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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR MULTIPOINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper deals with the multipoint boundary value problem for one dimensional p-Laplacian

$$(\phi_p(u'))'(t) + f(t, u(t)) = 0,\ t \in (0, 1),$$

subject to the boundary value conditions:

$$u'(0) - \sum_{i=1}^{n} \alpha_i u(\xi_i) = 0,\ \ u'(1) + \sum_{i=1}^{n} \alpha_i u(\eta_i) = 0.$$

Using a fixed point theorem for operators on a cone, we provide sufficient conditions for the existence of multiple (at least three) positive solutions to the above boundary value problem.

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1. Introduction

In this paper, we consider the existence and multiplicity of positive solutions for multipoint boundary value problem (BVP for short)

$$(\phi_p(u'))'(t) + f(t, u(t)) = 0,\ t \in (0, 1), \tag{1}$$

$$u'(0) - \sum_{i=1}^{n} \alpha_i u(\xi_i) = 0,\ \ u'(1) + \sum_{i=1}^{n} \alpha_i u(\eta_i) = 0, \tag{2}$$

where $\phi_p(s) = |s|^{p-2}\ s,\ p > 1,\ \ (\phi_p)^{-1} = \phi_q,\ \ \frac{1}{p} + \frac{1}{q} = 1.$

In this article, we assume that:

(H_1) $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1,\ 0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1,\ \ \xi_i < \eta_i,\ \alpha_i > 0$ for $i = 1, 2, \cdots, n;\ \sum_{i=1}^{n} \alpha_i \xi_i < 1,\ \sum_{i=1}^{n} \alpha_i (1 - \eta_i) < 1;$

(H_2) $f \in C([0, 1] \times [0, \infty), (0, \infty)).$

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The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseer [1][2]. Then Gupta [5] studied three-point boundary value problems for nonlinear ordinary differential equations. Since then, there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see [3-4,6,9,13,14]. In paper [5], the author studied the existence of solutions for the generalized multipoint boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + r(t), \ 0 \le t \le 1,$$
$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(1) = \sum_{i=1}^{n-2} \beta_i x'(\eta_i),$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1,\ \alpha_i, \beta_i \in R$ and $(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \sum_{i=1}^{n-2} \beta_i) \neq 0$. The author established some existence results for the above BVP.

Recently, Wang [12] considered the multipoint BVP for the one dimensional p-Laplacian

$$(\phi_p(u'))'(t) + f(t, u(t)) = 0, \ t \in (0, 1),$$
$$\phi_p(u'(0)) = \sum_{i=1}^{n-2} a_i \phi_p(u'(\xi_i)), \quad u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i),$$

where $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$, and a_i, b_i satisfy a_i, $b_i \in [0, \infty)$, $0 < \sum_{i=1}^{n-2} a_i < 1$, and $\sum_{i=1}^{n-2} b_i < 1$. Using a fixed point theorem in a cone, the authors provided sufficient conditions for the existence of multiple positive solutions to the above boundary value problem. Ma [7] obtained the existence of monotone positive solutions for the following BVP

$$(\phi_p(u'))' + q(t) f(t, u) = 0, \qquad t \in (0, 1), \tag{3}$$

$$u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i), \tag{4}$$

where $\xi_i \in (0, 1)$ and $0 \le \alpha_i, \beta_i < 1$ satisfy $0 \le \sum_{i=1}^{n} \alpha_i,\ \sum_{i=1}^{n} \beta_i < 1$. The main tool is the monotone iterative technique.

However, to the best of our knowledge, no work has been done for BVP (1), (2). The aim of this paper is to fill the gap in the relevant literatures. In this paper, a positive solution $u(t)$ of BVP (1),(2) means a solution $u(t)$ of BVP (1),(2) satisfying $u(t) > 0,\ 0 < t < 1$.

To obtain positive solutions of (1) and (2), the following fixed point theorem in cones is fundamental.

Lemma 1. [10, 11] *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_k = D \cap K \neq \emptyset$ and $\overline{D_k} \neq K$. Assume that $T : \overline{D_k} \longrightarrow K$ is a compact map such that $x \neq Tx$ for $x \in \partial D_k$. Then the following results hold:*

(1) *If $\|Tx\| \le \|x\|,\ \ x \in \partial D_k$, then $i_k(T, D_k) = 1$.*

(2) *If there exists* $e \in K \setminus \{0\}$ *such that* $x \neq Tx + \lambda e$ *for all* $x \in \partial D_k$ *and all* $\lambda > 0$, *then* $i_k(T, D_k) = 0$.

