Bull. Korean Math. Soc. **46** (2009), No. 6, pp. 1229–1236
DOI 10.4134/BKMS.2009.46.6.1229

QUASI-INNER FUNCTIONS OF A GENERALIZED BEURLING'S THEOREM

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Abstract. We introduce two kinds of quasi-inner functions. Since every rationally invariant subspace for a shift operator S_K on a vector-valued Hardy space $H^2(\Omega, K)$ is generated by a quasi-inner function, we also provide relationships of quasi-inner functions by comparing rationally invariant subspaces generated by them. Furthermore, we discuss fundamental properties of quasi-inner functions and quasi-inner divisors.

1. Introduction

Beurling characterized all invariant subspaces for the shift operator on the Hardy space H^2 in terms of inner functions [2]. If φ and ϕ are inner functions such that $\varphi H^2 \subset \phi H^2$, then we have φ is divisible by ϕ [3]. In fact, the converse is also true [1].

In this paper, Ω denotes a bounded finitely connected region in the complex plane and $R(\Omega)$ denotes the algebra of rational functions with poles off $\overline{\Omega}$.

For a Hilbert space K and a shift operator S_K on a vector-valued Hardy space $H^2(\Omega, K)$, every $R(\Omega)$-invariant (rationally invariant) subspace M for the operator S_K is characterized in terms of quasi-inner functions [4]; $M = \psi H^2(\Omega, K')$ for some quasi-inner function $\psi : \Omega \to L(K', K)$ and a Hilbert space K'. Even though a quasi-inner function is defined as an operator-valued function in [4], by the Riesz representation theorem, we also provide a definition of a scalar-valued quasi-inner function.

For quasi-inner functions $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $u \in H^\infty(\Omega)$, we discuss some relationships between operator-valued and scalar-valued quasi-inner functions (Theorem 3.4). In addition, by using a multiplication operator on a vector-valued Hardy space, we study quasi-inner functions (Corollary 3.6).

For quasi-inner functions $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, K)$, we provide definitions of the following two cases;

(1) θ is divisible by φ.

Received October 14, 2008.

2000 *Mathematics Subject Classification.* 47A15, 47A56, 47B37, 47B38.

Key words and phrases. a generalized Beurling's theorem, Hardy spaces, quasi-inner functions, rationally invariant subspaces.

(2) φ is divisible by θ.

With these definitions, we characterize those divisibilities by comparing $R(\Omega)$-invariant subspaces, $\theta H^2(\Omega, K)$ and $\varphi H^2(\Omega, K)$ (Theorem 4.3 and Theorem 4.4); for any quasi-inner functions $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, the following assertions are equivalent:

(a) $\theta|\varphi$.

(b) $\varphi H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K)$.

(c) $\varphi H^2(\Omega, K) \subset \theta H^2(\Omega, K)$.

(d) There is a $\lambda > 0$ such that $\varphi(z)\varphi(z)^* \leq \lambda^2|\theta(z)|^2 I_K$ for any $z \in \Omega$.

2. Preliminaries and notation

In this paper, $\mathbb{C}$, $\overline{M}$, and $L(H)$ denote the set of complex numbers, the (norm) closure of a set M, and the set of bounded linear operators from H to H where H is a Hilbert space, respectively.

2.1. Inner functions

Let $\mathbf{D}$ be the open unit disc. We denote by H^∞ the Banach space of all bounded analytic functions $\phi : \mathbf{D} \to \mathbb{C}$ with the norm $\|\phi\|_\infty = \sup\{|\phi(z)| : z \in \mathbf{D}\}$.

Let θ and θ' be two functions in H^∞. We say that θ *divides* θ'(or $\theta|\theta'$) if θ' can be written as $\theta' = \theta \cdot \phi$ for some $\phi \in H^\infty$. We will use the notation $\theta \equiv \theta'$ if $\theta|\theta'$ and $\theta'|\theta$.

