APPLICATIONS OF GENERALIZED KUMMER'S SUMMATION THEOREM FOR THE SERIES $_2F_1$

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ABSTRACT. The aim of this research paper is to establish generalizations of classical Dixon's theorem for the series $_3F_2$, a result due to Bailey involving product of generalized hypergeometric series and certain very interesting summations due to Ramanujan. The results are derived with the help of generalized Kummer's summation theorem for the series $_2F_1$ obtained earlier by Lavoie, Grondin, and Rathie.

1. Introduction

The generalized hypergeometric functions with p numerator and q denominator parameters is defined by [7]

(1.1)
$$_pF_q \begin{bmatrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{bmatrix} = {_pF_q}\left[\alpha_1, \ldots, \alpha_p; \ \beta_1, \ldots, \beta_q; z\right]$$
$$:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number α by

(1.2)
$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, \ldots\} \\ 1, & \text{if } n = 0 \end{cases}$$

using the fundamental relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written in the form

(1.3)
$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (n \in \mathbb{N} \cup \{0\}),$$

where Γ is the well-known Gamma function.

It is well-known that whenever a generalized hypergeometric function reduces to the Gamma function, the results are very important from the application point of view. We mention here some of special cases of (1.1).

Gauss's summation theorem [3]:

$$(1.4) \qquad {}_2F_1 \left[\begin{array}{cc} a, & b \\ & c \end{array} ; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

provided $\Re(c-a-b) > 0$.

Kummer's summation theorem [3]:

$$(1.5) \qquad {}_2F_1 \left[\begin{array}{c} a, \ b \\ 1+a-b \end{array} ; -1 \right] = \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b)}{2^a\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(1+\frac{1}{2}a-b)}.$$

Dixon's summation theorem [3]:

$$(1.6)$$
$$
{}_3F_2 \left[\begin{array}{ccc} a, & b, & c \\ 1+a-b, & 1+a-c \end{array} ; 1 \right]
$$
$$
= \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}
$$

provided $\Re(a-2b-2c) > -2$.

On the other hand, an interesting result due to Bailey [2] involving product of generalized hypergeometric series is

$$(1.7) \qquad {}_0F_1 \left[\begin{array}{c} - \\ \rho \end{array} ; x \right] \times {}_0F_1 \left[\begin{array}{c} - \\ \rho \end{array} ; -x \right] = {}_0F_3 \left[\begin{array}{c} - \\ \rho, \ \frac{1}{2}\rho, \ \frac{1}{2}\rho+\frac{1}{2} \end{array} ; -\frac{x^2}{4} \right].$$

The followings are the summations due to Ramanujan [4]:

$$(1.8) \qquad 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 - \cdots + (-1)^n \left(\frac{(\frac{1}{2})_n}{n!}\right)^2 + \cdots = \frac{\sqrt{\pi}}{\sqrt{2}\Gamma^2(3/4)}$$

and for $\Re(x) > 0$
$$(1.9)$$
$$1 + \frac{(x-1)}{(x+1)} + \frac{(x-1)(x-2)}{(x+1)(x+2)} + \cdots + (-1)^n \frac{(1-x)_n}{(x+1)_n} + \cdots = \frac{2^{2x-1}\Gamma^2(x+1)}{\Gamma(2x+1)}.$$

In the theory of generalized hypergeometric series, classical summation theorems for the series ${}_2F_1$, ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$ play an important role. The Dixon's theorem (1.6) can be obtained with the help of Gauss's summation theorem (1.4) and Kummer's summation theorem (1.5). Bailey [2] derived the result (1.7) by utilizing Kummer's summation theorem (1.7). Berndt [4] pointed out that the summation (1.8) and (1.9) due to Ramanujan can be obtained quite simply from Kummer's theorem (1.5) by taking $a = b = \frac{1}{2}$ and $a = 1$, $b = 1-x$, respectively.

