}) \leq (y_1, \cdots, y_{n-2})$ if and only if $x_i \leq y_i$ for all $i \in [1, n-2]$. By using upper and lower solution methods, it was proved that BVP(1) has at least one solution.

In [8,10], the authors studied the Discrete Sturm-Liouville problem

$$\begin{cases} \Delta[p(t-1)\Delta y(t-1)] + q(t)y(t) + \lambda k y(t) = f(t, y(t)) + h(t), \\ a_{11}y(a) + a_{12}\Delta y(a) = 0, \\ a_{21}y(b+1) + a_{22}\Delta y(b+1) = 0 \end{cases} \tag{2}$$

f is subject to the sublinear growth condition

$$|f(t, s)| \leq A|s|^{\alpha} + B, \ t \in [a+1, b+1], \ s \in R$$

for some $0 \leq \alpha < 1$ and $A, B \in (0, +\infty)$. It was proved that BVP(2) has solutions by using the connectivity properties of solution sets of parameterized families of compact vector fields.

In [9], Liu studied the existence of solutions of the following problem

$$\begin{cases} \Delta^2 x(n) = f(n, x(n+1), x(n-\tau_1(n)), \cdots, x(n-\tau_m(n)), \ n \in [0, T-1], \\ ax(0) - b\Delta x(0) = 0, \\ cx(T+1) + d\Delta x(T) = 0, \\ x(i) = \phi(i), \ i \in [-\tau, -1], \\ x(i) = \psi(i), \ i \in [T+2, T+\delta], \end{cases} \tag{3}$$

where $T \geq 1$, $a, b, c, d \in R$ with $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$, $\tau_i(n)$, $i = 1, \cdots, m$, are sequences,

$$\tau = \max\{0, \max_{n \in [0, T-1]} \{\tau_i(n)\} : \ i = 1, \cdots, m\},$$

$$\delta = -\min\{0, \min_{n \in [0, T-1]} \{\tau_i(n)\} : \ i = 1, \cdots, m\},$$

$f(n, u)$ is continuous in $u = (x_0, \cdots, x_m, x_{m+1})$ for each n. We note that the boundary conditions in (3) are different from those in (1) and (2).

Motivated by the papers [6,9], the purposes of this paper are to establish sufficient conditions for the existence of at least one solutions of the problem

$$\begin{cases} \Delta^n x(i-1) = f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2}x(i)),\ i \in [1, T+1], \\ \Delta^m x(0) = 0,\ m \in [0, n-3], \\ \Delta^{n-2}x(0) = \phi(\Delta^{n-1}(0)), \\ \Delta^{n-1}x(T+1) = -\psi(\Delta^{n-2}x(T+1)). \end{cases} \tag{4}$$

The following assumptions are supposed:

(A_1). $f \in C([1, T+1] \times R^{n-1}, R)$, ϕ and ψ are continuous and satisfy $x\phi(x) \geq 0$, $x\psi(x) \geq 0$ for all $x \in R$;

(A_2). There exist constants $\beta > 0, \theta > 1$, nonnegative sequences $p_i(k), r(k) (i = 0, \cdots, n-1)$, functions $g(k, x_1, \cdots, x_{n-1})$, $h(k, x_1, \cdots, x_{n-1})$ such that

$$f(k, x_1, \cdots, x_{n-1}) = g(k, x_1, \cdots, x_{n-1}) + h(k, x_1, \cdots, x_{n-1})$$

$$g(k, x_1, \cdots, x_{n-1})x_{n-1} \geq \beta |x_{n-1}|^{\theta+1},$$

and

$$|h(k, x_1, \cdots, x_{n-1})| \leq \sum_{i=1}^{n-1} p_i(k)|x_i|^\theta + r(k),$$

for all $k \in [1, T+1]$, $(x_1, \cdots, x_{n-1}) \in R^{n-1}$.

BVP(4) is the discrete analogue of the well known Sturm-Liouville BVP of higher order differential equation

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \cdots, x^{(n-2)}(t)) = 0, \quad t \in (0, 1), \\ x^{(i)}(0) = 0, \quad i = 0, \cdots, n-3, \\ x^{(n-2)}(0) = \phi(x^{(n-1)}(0)), \\ x^{(n-1)}(1) = -\psi(x^{(n-2)}(1)), \end{cases}$$

whose special case were studied extensively in [7,11] and the references therein.

This paper is organized as follows. In Section 2, we give the main results, and in Section 3, examples to illustrate the main results will be presented.