Recall that a function $u \in H^\infty$ is *inner* if $|u(e^{it})|= 1$ almost everywhere on $\partial\mathbf{D}$. By Beurling's theorem on invariant subspaces of the Hardy spaces, for any inner function $\theta \in H^\infty$, we have that θH^2 is an invariant subspace for the shift operator $S : H^2 \to H^2$ defined by $(Sf)(z) = zf(z)$ for $f \in H^2$.

2.2. Hardy spaces

We refer to [5] for basic facts about Hardy space, and recall here the basic definitions. Let Ω be a bounded finitely connected region in the complex plane.

Definition 2.1. The space $H^2(\Omega)$ is defined to be the space of analytic functions f on Ω such that the subharmonic function $|f|^2$ has a harmonic majorant on Ω. For a fixed $z_0 \in \Omega$, there is a norm on $H^2(\Omega)$ defined by

$$\|f\| = \inf\{u(z_0)^{1/2} : u \text{ is a harmonic majorant of } |f|^2\}.$$

Let m be the harmonic measure for the point z_0, let $L^2(\partial\Omega)$ be the L^2-space of complex valued functions on the boundary of Ω defined with respect to m, and let $H^2(\partial\Omega)$ be the set of functions f in $L^2(\partial\Omega)$ such that $\int_{\partial\Omega} f(z)g(z)dz = 0$ for every g that is analytic in a neighborhood of the closure of Ω. If f is in $H^2(\Omega)$, then there is a function f^* in $H^2(\partial\Omega)$ such that $f(z)$ approaches $f^*(\lambda_0)$ as z approaches λ_0 nontangentially, for almost every λ_0 relative to m. The map $f \to f^*$ is an isometry from $H^2(\Omega)$ onto $H^2(\partial\Omega)$.

A function f defined on Ω is in $H^\infty(\Omega)$ if it is holomorphic and bounded. Then, $H^\infty(\Omega)$ is a closed subspace of $L^\infty(\Omega)$ and it is a Banach algebra if endowed with the supremum norm. Finally, the mapping $f \to f^*$ is an isometry of $H^\infty(\Omega)$ onto a week*-closed subalgebra of $L^\infty(\partial\Omega)$.

Definition 2.2. If K is a Hilbert space, then $H^2(\Omega,K)$ is defined to be the space of analytic functions $f : \Omega \to K$ such that the subharmonic function $\|f\|^2$ is majorized by a harmonic function ν. Fix a point z_0 in Ω and define a norm on $H^2(\Omega,K)$ by

$$\|f\| = \inf\{\nu(z_0)^{1/2} \ : \ \nu \text{ is a harmonic majorant of } \|f\|^2\}.$$

We will work on this vector-valued Hardy space $H^2(\Omega,K)$. Define a shift operator $S_K : H^2(\Omega, K) \to H^2(\Omega, K)$ by

$$(S_K f)(z) = zf(z).$$

3. Quasi-inner functions

Let $R(\Omega)$ denote the algebra of rational functions with poles off $\overline{\Omega}$, and T be an operator in $L(H)$ such that $\sigma(T) \subset \overline{\Omega}$. Then a closed subspace M is said to be $R(\Omega)$-*invariant* (*rationally invariant*) for the operator T, if it is invariant under $u(T)$ for any function $u \in R(\Omega)$.

To characterize every $R(\Omega)$-invariant subspace for the shift operator S_K, quasi-inner function was defined in [4].

Definition 3.1. Let K and K' be Hilbert spaces and let $H^\infty(\Omega, L(K, K'))$ be the Banach space of all analytic functions $\Phi : \Omega \to L(K, K')$ with the supremum norm. For $\varphi \in H^\infty(\Omega, L(K, K'))$, we will say that φ is *quasi-inner* if there exists a constant $c > 0$ such that

$$\|\varphi(z)k\| \geq c\,\|k\|$$

for every $k \in K$ and almost every $z \in \partial\Omega$.