In 1996, Grodin and Rathie [6] generalized the Kummer's theorem (1.5) in the form

(1.10)

$$
{}_2F_1 \left[\begin{matrix} a, \ b \\ 1 + a - b + i \end{matrix} ; -1 \right]
$$

$$
= \frac{\Gamma(\frac{1}{2})\Gamma(1 + a - b + i)\Gamma(1 - b)}{2^a \Gamma(1 - b + \frac{1}{2}(i + |i|))}
$$

$$
\times \left\{ \frac{A_i}{\Gamma(\frac{1}{2}a + \frac{1}{2}i + \frac{1}{2} - [\frac{1+i}{2}])\Gamma(1 + \frac{1}{2}a - b + \frac{1}{2}i)} + \frac{B_i}{\Gamma(\frac{1}{2}a + \frac{1}{2}i - [\frac{i}{2}])\Gamma(\frac{1}{2} + \frac{1}{2}a - b + \frac{i}{2})} \right\}
$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

As usual, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients A_i and B_i are given respectively in [6, p. 298]. The paper is organized as follows. In Section 2, explicit expression for the series

$$
{}_3F_2 \left[\begin{matrix} a, \quad b, \quad c \\ 1 + a - b + i, \ 1 + a - c \end{matrix} ; 1 \right]
$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ are given. The results will be derived with the help of generalized Kummer's theorem (1.10) and Gauss's theorem (1.4). As an applications generalizations of certain summations due to Ramanujan are also given. In Section 3, single series expressions of

$$
{}_0F_1 \left[\begin{matrix} - \\ \rho \end{matrix} ; x \right] \times {}_0F_1 \left[\begin{matrix} - \\ \rho + i \end{matrix} ; -x \right]
$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ are given. As special cases, some interesting results are also mentioned. In Section 4, summations for the series

$$
1 - \left(\frac{1}{2}\right)^2 \frac{1}{(1 + i)} \frac{1}{1!} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{1}{(1 + i)(2 + i)} \frac{1}{2!} - \cdots + (-1)^n \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{n!(1 + i)_n} + \cdots
$$

and for $\Re(x) > 0$

$$
1 + \frac{(x - 1)}{(x + 1 + i)} + \frac{(x - 1)(x - 2)}{(x + 1 + i)(x + 2 + i)} + \cdots + (-1)^n \frac{(1 - x)_n}{(x + 1 + i)_n} + \cdots
$$

each for $i = 0, 1, 2, 3, 4, 5$ are given. The results given in Sections 3 and 4 are derived with the help of generalized Kummer's theorem (1.10).

2. Generalization of Dixon's theorem

In this section, the result to be proved is given in the following theorem.

Theorem 1. *For $\Re(a - 2b - 2c + i) > -2$, we have*

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ 1 + a - b + i, & 1 + a - c \end{matrix}; 1\right]
$$

$$
= \frac{\Gamma(\frac{1}{2})\Gamma(1 - b)\Gamma(1 + a - b + i)}{\Gamma(1 - b + \frac{1}{2}(i + |i|))} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}a)_m(\frac{1}{2}a + \frac{1}{2})_m(b - \frac{1}{2}(i + |i|))_m}{(1 + a - c)_m}
$$

$$
\times \left\{ \frac{A_i{}^{'}}{\Gamma(\frac{1}{2}a - b + \frac{1}{2}i + 1)\Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}i - [\frac{1+i}{2}])(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}i - [\frac{1+i}{2}])_m} \right.
$$

$$
\left. + \frac{B_i{}^{'}}{\Gamma(\frac{1}{2}a - b + \frac{1}{2}i + \frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}i - [\frac{i}{2}])(\frac{1}{2}a + \frac{1}{2} - [\frac{i}{2}])_m} \right\}
$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. The coefficients $A_i{}^{'}$ and $B_i{}^{'}$ can be obtained from the tables of A_i and B_i by changing a by $a + 2m$ and b by $b + m$ respectively in [6, p. 298].

Proof. Denoting the left-hand side by $\mathcal{L}$, we have

$$
\mathcal{L} = {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1 + a - b + i, & 1 + a - c \end{matrix}; 1\right].
$$

Expressing as a series,

$$
\mathcal{L} = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{(1 + a - b + i)_k(1 + a - c)_k k!}.
$$

We can write this in the form

$$
= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(1 + a - b + i)_k(-1)_k k!} \left\{ \frac{(-1)^k(c)_k}{(1 + a - c)_k} \right\}.
$$

Using a result [7, p. 69, Ex. 5]

$$
{}_2F_1\left[\begin{matrix} -n, & a + n \\ & b \end{matrix}; 1\right] = \frac{(-1)^n(1 + a - b)_n}{(b)_n},
$$

$$
\mathcal{L} = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(1 + a - b + i)_k(-1)^k k!} {}_2F_1\left[\begin{matrix} -k, & a + k \\ & 1 + a - c \end{matrix}; 1\right].
$$