(3) *Let* U *be open in* X *such that* $\overline{U} \subset D_k$. *If* $i_k(T, D_k) = 1$ *and* $i_k(T, U_k) = 0$, *then* T *has a fixed point in* $D_k \setminus \overline{U_k}$. *The same result holds if* $i_k(T, D_k) = 0$ *and* $i_k(T, U_k) = 1$.

2. Preliminary

Let $E = C[0,1]$, then E is a Banach space with the norm $\|u\| := \max_{0 \le t \le 1} |u(t)|$. We denote

$$C^+[0,1] = \{u \in C[0,1] : \ u(t) \ge 0, \ t \in [0,1]\}.$$

Lemma 2. *Let* $(H_1), (H_2)$ *hold. Then for* $x \in C^+[0,1]$, *the problem*

$$(\phi_p(u'))'(t) + f(t, x(t)) = 0, \ t \in (0,1), \tag{5}$$

$$u'(0) - \sum_{i=1}^{n} \alpha_i u(\xi_i) = 0, \quad u'(1) + \sum_{i=1}^{n} \alpha_i u(\eta_i) = 0, \tag{6}$$

has a solution

$$\begin{aligned} u(t) \ &= \frac{\phi_q(A_x) + \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{1} \phi_q(A_x - \int_0^s f(\tau, x(\tau))d\tau) ds}{\sum_{i=1}^{n} \alpha_i} \\ &\quad - \int_t^1 \phi_q\left(A_x - \int_0^s f(\tau, x(\tau))d\tau\right) ds, \end{aligned} \tag{7}$$

where A_x *satisfies*

$$\phi_q(A_x) + \phi_q(A_x - \int_0^1 f(\tau, x(\tau))d\tau) + \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q(A_x - \int_0^s f(\tau, x(\tau))d\tau) ds = 0, \tag{8}$$

then there exists a unique $A_x \in (0, \int_0^1 f(\tau, x(\tau))d\tau)$ *satisfying (8). This implies that there is a unique* $\sigma_x \in (0,1)$ *such that* $A_x = \int_0^{\sigma_x} f(\tau, x(\tau))d\tau$.

Proof. For any $x \in C^+[0,1]$, define

$$H_x(c) = \phi_q(c) + \phi_q(c - \int_0^1 f(\tau, x(\tau))d\tau) + \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q(c - \int_0^s f(\tau, x(\tau))d\tau) ds.$$

Then, $H_x : R \to R$ is continuous and strictly increasing.

$$H_x(0) < 0, \quad H_x\left(\int_0^1 f(\tau, x(\tau))d\tau\right) > 0,$$

imply the existence of a unique $c = A_x \in (0, \int_0^1 f(\tau, x(\tau))d\tau)$ such that $H_x(A_x) = 0$. Then the existence of $\sigma_x \in (0,1)$ is obvious. □

Lemma 3. *Let $(H_1),(H_2)$ hold. Then for $x \in C^+[0,1]$, the solution of BVP (5),(6) can also be expressed:*

$$u(t) = \frac{\phi_q(B_x + \int_0^1 f(\tau,x(\tau))d\tau) - \sum_{i=1}^n \alpha_i \int_0^{\xi_i} \phi_q(B_x + \int_s^1 f(\tau,x(\tau))d\tau)ds}{\sum_{i=1}^n \alpha_i} + \int_0^t \phi_q(B_x + \int_s^1 f(\tau,x(\tau))d\tau)ds, \quad (9)$$

where B_x satisfies

$$\phi_q(B_x) + \phi_q(B_x + \int_0^1 f(\tau,x(\tau))d\tau) + \sum_{i=1}^n \alpha_i \int_{\xi_i}^{\eta_i} \phi_q(B_x + \int_s^1 f(\tau,x(\tau))d\tau)ds = 0, \quad (10)$$

then there exists a unique $B_x \in (-\int_0^1 f(\tau,x(\tau))d\tau, 0)$ satisfying (10). This implies that there is a unique $\tau_x \in (0,1)$ such that $B_x = -\int_{\tau_x}^1 f(\tau,x(\tau))d\tau$.