2. Main results

Let $X = R^{T+n+1}$ be endowed with the norm

$$||x|| = \max_{n \in [0, T+n]} |x(n)| \text{ for } x \in X,$$

$Y = R^{T+3}$ be endowed with the norm

$$||y|| = \max\{\max_{n \in [1, T+1]} |y(n)|,\ |a|, |b|\} \text{ for } (y, a, b) \in Y.$$

It is easy to see that X and Y are real Banach spaces.

Choose

$$D(L) = \{x \in X :\ \Delta^i x(0) = 0,\ i \in [0, n-3]\}.$$

Set

$$L : D(L) \to Y, \quad (Lx)(i) = \begin{pmatrix} \Delta^n x(i-1), \ i \in [1, T+1] \\ \Delta^{n-2} x(0) \\ \Delta^{n-1} x(T+1) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(i) = \begin{pmatrix} f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2} x(i)), \ i \in [1, T+1], \\ \phi(\Delta^{n-1} x(0)) \\ -\psi(\Delta^{n-2} x(T+1)) \end{pmatrix}$$

for all $x \in X$.

It follows from (A_1) and the definitions of L and N that the followings hold.

(i). $\text{Ker}L = \{x(k) \equiv 0, \ k \in [0, N+n-1]\}$.

(ii). L is a Fredholm operator of index zero.

(iii). Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$.

(iv). $x \in D(L)$ is a solution of $Lx = Nx$ implies that x is a solution of BVP(4).

Theorem L1. *Suppose that (A_1) and (A_2) hold. Then BVP (4) has at least one solution if*

$$(T+1)^\theta \sum_{i=0}^{n-3} ||p_i||(T+1)^{(n-2-i)\theta+1} + ||p_{n-2}|| < \beta. \tag{5}$$

Proof. Let $\Omega_1 = \{x : Lx = \lambda Nx, \ \text{for some } \lambda \in (0,1)\}$. For $x \in \Omega_1$, we have $Lx = \lambda Nx$, $\lambda \in (0,1)$, so

$$\begin{cases} \Delta^n x(i-1) = \lambda f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2} x(i)), \ i \in [1, T+1], \\ \Delta^{n-2} x(0) = \lambda \phi(\Delta^{n-1}(0)), \\ \Delta^{n-1} x(T+1) = -\lambda \psi(\Delta^{n-2} x(T+1)). \end{cases} \tag{6}$$

So for $i \in [1, T+1]$, one has that

$$\sum_{i=1}^{T+1} [\Delta^n x(i-1)] \Delta^{n-2} x(i) = \lambda \sum_{i=1}^{T+1} f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2} x(i)) \Delta^{n-2} x(i).$$

Since $x\phi(x) \geq 0$ and $x\psi(x) \geq 0$ for all $x \in R$, we get

$$\begin{aligned}
&\sum_{i=1}^{T+1}[\Delta^n x(i-1)]\Delta^{n-2}x(i) \\
= &\sum_{i=1}^{T+1}[\Delta^{n-1}x(i) - \Delta^{n-1}x(i-1)][\Delta^{n-1}x(i-1) + \Delta^{n-2}x(i-1)] \\
= &\sum_{i=1}^{T+1}\Big(\Delta^{n-1}x(i)\Delta^{n-2}x(i) - \Delta^{n-1}x(i-1)\Delta^{n-2}x(i-1) - [\Delta^{n-1}x(i-1)]^2\Big) \\
= &-\lambda\psi(\Delta^{n-2}x(T+1))\Delta^{n-2}x(T+1) - \lambda\Delta^{n-1}x(0)\phi(\Delta^{n-1}x(0)) \\
&-\sum_{i=1}^{T+1}[\Delta^{n-1}x(i-1)]^2 \\
\leq &-\sum_{i=1}^{T+1}[\Delta^{n-1}x(i-1)]^2 \leq 0.
\end{aligned}$$

Then $\sum_{i=1}^{T+1} f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2}x(i))\Delta^{n-2}x(i) \leq 0$. It follows that

$$\begin{aligned}
&\beta\sum_{k=1}^{T+1}|\Delta^{n-2}x(k)|^{\theta+1} \\
\leq &\sum_{k=1}^{T+1} g(k, x(k), \Delta x(k), \cdots, \Delta^{n-2}x(k))\Delta^{n-2}x(k) \\
\leq &-\sum_{k=1}^{T+1} h(k, x(k), \Delta x(k), \cdots, \Delta^{n-2}x(k))\Delta^{n-2}x(k) \\
\leq &\sum_{k=1}^{T+1} |h(k, x(k), \Delta x(k), \cdots, \Delta^{n-2}x(k))||\Delta^{n-2}x(k)| \\
\leq &\sum_{i=0}^{n-2}\sum_{k=1}^{T+1} p_i(k)|\Delta^i x(k)|^\theta|\Delta^{n-2}x(k)| + \sum_{k=1}^{T+1}|r(k)||\Delta^{n-2}x(k)| \\
\leq &\sum_{i=0}^{n-3}||p_i||\sum_{k=1}^{T+1}|\Delta^i x(k)|^\theta|\Delta^{n-2}x(k)| + ||r||\sum_{k=1}^{T+1}|\Delta^{n-2}x(k)| \\
&+||p_{n-2}||\sum_{k=1}^{T+1}|\Delta^{n-2}x(k)|^{\theta+1}.
\end{aligned}$$