Expressing ${}_2F_1$ as a series, we have, after a little simplification

$$
\mathcal{L} = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(a)_k(b)_k(-1)^k(-k)_m(a + k)_m}{(1 + a - b + i)_k k!(1 + a - c)_m m!}.
$$

Using the following identities

$$
(a)_k(a + k)_m = (a)_{k+m},
$$

(2.1)

$$
(-k)_m = \frac{(-1)^m k!}{(k - m)!},
$$

we have

$$\mathcal{L} = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(a)_{k+m}(b)_k(-1)^{k+m}}{(1+a-b+i)_k(1+a-c)_m m!(k-m)!}.$$

Using a known result [7, p. 57, Eq. (2)]

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{B}(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{B}(k,n+k),$$

we have

$$\mathcal{L} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_{k+2m}(b)_{k+m}(-1)^k}{(1+a-b+i)_{k+m}(1+a-c)_m m! k!}.$$

Using the identities

(2.2)
$$(a)_{k+2m} = (a)_{2m}(a+2m)_k,$$
$$(b)_{k+m} = (b)_m(b+m)_k,$$

we have

$$\mathcal{L} = \sum_{m=0}^{\infty} \frac{(a)_{2m}(b)_m}{(1+a-b+i)_m(1+a-c)_m m!} \sum_{k=0}^{\infty} \frac{(a+2m)_k(b+m)_k(-1)^k}{(1+a-b+i+m)_k k!}.$$

Summing up the inner series, we get

$$\mathcal{L} = \sum_{m=0}^{\infty} \frac{(a)_{2m}(b)_m}{(1+a-b+i)_m(1+a-c)_m m!} {}_2F_1 \left[\begin{array}{cc} a+2m, & b+m \\ 1+a-b+i+m \end{array} ; -1 \right].$$

It is now easy to see that the $_2F_1$ can be summed up by the known result (1.10) and after a little algebra, we easily arrive at the right-hand side of the Theorem 1. This completes the proof of the Theorem 1. □

Corollary 2. (1) *In Theorem 1, if we put $i = 0$, we get after a little simplification, the classical Dixon's theorem* (1.6).

(2) *In Theorem 1, if we put $i = 1$, we get, after some simplification, the following summation formula*
(2.3)
$${}_3F_2 \left[\begin{array}{ccc} a, & b, & c \\ 2+a-b, & 1+a-c \end{array} ; 1 \right]$$
$$= \frac{2^{-2c+1}\Gamma(2+a-b)\Gamma(1+a-c)}{(b-1)\Gamma(a-2b+2)\Gamma(a-b-c+2)}$$
$$\times \left\{ \frac{\Gamma(\frac{1}{2}a-b+3/2)\Gamma(\frac{1}{2}a-b-c+2)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}a-c+1)} + \frac{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-b-c+3/2)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-c+\frac{1}{2})} \right\}$$

provided $\Re(a-2b-2c) > 1$.

Similarly, other results can also be obtained.

3. Generalization of Bailey's formula

In this section, the result to be proved involving product of generalized hypergeometric series is given in the following theorem.

Theorem 3.

$$
{}_0F_1\left[\begin{matrix} - \\ \rho \end{matrix}; x\right] \times {}_0F_1\left[\begin{matrix} - \\ \rho+i \end{matrix}; -x\right]
$$

$$
= \frac{\Gamma(\frac{1}{2})\Gamma(\rho)\Gamma(\rho+i)}{\Gamma(\rho+\frac{1}{2}(i+|i|))} \sum_{m=0}^{\infty} \frac{(-1)^m(x)^{2m}}{(\frac{1}{2})_m 2^{2m} m!(\frac{1}{2}\rho+\frac{1}{4}(i+|i|))_m(\frac{1}{2}\rho+\frac{1}{4}(i+|i|)+\frac{1}{2})_m}
$$

$$
\times \left\{ \frac{A_i^{'}}{\Gamma(\rho+\frac{1}{2}i)\Gamma(\frac{1}{2}+\frac{1}{2}i-[\frac{1+i}{2}])} \frac{(\frac{1}{2}-\frac{1}{2}i+[\frac{1+i}{2}])_m}{(\rho+\frac{1}{2}i)_m} \right.
$$