Proof. The proof is similar to lemma 2, we omit it here. □

Lemma 4. *Let $(H_1),(H_2)$ hold. Then for $x \in C^+[0,1]$, the unique solution $u(t)$ of BVP (5),(6) has the following properties:*
(i) $u(t)$ is concave on $(0,1)$;
(ii) $u(t) > 0$;
(iii) there exists a unique $t_0 \in (0,1)$, st $u'(t_0) = 0$;
(iv) $\sigma_x = \tau_x = t_0$.

Proof. Suppose that $u(t)$ is a solution of BVP (5),(6), then

(*i*) $(\phi_p(u'))'(t) = -f(t,x(t)) \le 0$, $\phi_p(u')$ is nonincreasing, so $u'(t)$ is nonincreasing, this implies that $u(t)$ is concave.

(*ii*) From lemma 2 and 3, we can get $u'(0) > 0, \quad u'(1) < 0$. Furthermore, since

$$\alpha_1 u(\xi_1) - \alpha_1 u(0) = \alpha_1 \int_0^{\xi_1} u'(s)ds \le \alpha_1 \xi_1 u'(0) = \alpha_1 \xi_1 \sum_{i=1}^n \alpha_i u(\xi_i),$$

$$\alpha_2 u(\xi_2) - \alpha_2 u(0) = \alpha_2 \int_0^{\xi_2} u'(s)ds \le \alpha_2 \xi_2 u'(0) = \alpha_2 \xi_2 \sum_{i=1}^n \alpha_i u(\xi_i),$$

$$\cdots \qquad \cdots$$

$$\alpha_n u(\xi_n) - \alpha_n u(0) = \alpha_n \int_0^{\xi_n} u'(s)ds \le \alpha_n \xi_n u'(0) = \alpha_n \xi_n \sum_{i=1}^n \alpha_i u(\xi_i),$$

using (H_1) and $u'(0) = \sum_{i=1}^n \alpha_i u(\xi_i) > 0, \quad u'(1) = -\sum_{i=1}^n \alpha_i u(\eta_i) < 0$,we have

$$\sum_{i=1}^n \alpha_i u(\xi_i) - \sum_{i=1}^n \alpha_i u(0) \le \sum_{i=1}^n \alpha_i u(\xi_i) \sum_{i=1}^n \alpha_i \xi_i < \sum_{i=1}^n \alpha_i u(\xi_i),$$

i.e., $\sum_{i=1}^n \alpha_i u(\xi_i) - \sum_{i=1}^n \alpha_i u(0) < \sum_{i=1}^n \alpha_i u(\xi_i)$, which implies that $u(0) > 0$. Similarly, we can prove that $u(1) > 0$. Therefore, we get $u(t) > 0, \quad t \in [0,1]$.

(*iii*) $u'(0) > 0$, $u'(1) < 0$ imply that there is a $t^* \in (0,1)$, such that $u'(t^*) = 0$.

If there exist $t_1,\ t_2 \in (0,1),\ t_1 < t_2$, such that $u'(t_1) = 0 = u'(t_2)$, then

$$0 = \phi_p(u'(t_2)) - \phi_p(u'(t_1)) = -\int_{t_1}^{t_2} f(s, x(s))ds < 0,$$

this is a contradiction.

(*iv*) From lemma 2 and 3, we have $u'(t) = \phi_q(\int_t^{\sigma_x} f(\tau, x(\tau))d\tau)$ and $u'(t) = \phi_q(\int_t^{\tau_x} f(\tau, x(\tau))d\tau)$, so $u'(\sigma_x) = u'(\tau_x) = u'(t_0) = 0$. Therefore, $\sigma_x = \tau_x = t_0$. □

Lemma 5. *Let*

$$K = \{u \in E :\ u(t) \geq 0,\ u \text{ is concave on } [0,1]\}.$$

Define an operator $T : K \to E$ *by*

$$(Tu)(t) = \begin{cases} \dfrac{\phi_q(\int_0^{\sigma_u} f(\tau, u(\tau))d\tau) + \sum_{i=1}^n \alpha_i \int_0^{\xi_i} \phi_q(\int_{\sigma_u}^s f(\tau, u(\tau))d\tau)ds}{\sum_{i=1}^n \alpha_i} \\ + \int_0^t \phi_q(\int_s^{\sigma_u} f(\tau, u(\tau))d\tau)ds, \\ 0 \leq t \leq \sigma_u, \\ \dfrac{\phi_q(\int_0^{\sigma_u} f(\tau, u(\tau))d\tau) + \sum_{i=1}^n \alpha_i \int_{\xi_i}^1 \phi_q(\int_s^{\sigma_u} f(\tau, u(\tau))d\tau)ds}{\sum_{i=1}^n \alpha_i} \\ + \int_t^1 \phi_q(\int_{\sigma_u}^s f(\tau, u(\tau))d\tau)ds, \\ \sigma_u \leq t \leq 1. \end{cases} \tag{11}$$