For $x_i \geq 0, y_i \geq 0$, Holder inequality implies

$$\sum_{i=1}^{s} x_i y_i \leq \left(\sum_{i=1}^{s} x_i^p\right)^{1/p}\left(\sum_{i=1}^{s} y_i^q\right)^{1/q}, \quad 1/p + 1/q = 1, \quad q > 0, \ p > 0.$$

From $\Delta^i x(0) = 0 (i = 0, \cdots, n-3)$, we get

$$|\Delta^i x(k)| = \left|\sum_{j=0}^{k-1} \Delta^{i+1} x(j)\right| \leq (T+1) \max_{j\in[1,T+1]} |\Delta^{i+1} x(j)|, \quad i = 0, \cdots, n-3.$$

It follows that

$$|\Delta^i x(k)| \leq (T+1)^{n-2-i} \sum_{j=1}^{T+1} |\Delta^{n-2} x(j)|, \quad i = 0, \cdots, n-3.$$

It follows that

$$\begin{aligned}
&\beta \sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1} \\
\leq\ & ||p_{n-2}|| \sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1} + ||r||(T+1)^{\frac{\theta}{\theta+1}} \left(\sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}} \\
& + \sum_{i=0}^{n-3} ||p_i||(T+1)^{(n-2-i)\theta} \left(\sum_{j=1}^{T+1} |\Delta^{n-2} x(j)|\right)^{\theta+1} \\
\leq\ & ||p_{n-2}|| \sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1} + ||r||(T+1)^{\frac{\theta}{\theta+1}} \left(\sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}} \\
& + (T+1)^{\theta} \sum_{i=0}^{n-3} ||p_i||(T+1)^{(n-2-i)\theta} \sum_{j=1}^{T+1} |\Delta^{n-2} x(j)|^{\theta+1}.
\end{aligned}$$

We get

$$\begin{aligned}
&\left(\beta - (T+1)^{\theta} \sum_{i=0}^{n-3} ||p_i||(T+1)^{(n-2-i)\theta} - ||p_{n-2}||\right) \sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1} \\
&\leq ||r||(T+1)^{\frac{\theta}{\theta+1}} \left(\sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}}.
\end{aligned}$$

It follows that there is $M > 0$ such that $\sum_{k=1}^{T+1} |\Delta^{n-2} x(k)|^{\theta+1} \leq M$. Hence $|\Delta^{n-2} x(k)| \leq M^{1/(\theta+1)}\}$ for all $k \in [1, T+1]$. Thus

$$|x(k)| \leq (T+1)^{n-2} \sum_{j=1}^{T+1} |\Delta^{n-2} x(j)| \leq (T+1)^{n-1} M^{1/(\theta+1)}.$$

It is easy to see that Ω_1 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \Omega_1$ centered at zero. It is easy to see that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. Thus $Lx = Nx$ has at least one solution $x \in D(L) \cap \overline{\Omega}$, So x is a solution of BVP(4). The proof is complete. □

Theorem L2. *Suppose that*

(A_3) $f \in C([1,T+1]\times R^{n-1}, R)$, *$\phi$ and ψ are continuous and satisfy* $x\phi(x) \geq 0$, $x\psi(x) \geq 0$ *for all* $x \in R$;

(A_4) *There exist constants nonnegative sequences* $p_i(k), r(k) (i = 0, \cdots, n-2)$ *such that*

$$|f(i, x_0, \cdots, x_{n-2})| \leq \sum_{k=0}^{n-2} p_k(i)|x_k| + r(i),$$

for all $i \in [1, T+1]$, $(x_0, x_1, \cdots, x_{n-2}) \in R^{n-1}$;

(A_5) *There exist a constant* $K > 0$ *such that* $|\phi(x)| \leq K|x|$ *for all* $x \in R$.
Then BVP(4) has at least one solution if

$$(T+2)(T+K+1)\sum_{k=0}^{n-2} ||p_k||(T+1)^{n-2-k} < 1. \tag{7}$$

Proof. *We consider the set* $\Omega_1 = \{x : Lx = \lambda Nx, \text{ for some } \lambda \in (0,1)\}$. *For* $x \in \Omega_1$, *one sees* $Lx = \lambda Nx$, $\lambda \in (0,1)$. *So we get (6).*

Case 1. $\Delta^{n-1}x(i) > 0$ *for all* $i \in [1, T+1]$.