$$
\left. + \frac{B_i^{'}}{\Gamma(\rho+\frac{1}{2}i-\frac{1}{2})\Gamma(\frac{1}{2}i-[\frac{i}{2}])} \frac{(1-\frac{1}{2}i+[\frac{i}{2}])_m}{(\rho+\frac{1}{2}i-\frac{1}{2})_m} \right\}
$$

$$
+ \frac{\Gamma(\frac{1}{2})\Gamma(\rho)\Gamma(\rho+i)}{\Gamma(1+\rho+\frac{1}{2}(i+|i|))} \sum_{m=0}^{\infty} \frac{(-1)^m(x)^{2m+1}}{(\frac{3}{2})_m 2^{2m-1} m!(\frac{1}{2}\rho+\frac{1}{2}+\frac{1}{4}(i+|i|))_m(\frac{1}{2}\rho+1+\frac{1}{4}(i+|i|))_m}
$$

$$
\times \left\{ \frac{A_i^{''}}{\Gamma(\rho+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}i-[\frac{1+i}{2}])} \frac{(1-\frac{1}{2}i+[\frac{1+i}{2}])_m}{(\rho+\frac{1}{2}i+\frac{1}{2})_m} \right.
$$

$$
\left. + \frac{B_i^{''}}{\Gamma(\rho+\frac{1}{2}i)\Gamma(\frac{1}{2}i-\frac{1}{2}-[\frac{i}{2}])} \frac{(\frac{3}{2}-\frac{1}{2}i+[\frac{i}{2}])_m}{(\rho+\frac{1}{2}i)_m} \right\}
$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

The coefficients $A_i^{'}$ and $B_i^{'}$ can be obtained from the table of A_i and B_i by changing a by $-2m$ and b by $1-\rho-2m$ and the coefficients $A_i^{''}$ and $B_i^{''}$ can be obtained from the table of A_i and B_i by changing a by $-2m-1$ and b by $-\rho-2m$ respectively in [6, p. 298].

Proof. Denoting the left-hand side by $\mathcal{S}$, we have

$$
\mathcal{S} := {}_0F_1\left[\begin{matrix} - \\ \rho \end{matrix}; x\right] \times {}_0F_1\left[\begin{matrix} - \\ \rho+i \end{matrix}; -x\right].
$$

Expressing both ${}_0F_1$ as series, we have

$$
\mathcal{S} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n}{(\rho)_m(\rho+i)_n m! n!} x^{m+n},
$$

which, by virtue of the following well-known result [7, p. 56, Eq. (1)]

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathcal{B}(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{B}(k,n-k),
$$

yields

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^n x^m}{(\rho)_{m-n}(\rho+i)_n(m-n)!n!}.$$

Indeed, using the result (2.1) and after a little simplification, we get

$$S = \sum_{m=0}^{\infty} \frac{x^m}{(\rho)_m m!} \sum_{n=0}^{m} \frac{(-m)_n(1-\rho-m)_n(-1)^n}{(\rho+i)_n n!}.$$

Summing up the inner series, we have

$$S = \sum_{m=0}^{\infty} \frac{x^m}{(\rho)_m m!} {}_2F_1 \left[\begin{array}{c} -m, \ 1-\rho-m \\ \rho+i \end{array} ; -1 \right].$$

Separating even and odd powers of x, we have on using elementary identities [5]

$$(a)_{2m} = 2^{2m}(\tfrac{1}{2}a)_m(\tfrac{1}{2}a+\tfrac{1}{2})_m,$$

$$(a)_{2m+1} = a2^{2m}(\tfrac{1}{2}a+\tfrac{1}{2})_m(\tfrac{1}{2}a+1)_m,$$

(3.1)

$$(2m)! = 2^{2m}m!(\tfrac{1}{2})_m,$$

$$(2m+1)! = 2^{2m}m!(3/2)_m,$$

$$S = \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{4m}(\tfrac{1}{2}\rho)_m(\tfrac{1}{2}\rho+\tfrac{1}{2})_m(\tfrac{1}{2})_m m!} {}_2F_1 \left[\begin{array}{c} -2m, \ 1-\rho-2m \\ \rho+i \end{array} ; -1 \right]$$

$$+ \sum_{m=0}^{\infty} \frac{x^{2m+1}}{\rho 2^{4m}(\tfrac{1}{2}\rho+\tfrac{1}{2})_m(\tfrac{1}{2}\rho+1)_m(3/2)_m m!} {}_2F_1 \left[\begin{array}{c} -2m-1, \ -\rho-2m \\ \rho+i \end{array} ; -1 \right].$$