Even though a quasi-inner function is defined as an operator-valued function, by the Riesz representation theorem, we can identify $L(\mathbb{C})$ with $\mathbb{C}$. Thus we have the following definition of a scalar-valued quasi-inner function:

Definition 3.2. For $\theta \in H^\infty(\Omega)$, we will say that θ is *quasi-inner* if there exists a constant $c > 0$ such that

$$|\theta(z)| \geq c$$

for almost every $z \in \partial\Omega$.

Proposition 3.3. *Let K and K' be Hilbert spaces with* $\dim K = \dim K' = n(< \infty)$.

If $\varphi \in H^\infty(\Omega, L(K, K'))$ is a quasi-inner function, then $\varphi(z)$ is invertible a.e. on $\partial\Omega$.

Proof. Since $\varphi \in H^\infty(\Omega, L(K, K'))$ is quasi-inner, there is a set $A \subset \partial\Omega$ with $m(A) = 0$ such that the range of $\varphi(z_0)$ is closed, and $\varphi(z_0)$ is one-to-one for any $z_0 \in \partial\Omega \setminus A$.

Thus, K and the range of $\varphi(z_0)$ have the same dimension.

Since dim $K =$ dim K', we conclude that the range of $\varphi(z_0)$ is K'. Thus $\varphi(z)$ is invertible for $z \in \partial\Omega\backslash A$. □

Theorem 3.4. (a) *If $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $u \in H^\infty(\Omega)$ are quasi-inner functions such that*

$$\varphi(z)\psi(z) = u(z)I_{\mathbb{C}^n},$$

where $\psi \in H^\infty(\Omega, L(\mathbb{C}^n))$, then ψ is also quasi-inner.

(b) *Conversely, if $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $\psi \in H^\infty(\Omega, L(\mathbb{C}^n))$ are quasi-inner functions such that*

$$\varphi(z)\psi(z) = u(z)I_{\mathbb{C}^n} \quad or \quad \psi(z)\varphi(z) = u(z)I_{\mathbb{C}^n}$$

for some $u \in H^\infty(\Omega)(u \neq 0)$, then u is quasi-inner.

Proof. (a) Since φ and u are quasi-inner functions, there are constants $m_1(>0)$ and $c_i(>0)(i = 1, 2)$ such that

(i) $m_1 \leq |u(z)|$ a.e. on $\partial\Omega$, and

(ii) for $h \in \mathbb{C}^n$, $c_1 \|h\| \leq \|\varphi(z)h\| \leq c_2 \|h\|$ a.e. on $\partial\Omega$.

Since $\varphi(z)\psi(z) = u(z)I_{\mathbb{C}^n}$ for $h \in \mathbb{C}^n$, $m_1 \|h\| \leq |u(z)| \|h\| = \|\varphi(z)\psi(z)h\| \leq c_2 \|\psi(z)h\|$ a.e. on $\partial\Omega$. Thus, for $h \in \mathbb{C}^n$,

$$\frac{m_1}{c_2} \|h\| \leq \|\psi(z)h\| \tag{3.1}$$

a.e. on $\partial\Omega$.

From (3.1), we conclude that ψ is also quasi-inner.

(b) Since $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $\psi \in H^\infty(\Omega, L(\mathbb{C}^n))$ are quasi-inner functions, there exist $m_1(>0)$ and $m_2(>0)$ such that, for $h \in \mathbb{C}^n$, $m_1 \|h\| \leq \|\varphi(z)h\|$ a.e. on $\partial\Omega$ and $m_2 \|h\| \leq \|\psi(z)h\|$ a.e. on $\partial\Omega$. Then

$$\|\varphi(z)\psi(z)h\| \geq m_1 \|\psi(z)h\| \geq m_1 m_2 \|h\|$$

and so $\|\varphi(z)\psi(z)\| \geq m_1 m_2$ a.e. on $\partial\Omega$. Since $|u(z)| = \|\varphi(z)\psi(z)\|$, it is proven. □