Then $T : K \to K$ *is completely continuous.*

Proof. It is easy to check that K is a cone in E. From Lemma 4, it is easy to see that $(Tu)(t) \geq 0,\ (Tu)'(0) - \sum_{i=1}^n \alpha_i (Tu)(\xi_i) = 0,\ (Tu)'(1) + \sum_{i=1}^n \alpha_i (Tu)(\eta_i) = 0$, and $(\phi_p(Tu)'(t))' = -f(t, u(t)) \leq 0$, this shows $T(K) \subset K$. It is easy to see that $T : K \to K$ is completely continuous. □

Lemma 6. [8] *Let* $u \in K$, *and* $\theta \in (0, \frac{1}{2})$ *is a constant. Then*

$$u(t) \geq \theta \|u\|,\quad t \in [\theta, 1-\theta].$$

Let $\theta \in (0, \frac{1}{2})$ is a constant, and we define

$$\begin{aligned} \gamma_1 &= \left[\frac{\theta(\frac{1}{2} - \theta)^q}{q}\right] \div \left[\frac{q + \sum_{i=1}^n \alpha_i \xi_i^q + \sum_{i=1}^n \alpha_i}{q \sum_{i=1}^n \alpha_i}\right], \\ \gamma &= \theta \left\{\left[\frac{\theta(\frac{1}{2} - \theta)^q}{q}\right] \div \left[\frac{q + \sum_{i=1}^n \alpha_i \xi_i^q + \sum_{i=1}^n \alpha_i}{q \sum_{i=1}^n \alpha_i}\right]\right\}, \\ K_\rho &= \{u \in K :\ \|u\| < \rho\}, \\ \Omega_\rho &= \{u \in K :\ \min_{\theta \leq t \leq 1-\theta} u(t) < \gamma\rho\} \\ &= \{u :\ u \in C[0,1],\ u(t) \geq 0,\ u \text{ is concave on } [0,1], \\ &\qquad \gamma\|u\| \leq \min_{\theta \leq t \leq 1-\theta} u(t) < \gamma\rho\}. \end{aligned}$$

Lemma 7. [11] Ω_ρ *has the following properties:*

(a) Ω_ρ *is open relative to* K.
(b) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$.
(c) $u \in \partial\Omega_\rho$ *if and only if* $\min_{\theta \le t \le 1-\theta} u(t) = \gamma\rho$.
(d) *If* $u \in \partial\Omega_\rho$, *then* $\gamma\rho \le u(t) \le \rho$ *for* $t \in [\theta, 1-\theta]$.

Now for convenience we introduce the following notations. Let

$$\begin{aligned} f\,{}^{\rho}_{\gamma\rho} &= \min\left\{\frac{f(t,u)}{\phi_p(\rho)}: \ t \in [\theta, 1-\theta],\ u \in [\gamma\rho, \rho]\right\}, \\ f_0^\rho &= \max\left\{\frac{f(t,u)}{\phi_p(\rho)}: \ t \in [0,1],\ u \in [0,\rho]\right\}, \\ f^\alpha &= \lim_{u\to\alpha} \max\left\{\frac{f(t,u)}{\phi_p(u)}: \ t \in [0,1]\right\}, \\ f_\alpha &= \lim_{u\to\alpha} \min\left\{\frac{f(t,u)}{\phi_p(u)}: \ t \in [\theta, 1-\theta]\right\} \qquad (\alpha := \infty \text{ or } 0^+), \\ \frac{1}{m} &= \frac{q + \sum_{i=1}^n \alpha_i \xi_i^q + \sum_{i=1}^n \alpha_i}{q\sum_{i=1}^n \alpha_i}, \quad \frac{1}{M} = \frac{\theta(\frac{1}{2}-\theta)^q}{q}. \end{aligned}$$

Lemma 8. *It is easy to see that* $0 < m, M < \infty$ *and* $M\gamma = M\theta\gamma_1 = \theta m < m$.