Finally using generalized Kummer's summation formula in each ${}_2F_1$ and after much simplification, we arrive at the right-hand side of the Theorem 3. This completes the proof of Theorem 3. □

Corollary 4. *We give some explicit special cases of Theorem 3.*
    For $i = 0$

(3.2) $${}_0F_1 \left[\begin{array}{c} - \\ \rho \end{array} ; x \right] \times {}_0F_1 \left[\begin{array}{c} - \\ \rho \end{array} ; -x \right] = {}_0F_3 \left[\begin{array}{c} - \\ \rho, \ \tfrac{1}{2}\rho, \ \tfrac{1}{2}\rho+\tfrac{1}{2} \end{array} ; -\frac{x^2}{4} \right].$$

    For $i = 1$
(3.3)
$${}_0F_1 \left[\begin{array}{c} - \\ \rho \end{array} ; x \right] \times {}_0F_1 \left[\begin{array}{c} - \\ \rho+1 \end{array} ; -x \right]$$

$$= {}_0F_3 \left[\begin{array}{c} - \\ \rho, \ \tfrac{1}{2}\rho+\tfrac{1}{2}, \ \tfrac{1}{2}\rho+1 \end{array} ; -\frac{x^2}{4} \right] + \frac{x}{\rho(\rho+1)} {}_0F_3 \left[\begin{array}{c} - \\ \rho+1, \ \tfrac{1}{2}\rho+1, \ \tfrac{1}{2}\rho+\tfrac{3}{2} \end{array} ; -\frac{x^2}{4} \right].$$

For $i = -1$

(3.4)

$$_0F_1\left[\begin{matrix} - \\ \rho \end{matrix}; x\right] \times {}_0F_1\left[\begin{matrix} - \\ \rho - 1 \end{matrix}; -x\right]$$

$$= {}_0F_3\left[\begin{matrix} - \\ \rho - 1, \ \frac{1}{2}\rho, \ \frac{1}{2}\rho + \frac{1}{2} \end{matrix}; -\frac{x^2}{4}\right] - \frac{x}{\rho(\rho - 1)}{}_0F_3\left[\begin{matrix} - \\ \rho, \ \frac{1}{2}\rho + \frac{1}{2}, \ \frac{1}{2}\rho + 1 \end{matrix}; -\frac{x^2}{4}\right].$$

For $i = 2$

(3.5)

$$_0F_1\left[\begin{matrix} - \\ \rho \end{matrix}; x\right] \times {}_0F_1\left[\begin{matrix} - \\ \rho + 2 \end{matrix}; -x\right]$$

$$= {}_0F_3\left[\begin{matrix} - \\ \rho, \ \frac{1}{2}\rho + 1, \ \frac{1}{2}\rho + \frac{3}{2} \end{matrix}; -\frac{x^2}{4}\right] + \frac{2x}{\rho(\rho + 2)}{}_0F_3\left[\begin{matrix} - \\ \rho + 1, \ \frac{1}{2}\rho + 2, \ \frac{1}{2}\rho + \frac{3}{2} \end{matrix}; -\frac{x^2}{4}\right].$$

For $i = -2$

(3.6)

$$_0F_1\left[\begin{matrix} - \\ \rho \end{matrix}; x\right] \times {}_0F_1\left[\begin{matrix} - \\ \rho - 2 \end{matrix}; -x\right]$$

$$= {}_0F_3\left[\begin{matrix} - \\ \rho - 1, \ \frac{1}{2}\rho, \ \frac{1}{2}\rho + \frac{1}{2} \end{matrix}; -\frac{x^2}{4}\right] - \frac{2x}{\rho(\rho - 2)}{}_0F_3\left[\begin{matrix} - \\ \rho - 1, \ \frac{1}{2}\rho + \frac{1}{2}, \ \frac{1}{2}\rho + 1 \end{matrix}; -\frac{x^2}{4}\right].$$