Furthermore, by using these quasi-inner functions, we have a generalization of Beurling's theorem as following:

Theorem A (Theorem 1.5 in [4])**.** *Let K be a Hilbert space. Then a closed subspace M of $H^2(\Omega, K)$ is $R(\Omega)$-invariant for S_K if and only if there is a Hilbert space K' and a quasi-inner function $\varphi : \Omega \to L(K', K)$ such that $M = \varphi H^2(\Omega, K')$.*

Since we have two kinds of quasi-inner functions, we have two kinds of $R(\Omega)$-invariant subspaces for S_K. One of them is generated by a scalar-valued quasi-inner function, and the other one is generated by an operator-valued quasi-inner function. We will also compare these two $R(\Omega)$-invariant subspaces for S_K in Theorem 4.3.

Let K_1 and K_2 be separable Hilbert spaces. To discuss quasi-inner functions, we define a *multiplication operator* for a given function $\psi \in H^\infty(\Omega, L(K_1, K_2))$. A *multiplication operator* $M_\psi : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ is defined by

$$M_\psi(g)(z) = \psi(z)g(z)$$

for all g in $H^2(\Omega, K_1)$. We can easily check that $\|M_\psi\| = \|\psi\|_\infty$.

Recall an important property of this multiplication operator:

Proposition 3.5 ([4]). *Let K_1 and K_2 be separable Hilbert spaces. If $T : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ is a bounded linear operator such that $TS_{K_1} = S_{K_2}T$, then there is a function $\psi \in H^\infty(\Omega, L(K_1, K_2))$ such that $T = M_\psi$.*

Theorem 3.6. *Let $\varphi \in H^\infty(\Omega, L(K_1, K_2))$.*

(a) *If φ is quasi-inner, then M_φ is one-to-one and has closed range.*

(b) *If $M_\varphi : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ is invertible, then φ is quasi-inner.*

Proof. (a) By Theorem A, $M_\varphi H^2(\Omega, K_1) = \varphi H^2(\Omega, K_1)$ is closed.

Since $\varphi(z)$ is a bounded below operator a.e. on $\partial\Omega$, $f \in \ker M_\varphi = \{f \in H^2(\Omega, K_1) : \varphi(z)f(z) = 0 (z \in \Omega)\}$ if and only if $f^* \equiv 0$ in $H^2(\partial\Omega, K_1)$ if and only if $f \equiv 0$.

(b) Since $\varphi H^2(\Omega, K_1)$ is $R(\Omega)$-invariant for S_{K_2}, by Theorem A,

$$\varphi H^2(\Omega, K_1) = \varphi_1 H^2(\Omega, K_0)$$

for a Hilbert space K_0 and a quasi-inner function $\varphi_1 : \Omega \to L(K_0, K_2)$.

Define a linear operator $T : H^2(\Omega, K_1) \to H^2(\Omega, K_0)$ as follows. For $f \in H^2(\Omega, K_1)$, $Tf = g$ such that $\varphi f = \varphi_1 g$. Since φ_1 is a quasi-inner function, by (a), T is well-defined and T is bounded. Since $S_{K_0}T = TS_{K_1}$, by Proposition 3.5, $T = M_{\varphi_2}$ for a function $\varphi_2 \in H^\infty(\Omega, L(K_1, K_2))$. It follows that

$$\varphi(z) = \varphi_1(z)\varphi_2(z) \tag{3.2}$$

for any $z \in \Omega$.

Since M_φ is onto, so is M_{φ_1}. By (a), M_{φ_1} is one-to-one, and so M_{φ_1} is invertible. Since M_φ and M_{φ_1} are invertible, so is $T = M_{\varphi_2}$. Note that the invertibility of M_φ is equivalent to the invertibility of $\varphi(z)$ for any z in Ω. It follows that $\varphi_2(z)$ is bounded below for any z in Ω.