Lemma 9. *If* f *satisfies the condition:*

$$f_0^\rho < \phi_p(m), \tag{12}$$

then $i_k(T, K_\rho) = 1$.

Proof. By (11) and (12), we have for $u(t) \in \partial K_\rho$,

$$\begin{aligned} (Tu)(t) &\le \frac{\phi_q(\int_0^1 f(\tau,u(\tau))d\tau) + \sum_{i=1}^n \alpha_i \int_0^{\xi_i} \phi_q(\int_0^s f(\tau,u(\tau))d\tau)ds}{\sum_{i=1}^n \alpha_i} \\ &\quad + \int_0^1 \phi_q(\int_s^1 f(\tau,u(\tau))d\tau)ds \\ &< m\rho\left(\frac{\phi_q(\int_0^1 1d\tau) + \sum_{i=1}^n \alpha_i \int_0^{\xi_i} \phi_q(\int_0^s 1d\tau)ds}{\sum_{i=1}^n \alpha_i} + \int_0^1 \phi_q(\int_s^1 1d\tau)ds\right) \\ &= m\rho\frac{q + \sum_{i=1}^n \alpha_i \xi_i^q + \sum_{i=1}^n \alpha_i}{q\sum_{i=1}^n \alpha_i} = \rho. \end{aligned}$$

This implies that $\|Tu\| < \|u\|$ for $u(t) \in \partial K_\rho$. By lemma 1.1 (1), we have $i_k(T, K_\rho) = 1$. □

Lemma 10. *If* f *satisfies the condition:*

$$f^\rho_{\gamma\rho} > \phi_p(M\gamma), \tag{13}$$

then $i_k(T, \Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $e \in \partial K_1$, we claim that

$$u \neq Tu + \lambda e, \quad u \in \partial\Omega_\rho, \quad \lambda \geq 0.$$

In fact, if not, there exist $u_0 \in \partial\Omega_\rho$ and $\lambda_0 \geq 0$ such that $u_0 = Tu_0 + \lambda_0 e$. In the following, we shall discuss it from two perspectives.

(i) If $\sigma_u \in [\frac{1}{2}, 1)$. By (11), (13) and lemma 7 (d), we have that for $t \in [\theta, 1-\theta]$,

$$\begin{aligned} u_0(t) &= Tu_0(t) + \lambda_0 e(t) \geq \theta \|Tu_0(t)\| + \lambda_0 \\ &\geq \theta \int_0^{\sigma_u} \phi_q(\int_s^{\sigma_u} f(\tau, u_0(\tau))d\tau)ds + \lambda_0 \\ &\geq \theta \int_\theta^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} f(\tau, u_0(\tau))d\tau)ds + \lambda_0 > \theta\rho M\gamma \int_\theta^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} d\tau)ds + \lambda_0 \\ &= \theta\rho M\gamma \frac{(\frac{1}{2}-\theta)^q}{q} + \lambda_0 = \gamma\rho + \lambda_0. \end{aligned}$$

(ii) If $\sigma_u \in (0, \frac{1}{2})$, By (11), (13) and lemma 7 (d), we have that for $t \in [\theta, 1-\theta]$,

$$\begin{aligned} u_0(t) &= Tu_0(t) + \lambda_0 e(t) \geq \theta \|Tu_0(t)\| + \lambda_0 \\ &\geq \theta \int_{\sigma_u}^{1} \phi_q(\int_{\sigma_u}^{s} f(\tau, u_0(\tau))d\tau)ds + \lambda_0 \\ &\geq \theta \int_{\frac{1}{2}}^{1-\theta} \phi_q(\int_{\frac{1}{2}}^{s} f(\tau, u_0(\tau))d\tau)ds + \lambda_0 > \theta\rho M\gamma \int_{\frac{1}{2}}^{1-\theta} \phi_q(\int_{\frac{1}{2}}^{s} d\tau)ds + \lambda_0 \\ &= \theta\rho M\gamma \frac{(\frac{1}{2}-\theta)^q}{q} + \lambda_0 = \gamma\rho + \lambda_0. \end{aligned}$$

Thus, in all cases, this imply that $\gamma\rho > \gamma\rho + \lambda_0$, which is a contradiction. Hence, by lemma 1 (2), it follows that $i_k(T, \Omega_\rho) = 0$. □

3. Main result

We now give our results on the existence of multiple positive solutions of BVP (1) and (2).