For $i = 3$

(3.7)

$$_0F_1\left[\begin{matrix} - \\ \rho \end{matrix}; x\right] \times {}_0F_1\left[\begin{matrix} - \\ \rho + 3 \end{matrix}; -x\right]$$

$$= {}_0F_3\left[\begin{matrix} - \\ \rho + 1, \ \frac{1}{2}\rho + 2, \ \frac{1}{2}\rho + \frac{3}{2} \end{matrix}; -\frac{x^2}{4}\right] + \frac{3x}{\rho(\rho + 3)}{}_0F_3\left[\begin{matrix} - \\ \rho + 2, \ \frac{1}{2}\rho + 2, \ \frac{1}{2}\rho + \frac{5}{2} \end{matrix}; -\frac{x^2}{4}\right]$$

$$+ \frac{2x^2}{\rho(\rho + 1)(\rho + 3)(\rho + 4)}{}_0F_3\left[\begin{matrix} - \\ \rho + 2, \ \frac{1}{2}\rho + 3, \ \frac{1}{2}\rho + \frac{5}{2} \end{matrix}; -\frac{x^2}{4}\right]$$

$$- \frac{2x^3}{\rho(\rho + 1)(\rho + 2)(\rho + 3)(\rho + 4)(\rho + 5)}{}_0F_3\left[\begin{matrix} - \\ \rho + 3, \ \frac{1}{2}\rho + 3, \ \frac{1}{2}\rho + \frac{7}{2} \end{matrix}; -\frac{x^2}{4}\right].$$

For $i = -3$

(3.8)

$$_0F_1 \left[\begin{matrix} - \\ \rho \end{matrix} ; x \right] \times {_0F_1} \left[\begin{matrix} - \\ \rho - 3 \end{matrix} ; -x \right]$$

$$= {_0F_3} \left[\begin{matrix} - \\ \rho - 2, \ \frac{1}{2}\rho, \ \frac{1}{2}\rho + \frac{1}{2} \end{matrix} ; -\frac{x^2}{4} \right] - \frac{3x}{\rho(\rho - 3)} {_0F_3} \left[\begin{matrix} - \\ \rho - 1, \ \frac{1}{2}\rho + 1, \ \frac{1}{2}\rho + \frac{1}{2} \end{matrix} ; -\frac{x^2}{4} \right]$$

$$+ \frac{2x^2}{\rho(\rho + 1)(\rho - 2)(\rho - 3)} {_0F_3} \left[\begin{matrix} - \\ \rho - 1, \ \frac{1}{2}\rho + 1, \ \frac{1}{2}\rho + \frac{3}{2} \end{matrix} ; -\frac{x^2}{4} \right]$$

$$- \frac{2x^3}{\rho(\rho + 1)(\rho + 2)(\rho - 1)(\rho - 2)(\rho - 3)} {_0F_3} \left[\begin{matrix} - \\ \rho, \ \frac{1}{2}\rho + 2, \ \frac{1}{2}\rho + \frac{3}{2} \end{matrix} ; -\frac{x^2}{4} \right].$$

The result (3.2) is due to Bailey [2] and the results (3.3) to (3.8) are closely related to Bailey's result.

4. Generalizations of the summations due to Ramanujan

In this section, generalizations of the summations due to Ramanujan are given in the following theorem.

Theorem 5.

$$1 - \left(\frac{1}{2} \right)^2 \frac{1}{1+i} \frac{1}{1!} + \left(\frac{1 \cdot 3}{2 \cdot 2} \right)^2 \frac{1}{(1+i)(2+i)} \frac{1}{2!} - \cdots + (-1)^n \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{n!(1+i)_n} + \cdots$$

$$= \frac{\pi \, \Gamma(1+i)}{\sqrt{2} \, \Gamma(\frac{1}{2} + i)} \left\{ \frac{C_i}{\Gamma(\frac{3}{4} + \frac{1}{2}i)\Gamma(\frac{3}{4} + \frac{1}{2}i - [\frac{1+i}{2}])} + \frac{D_i}{\Gamma(\frac{1}{4} + \frac{1}{2}i)\Gamma(\frac{1}{4} + \frac{1}{2}i - [\frac{i}{2}])} \right\}$$

for $i = 0, 1, 2, 3, 4, 5$.