Since φ_1 is quasi-inner, $\varphi_1(z)$ is also bounded below a.e. on $\partial\Omega$.

Therefore, by equation (3.2), for any $a \in K_1$, there is a constant $c > 0$ such that

$$\|\varphi(z)a\| \geq c\,\|a\|$$

a.e. on $\partial\Omega$. □

4. Quasi-inner divisors

Let K be a Hilbert space. The time has come to consider divisibilities between a function in $H^\infty(\Omega)$ and a function in $H^\infty(\Omega, L(K))$.

Definition 4.1. If $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, then we say that θ *divides* φ (denoted $\theta|\varphi$) if φ can be written as

$$\varphi = \theta \cdot \phi'$$

for some $\phi' \in H^\infty(\Omega, L(K))$.

Definition 4.2. If $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, then we say that φ *divides* θ (denoted $\varphi|\theta$) if there exists $\psi \in H^\infty(\Omega, L(K))$ satisfying the following relations;

$$\varphi(z)\psi(z) = \theta(z)I_K$$

and

$$\psi(z)\varphi(z) = \theta(z)I_K$$

for $z \in \Omega$.

Theorem 4.3. *For any quasi-inner functions $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, the following assertions are equivalent*:

(a) $\theta|\varphi$.

(b) $\varphi H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K)$.

(c) $\varphi H^2(\Omega, K) \subset \theta H^2(\Omega, K)$.

(d) *There is a $\lambda > 0$ such that $\varphi(z)\varphi(z)^* \leq \lambda^2|\theta(z)|^2 I_K$ for any $z \in \Omega$, where I_K is the identity function on K.*

Proof. If $\theta|\varphi$, $\varphi = \theta\varphi_1$ for some $\varphi_1 \in H^\infty(\Omega, L(K))$. Then

$$\varphi H^\infty(\Omega, K) = \theta\varphi_1 H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K).$$

Thus (a) implies (b).

Conversely, suppose that $\varphi H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K)$. Then

$$\varphi^* H^\infty(\partial\Omega, K) \subset \theta^* H^\infty(\partial\Omega, K). \tag{4.1}$$

Let $\{b_i : i \in I\}$ be an orthonormal basis of K and $g_i \in H^\infty(\partial\Omega, K)$ defined by $g_i(z) = b_i (i \in I)$. By (4.1), there is $f_i \in H^\infty(\partial\Omega, K)$ such that $\varphi^* g_i = \theta^* f_i$, i.e., for $i \in I$,

$$\varphi^*(z)b_i = \theta^*(z)f_i(z). \tag{4.2}$$

Define $\varphi_1 : \partial\Omega \to L(K)$ by for $i \in I$,

$$\varphi_1(z)b_i = f_i(z). \tag{4.3}$$

For $i \in I$, define $\varphi_i \in H^\infty(\partial\Omega, L(K))$ by $\varphi_i(z)b_j = \delta_{ij} f_i(z) (j \in I)$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$ Then

$$\varphi_1 = \sum_{i \in I} \varphi_i. \tag{4.4}$$

By (4.2) and (4.3), for each $i \in I$, $\varphi^*(z)b_i = \theta^*(z)\varphi_1(z)b_i$, and so

$$\varphi^* = \theta^*\varphi_1.$$

To prove that (b) implies (a), we have to show that $\varphi_1 \in H^\infty(\partial\Omega, L(K))$. Since $\theta \in H^\infty$ is a quasi-inner function, there is $c > 0$ such that $|\theta(z)| \geq c$ for every $z \in A \subset \partial\Omega$ with $m(\partial\Omega \setminus A) = 0$. For any $x \in K$ with $\|x\| = 1$ and $z \in A$,

$$\|\varphi_1(z)x\| = \frac{\|\varphi^*(z)x\|}{|\theta^*(z)|} \leq \frac{\|\varphi\|_\infty}{c}. \tag{4.5}$$