Theorem 1. *Assume $(H_1), (H_2)$ hold. In addition, the following condition (H_3) holds:*

(H_3) *There exist $\rho_1,\ \rho_2,\ \rho_3 \in (0, \infty)$, with $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3$ such that*

$$f_0^{\rho_1} < \phi_p(m), \qquad f_{\gamma\rho_2}^{\rho_2} > \phi_p(M\gamma), \qquad f_0^{\rho_3} \leq \phi_p(m).$$

Then problem (1) and (2) has three positive solutions in K.

Assume $(H_1), (H_2)$ hold. In addition, the following condition

(H_4) *holds:*

(H_4) *There exist $\rho_1,\ \rho_2,\ \rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2 < \gamma\rho_3$ such that*

$$f_{\gamma\rho_1}^{\rho_1} > \phi_p(M\gamma), \quad f_0^{\rho_2} < \phi_p(m), \quad f_{\gamma\rho_3}^{\rho_3} \geq \phi_p(M\gamma).$$

Then problem (1) and (2) has two positive solutions in K.

The proof is similar to that given for Theorem 2.10 in [11], we omit it here. As a special case of Theorem 1, we obtain the following result.

Corollary 1. *Assume* $(H_1), (H_2)$ *hold. In addition, if there exists* $\rho \in (0, \infty)$, *such that the following condition* (H_5) *holds:*

$$0 \leq f^0 < \phi_p(m), \quad f^\rho_{\gamma\rho} > \phi_p(M\gamma), \quad 0 \leq f^\infty < \phi_p(m).$$

Then problem (1) and (2) has three positive solutions in K.
Assume $(H_1), (H_2)$ *hold. In addition, if there exists* $\rho \in (0, \infty)$, *such that the following condition* (H_6) *holds:*

$$\phi_p(M) < f_0 \leq \infty, \quad f^\rho_0 < \phi_p(m), \quad \phi_p(M) < f_\infty \leq \infty.$$

Then problem (1) and (2) has two positive solutions in K.

Proof. We show that (H_5) implies (H_3). It is easy to verify that $0 \leq f^0 < \phi_p(m)$ implies that there exists $\rho_1 \in (0, \gamma\rho)$ such that $f^{\rho_1}_0 < \phi_p(m)$. Let $k \in (f^\infty, \phi_p(m))$. Then there exists $r > \rho$ such that $\max_{t\in[0,1]} f(t,u) \leq k\phi_p(u)$ for $u \in [r, \infty)$ since $0 \leq f^\infty < \phi_p(m)$. Let

$$\beta = \max\left\{\max_{t\in[0,1]} f(t,u): \quad 0 \leq u \leq r\right\} \quad \text{and} \quad \rho_3 > \phi_q\left(\frac{\beta}{\phi_p(m) - k}\right).$$

Then we have

$$\max_{t\in[0,1]} f(t,u) \leq k\phi_p(u) + \beta \leq k\phi_p(\rho_3) + \beta < \phi_p(m)\phi_p(\rho_3) \quad \text{for } u \in [0, \rho_3].$$

This implies that $f^{\rho_3}_0 < \phi_p(m)$ and (H_3) holds. Similarly (H_6) implies (H_4). □

By an argument similar to that of Theorem 1 we obtain the following results.

Theorem 2. *Assume* $(H_1), (H_2)$ *hold. In addition, one of the following conditions holds:*
(H_7) *There exist* $\rho_1,\ \rho_2 \in (0, \infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that* $f^{\rho_1}_0 \leq \phi_p(m)$, $f^{\rho_2}_{\gamma\rho_2} \geq \phi_p(M\gamma)$,
(H_8) *There exist* $\rho_1,\ \rho_2 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *such that* $f^{\rho_1}_{\gamma\rho_1} \geq \phi_p(M\gamma)$, $f^{\rho_2}_0 \leq \phi_p(m)$.
Then problem (1) and (2) has a positive solution in K.

As a special case of Theorem 2 we obtain the following results.

Corollary 2. *Assume* $(H_1), (H_2)$ *hold. In addition, one of the following conditions holds:*
(H_9) $0 \leq f^0 < \phi_p(m), \quad \phi_p(M) < f_\infty \leq \infty$,
(H_{10}) $0 \leq f^\infty < \phi_p(m), \quad \phi_p(M) < f_0 \leq \infty$.
Then problem (1) and (2) has a positive solution in K.

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