At this case, we get $\Delta^{n-2}x(0) < \Delta^{n-2}x(T+1)$. *Since*

$$\Delta^{n-2}x(0) = \lambda\phi(\Delta^{n-1}(0)),$$
$$\Delta^{n-1}x(T+1) = -\lambda\psi(\Delta^{n-2}x(T+1)),$$
$$x\phi(x) \geq 0, \psi(x)x \geq 0, \ x \in R,$$

we get that $\Delta^{n-2}x(0) > 0$ *and* $\Delta^{n-2}x(T+1) < 0$, *a contradiction.*

Case 2. $\Delta^{n-1}x(i) < 0$ *for all* $i \in [1, T+1]$.

At this case, we get $\Delta^{n-2}x(0) > \Delta^{n-2}x(T+1)$. *Similar to above discussion, we get a contradiction.*

It follows that there exists $k \in [1, T]$ *such that* $\Delta^{n-1}x(k)\Delta^{n-1}x(k+1) \leq 0$. *Hence there is* $\xi \in [k, k+1]$ *such that*

$$\frac{\Delta^{n-1}x(k+1) - \Delta^{n-1}x(k)}{k+1-k} = \frac{0 - \Delta^{n-1}x(k)}{\xi - k}.$$

It follows that $|\Delta^{n-1}x(k)| \leq |\Delta^n x(k)|$. *Then for* $i \in [0, k-1]$ *we get*

$$\begin{aligned}
|\Delta^{n-1}x(i)| &= \left|\Delta^{n-1}x(k) - \sum_{j=i}^{k}\Delta^n x(j)\right| \\
&\leq |\Delta^n x(k)| + \sum_{j=i}^{k}\max_{i\in[1,T+1]}|\lambda f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2}x(i))| \\
&\leq (T+2)\max_{i\in[1,T+1]}|\lambda f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2}x(i))|.
\end{aligned}$$

Similar to above discussion, we get for $i \in [k+1, T+1]$ *that*

$$|\Delta^{n-1}x(i)| \leq (T+2)\max_{i\in[1,T+1]}|\lambda f(i, x(i), \Delta x(i), \cdots, \Delta^{n-2}x(i))|.$$

It follows from (A_4) *that*

$$\begin{aligned}|\Delta^{n-1}x(i)| &\leq (T+2)\max_{i\in[1,T+1]}\left(\sum_{k=0}^{n-2}p_k(i)|\Delta^i x(i)|+r(i)\right)\\ &\leq (T+2)\max_{i\in[1,T+1]}\sum_{k=0}^{n-2}||p_k|||\Delta^i x(i)|+||r||.\end{aligned}$$

From $\Delta^i x(0)=0(i=0,\cdots,n-3,$ *we get*

$$|\Delta^i x(i)|=\left|\sum_{j=0}^{i-1}\Delta^{i+1}x(j)\right|\leq (T+1)\max_{j\in[1,T+1]}|\Delta^{i+1}x(j)|,\ \ i=0,\cdots,n-3.$$

It follows that

$$|\Delta^i x(i)|\leq (T+1)^{n-2-i}\max_{j\in[1,T+1]}|\Delta^{n-2}x(j)|,\ \ i=0,\cdots,n-3.$$

It follows from (A_5) *that*

$$\begin{aligned}|\Delta^{n-2}x(i)| &= \left|\Delta^{n-2}x(0)+\sum_{i=0}^{i-1}\Delta^{n-1}x(i)\right|\\ &= \left|\lambda\phi(\Delta^{n-1}x(0))+\sum_{i=0}^{i-1}\Delta^{n-1}x(i)\right|\\ &\leq K|\Delta^{n-1}x(0)|+(T+1)\max_{j\in[1,T+1]}|\Delta^{n-1}x(j)|\\ &\leq (T+K+1)||\max_{j\in[1,T+1]}|\Delta^{n-1}x(j)|.\end{aligned}$$

Then

$$\begin{aligned}|\Delta^k x(i)| &\leq (T+1)^{n-2-k}\max_{j\in[1,T+1]}|\Delta^{n-2}x(j)|\\ &\leq (T+1)^{n-2-k}(T+K+1)\max_{j\in[1,T+1]}|\Delta^{n-1}x(j)|,\ \ k=0,\cdots,n-3.\end{aligned}$$