From (4.4) and (4.5), we conclude that

$$\varphi_1 \in H^\infty(\partial\Omega, L(K)).$$

By the same way as above, (a)⇔(c) is proven. We begin to prove (a)⇔(d). If $\theta|\varphi$, $\varphi = \theta\varphi_1$ for some $\varphi_1 \in H^\infty(\Omega, L(K))$. Then

$$\varphi(z)\varphi(z)^* = \theta(z)\varphi_1(z)\varphi_1(z)^*\overline{\theta(z)} \leq \|\varphi_1\|_\infty^2 |\theta(z)|^2 I_K.$$

Let $\lambda = \|\varphi_1\|_\infty$. Since φ is quasi-inner, $\varphi \neq 0$ and so $\lambda > 0$. Thus (a) implies (d).

Conversely, suppose that for any $z \in \Omega$,

$$\varphi(z)\varphi(z)^* \leq \lambda^2|\theta(z)|^2 I_K \tag{4.6}$$

for some $\lambda > 0$. For each $z \in \Omega$, we will define a linear mapping $F_z \in L(K)$. Let

$$A = \{z \in \Omega : \theta(z) = 0\}$$

and

$$B = \{z \in \Omega : \theta(z) \neq 0\}.$$

If $z \in A$, then let $F_z = 0$. If $z \in B$, then range of $\overline{\theta(z)}I_K$ is K and so we can define a linear mapping F_z from K to range of $\varphi(z)^*$ by

$$F_z(\overline{\theta(z)}f) = \varphi(z)^* f$$

for $f \in K$.

Since $\left\|F_z(\overline{\theta(z)}f)\right\|^2 = \|\varphi(z)^* f\|^2 = (\varphi(z)\varphi(z)^* f, f) \leq \lambda^2(|\theta(z)|^2 f, f) = \lambda^2 \|\theta(z)f\|^2$, that is,

$$\left\|F_z(\overline{\theta(z)}f)\right\| \leq \lambda \|\theta(z)f\|, \tag{4.7}$$

F_z is well-defined for $z \in B$. By definition of F_z, if $z \in B$,

$$\theta(z)F_z^* = \varphi(z). \tag{4.8}$$

If $z \in A$, by (4.6) $\|\varphi(z)\| = 0$ and so $\varphi(z) = 0 (z \in A)$. Thus $\theta(z)F_z^* = \varphi(z)$ for any $z \in \Omega$.

Define a function $F : \Omega \to L(K)$ by

$$F(z) = F_z^*.$$

Then by equation (4.8),

$$\varphi(z) = \theta(z)F(z)$$

for $z \in \Omega$. To finish this proof, we have to prove that $F \in H^\infty(\Omega, L(K))$. From inequality (4.7), we have

$$\|F\|_\infty \leq \lambda \tag{4.9}$$

and so $F = \frac{\varphi}{\theta}$ has only removable singularities in Ω. Thus F can be defined on $\{z \in \Omega : \theta(z) = 0\}$ so that F is analytic and

$$\varphi = \theta F.$$

From (4.9), $F \in H^\infty(\Omega, L(K))$ which proves (d)⇒(a). □

We have another result similar to Theorem 4.3.

Theorem 4.4. *For any quasi-inner functions $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, the following assertions are equivalent*:

(a) $\varphi | \theta$.
(b) $\theta H^\infty(\Omega, K) \subset \varphi H^\infty(\Omega, K)$.
(c) $\theta H^2(\Omega, K) \subset \varphi H^2(\Omega, K)$.
(d) *There is a* $\lambda > 0$ *such that* $|\theta(z)|^2 I_K \leq \lambda^2 \varphi(z)\varphi(z)^*$ *for any* $z \in \Omega$.

Proof. This theorem is proven by the same way as Theorem 4.3. □

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