The coefficients C_i and D_i can be obtained from the table of A_i and B_i by putting $a = b = \frac{1}{2}$ in [6, p. 298].

Theorem 6. *For $\Re(x) > 0$*

$$1 + \frac{(x - 1)}{(x + 1 + i)} + \frac{(x - 1)(x - 2)}{(x + 1 + i)(x + 2 + i)} + \cdots + (-1)^n \frac{(1 - x)_n}{(x + 1 + i)_n} + \cdots$$

$$= \frac{\sqrt{\pi}(x + i)\Gamma(x)}{2} \left\{ \frac{E_i}{\Gamma(x + \frac{1}{2} + \frac{1}{2}i)\Gamma(1 + \frac{1}{2}i - [\frac{1+i}{2}])} + \frac{F_i}{\Gamma(x + \frac{1}{2}i)\Gamma(\frac{1}{2} + \frac{1}{2}i - [\frac{i}{2}])} \right\}$$

for $i = 0, 1, 2, 3, 4, 5$.

The coefficients E_i and F_i can be obtained from the table of A_i and B_i by putting $a = 1$ and $b = 1 - x$ respectively in [6].

Proof. The proof of the Theorems 5 and 6 are straight forward. It is easy to see that the left-hand side expression of the Theorem 5 corresponds to

$$
{}_2F_1\left[\begin{array}{cc} \dfrac{1}{2},\ \dfrac{1}{2} \\ 1+i \end{array};-1\right]
$$

which is seen to be the special case of generalized Kummer's theorem (1.10), so it can easily be evaluated by (1.10) and we get the right-hand side of the Theorem 5. Similarly, it is easy to see that the left-hand side expression of the Theorem 6 corresponds to

$$
{}_2F_1\left[\begin{array}{cc} 1,\ 1-x \\ 1+x+i \end{array};-1\right]
$$

which is also seen to be the special case of (1.10) and so it can easily be evaluated by (1.10) and we get the right-hand side of the Theorem 6.   □

Corollary 7. *If in our Theorem 5, we take $i = 0, 1, 2$, we get the following summations.*

(1) $1 - \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1\cdot3}{2\cdot4}\right)^2 - \cdots = \dfrac{\sqrt{\pi}}{\sqrt{2}\Gamma^2(3/4)}.$

(2) $1 - \dfrac{1}{2}\left(\dfrac{1}{2}\right)^2 + \dfrac{1}{3}\left(\dfrac{1\cdot3}{2\cdot4}\right)^2 - \cdots = \dfrac{\sqrt{2\pi}}{\Gamma^2(3/4)} - \dfrac{2\sqrt{2}\Gamma^2(3/4)}{\pi^{3/2}}.$

(3) $1 - \dfrac{1}{3}\left(\dfrac{1}{2}\right)^2 + \dfrac{1}{6}\left(\dfrac{1\cdot3}{2\cdot4}\right)^2 - \cdots = \dfrac{16\sqrt{2\pi}}{9\Gamma^2(3/4)} - \dfrac{16\sqrt{2}\Gamma^2(3/4)}{3\pi^{3/2}}.$

Similarly, other results can be obtained from our Theorem 5.

Corollary 8. *If in our Theorem 6, we take $i = 0, 1, 2$, we get the following summations.*

(1) $1 + \dfrac{(x-1)}{(x+1)} + \dfrac{(x-1)(x-2)}{(x+1)(x+2)} + \cdots = \dfrac{\Gamma(3/2)\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}.$

(2) $1 + \dfrac{(x-1)}{(x+2)} + \dfrac{(x-1)(x-2)}{(x+2)(x+3)} + \cdots = (x+1)\Gamma(x)\Gamma(3/2)\left\{\dfrac{1}{\Gamma(x+\frac{1}{2})} - \dfrac{1}{\sqrt{\pi}\Gamma(x+1)}\right\}.$

(3) $1 + \dfrac{(x-1)}{(x+3)} + \dfrac{(x-1)(x-2)}{(x+3)(x+4)} + \cdots = (x+2)\Gamma(x)\Gamma(3/2)\left\{\dfrac{(x+2)}{\Gamma(x+\frac{3}{2})} - \dfrac{(3x+4)}{\sqrt{\pi}\Gamma(x+2)}\right\}.$

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