So

$$\begin{aligned}|\Delta^{n-1}x(i)| \leq\ & (T+2)(T+K+1)\sum_{k=0}^{n-2}||p_k||(T+1)^{n-2-k}\times\\ & \max_{j\in[1,T+1]}|\Delta^{n-1}x(j)|+||r||.\end{aligned}$$

We get

$$\begin{aligned}\max_{j\in[1,T+1]}|\Delta^{n-1}x(j)| \leq\ & (T+2)(T+K+1)\sum_{k=0}^{n-2}||p_k||(T+1)^{n-2-k}\times\\ & \max_{j\in[1,T+1]}|\Delta^{n-1}x(j)|+||r||.\end{aligned}$$

It follows from (7) that there exists a constant $M > 0$ such that

$$\max_{j\in[1,T+1]} |\Delta^{n-1}x(j)| \leq M.$$

Then

$$\begin{aligned} |x(i)| &\leq (T+1)^{n-2}(T+K+1) \max_{j\in[1,T+1]} |\Delta^{n-1}x(j)| \\ &\leq (T+1)^{n-2}(T+K+1)M, \; i \in [1, T+1]. \end{aligned}$$

So Ω_1 is bounded.

Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \Omega_1$ centered at zero. It is easy to see that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. Thus $Lx = Nx$ has at least one solution $x \in D(L) \cap \overline{\Omega}$, So x is a solution of $BVP(4)$. The proof is complete. □

3. Examples

In this section, we present examples, which can not be solved by known results, to illustrate the main results in Section 2.

Example 3.1. *Consider the following BVP*

$$\begin{cases} \Delta^n x(k-1) = \beta[\Delta^{n-2}x(k)]^{2m+1} + \sum_{i=0}^{n-2} p_i(k)[\Delta^i x(k)]^{2m+1} + r(k), \\ \qquad k \in [1, T+1], \\ \Delta^i x(0) = 0, \;\; i = 0, \cdots, n-3, \\ \Delta^{n-2}x(0) = 5[\Delta^{n-1}(0)]^5, \\ \Delta^{n-1}x(T+1) = -\frac{[\Delta^{n-2}x(T+1)]^3}{1+[\Delta^{n-2}x(T+1)]^4}, \end{cases} \tag{8}$$

where m, T, and $n \geq 2$ are a positive integer, $\beta > 0$, $p_i(n), r(n)$ are sequences. Corresponding to the assumptions of Theorem L1, we set

$$g(k, x_0, \cdots, x_{n-2}) = \beta[x_{n-2}]^{2m+1},$$

$$h(k, x_0, \cdots, x_{n-2}) = \sum_{i=0}^{n-2} p_i(k)x_i^{2m+1} + r(k)$$

with $\theta = 2m+1$, and

$$\phi(x) = 5x^5, \;\; \psi(x) = -\frac{x^3}{1+x^4}.$$

It is easy to see that conditions of Theorem L1 hold. It follows from Theorem L1 that (8) has at least one solution if

$$(T+1)^{\frac{\theta}{1+\theta}} \sum_{i=0}^{n-3} ||p_i||(T+1)^{(n-2-i)\theta+1} + ||p_{n-2}|| < \beta.$$

Example 3.2. *Consider the following BVP*

$$\begin{cases} \Delta^n x(k-1) = -\sum_{i=0}^{n-2} p_i(k)|\Delta^i x(k)| - r(k), \quad k \in [1, T+1], \\ \Delta^i x(0) = 0, \ i = 0, \cdots, n-3, \\ \Delta^{n-2} x(0) = 5\Delta^{n-1}(0), \\ \Delta^{n-1} x(T+1) = -\frac{[\Delta^{n-2} x(T+1)]^3}{1 + [\Delta^{n-2} x(T+1)]^4}, \end{cases} \tag{9}$$

where $T \geq 1$ and $n \geq 2$ are a positive integer, $p_i(n), r(n)$ are nonnegative sequences. Corresponding to the assumptions of Theorem L2, we set

$$f(k, x_0, \cdots, x_{n-2}) = -\sum_{i=0}^{n-2} p_i(k)|x_i| - r(k),$$

and

$$\phi(x) = 5x, \ \ \psi(x) = -\frac{x^3}{1+x^4}.$$

It is easy to see that $K = 5$ and the conditions of Theorem L2 hold. It follows from Theorem L2 that (9) has at least one solution if

$$(T+2)(T+5+1) \sum_{k=0}^{n-2} ||p_k||(T+1)^{n-2-k} < 1.$$

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