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TWO MEROMORPHIC FUNCTIONS SHARING SETS CONCERNING SMALL FUNCTIONS

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Abstract. The main purpose of this paper is to deal with the uniqueness of meromorphic functions sharing sets concerning small functions. We obtain two main theorems which improve and extend strongly some results due to R. Nevanlinna, Li-Qiao, Yao, Yi, Thai-Tan, and Cao-Yi.

1. Introduction

It is well known that two nonconstant polynomials f and g over an algebraic closed field of characteristic zero are identical if there exist two distinct values a and b such that $f(x)=a$ if and only if $g(x)=a$ and $f(x)=b$ if and only if $g(x)=b$. In 1926, R. Nevanlinna [3] proved his famous five-value theorem that for two nonconstant meromorphic functions f and g on the whole complex plane $\mathbb{C}$, if they have the same inverse images (ignoring multiplicities) for five distinct values, then $f(z)\equiv g(z)$. After this very work, the uniqueness of meromorphic functions with shared values on $\mathbb{C}$ attracted many investigations (for references, see [10]).

It is very interesting to consider distinct small functions instead of distinct complex numbers on $\mathbb{C}$. Over the last few years, there were several generalizations of Nevanlinna's five-value theorem. To state some of these results, we must introduce some notions.

Let h be a nonzero holomorphic function on $\mathbb{C}$, expanding h as $h(z)=\sum_{i=0}^{\infty} b_i(z-z_0)^i$ around z_0, then we define $\nu_h(z_0):=\min\{i : b_i\neq 0\}$. Let k and M be positive integers or $+\infty$. We set

$$\nu_h^M(z)=\min\{M,\nu_h(z)\},$$

$$\nu_{h,\leq k}^M(z)=\begin{cases} 0, & \text{if } \nu_{h,\leq k}(z)>k; \\ \nu_h^M(z), & \text{if } \nu_{h,\leq k}(z)\leq k. \end{cases}$$

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Let φ be a nonconstant meromorphic function on $\mathbb{C}$ with reduced representation $\varphi = (\varphi_0 : \varphi_1)$, where φ_0, φ_1 are holomorphic functions on $\mathbb{C}$ having no common zeros and $\varphi = \frac{\varphi_0}{\varphi_1}$. We define $\nu_\varphi := \nu_{\varphi_0}$ and $\nu_{\varphi,\leq k} := \nu_{\varphi_0,\leq k}$.

The characteristic function of φ is defined by

$$T_\varphi(r) = \frac{1}{2\pi}\int_0^{2\pi} \log\|\varphi(re^{i\theta})\|d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log\|\varphi(e^{i\theta})\|d\theta \quad (r>1),$$

where $\|\varphi\| = \left(|\varphi_0|^2 + |\varphi_1|^2\right)^{1/2}$.

For two meromorphic functions f and a on $\mathbb{C}$ with reduced representations $f = (f_0 : f_1)$, $a = (a_0 : a_1)$ respectively, we set $(f, a) = a_0 f_0 + a_1 f_1$. The meromorphic function a is said to be "small" with respect to f if $T_a(r) = o\left(T_f(r)\right)$ as $r \to \infty$. Let $R := \mathcal{R}(f)$ be the set of meromorphic functions on $\mathbb{C}$ which are small with respect to f. Then $\mathcal{R}(f)$ is a field. We define

$$\nu^M_{(f,R),\leq k}(z) = \bigcup_{a\in R} \nu^M_{(f,a),\leq k}(z).$$

In 1999, Li and Qiao [2] gave a generalization of the above Nevanlinna theorem that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ and five meromorphic functions $\{a_j\}_{j=1}^5$ in $\mathcal{R}(f)\cap\mathcal{R}(g)$ satisfy $\nu^1_{(f,a_j)} = \nu^1_{(g,a_j)}$ $(1\leq j\leq 5)$, then $f(z)\equiv g(z)$. In 2005, Thai and Tan [4] improved strongly the results of Li-Qiao [2], Yao [6], and Yi [7]. They obtained that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ and five meromorphic functions $\{a_j\}_{j=1}^5$ in $\mathcal{R}(f)\cap\mathcal{R}(g)$ satisfy $\nu^1_{(f,a_j),\leq k} = \nu^1_{(g,a_j),\leq k}$ $(1\leq j\leq 5)$, then $f(z)\equiv g(z)$ for each $k\geq 3$. Recently, Cao and Yi [1] obtained a more general theorem which improves and extends the above-mentioned result of Thai-Tan [4] and Yi [8].

In this paper we continue to investigate this subject. In 1986, Yi [9] obtained two theorems on the shared sets and uniqueness of meromorphic functions as follows. Nevanlinna's five-value theorem is the special case when $l = 1$.

Theorem A ([10] or [9]). *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$. Suppose that*

$$S_j = \{a_j, a_j + b, \ldots, a_j + (l-1)b\},$$

where a_j $(j = 1, 2, \ldots, q)$ and b are finite complex numbers such that $b \neq 0$, $q > 4$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If $\nu^1_{(f_1,S_j)} = \nu^1_{(f_2,S_j)}$ for $j = 1, 2, \ldots, q$, then $f_1(z) \equiv f_2(z)$.

Theorem B ([10] or [9]). *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\},$$

where a_j $(j = 1, 2, \ldots, q)$ and c are finite complex numbers such that $a_j \neq 0$, $(j = 1, 2, \ldots, q)$, $q > 2 + \frac{2}{l}$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If $\nu^1_{(f_1,S_j)} = \nu^1_{(f_2,S_j)}$ for $j = 1, 2, \ldots, q$, then $(f_1(z) - c)^l \equiv (f_2(z) - c)^l$.

It is natural to ask:

Question 1. *Do Theorem A and Theorem B still hold if a_j $(j = 1, 2, \dots, q)$ b and c are distinct elements in $\mathcal{R}(f_1) \cap \mathcal{R}(f_2)$ instead of distinct complex numbers?*

The purpose of this article is to answer this question by making use of the second main theorem for small functions due to Yamanoi [5]. Our main theorems which improve and extend the above mentioned results will be showed in Section 3. In the next section we introduce the basic notions in Nevanlinna theory and some lemmas.

2. Preliminaries and some lemmas

Let h be a nonzero holomorphic function on $\mathbb{C}$ and k be a positive integer or $k = \infty$. We define

$$N_{h,\le k}(r) = \int_1^r \frac{n_{\le k}(t)}{t}dt \quad \text{and} \quad \overline{N}_{h,\le k}(r) = \int_1^r \frac{\overline{n}_{\le k}(t)}{t}dt \quad (r > 1),$$

where $n_{\le k}(t) = \sum_{|z|\le t} \nu_{h,\le k}(z)$ and $\overline{n}_{\le k}(t) = \sum_{|z|\le t} \nu^1_{h,\le k}(z)$.

Let φ be a nonconstant meromorphic function on $\mathbb{C}$ with reduced representation $\varphi = (\varphi_0 : \varphi_1)$. We define $N_{\varphi,\le k}(r) := N_{\varphi_0,\le k}(r)$ and $\overline{N}_{\varphi,\le k}(r) := \overline{N}_{\varphi_0,\le k}(r)$. For brevity we write $N_{\varphi,\le\infty}(r)$ as $N_\varphi(r)$ or $N(r, \nu_\varphi)$; write $\overline{N}_{\varphi,\le\infty}(r)$ as $\overline{N}_\varphi(r)$ or $\overline{N}(r, \nu_\varphi)$; and write $N_{\varphi,\le k}(r)$ as $N_{\le k}(r, \nu_\varphi)$. Set

$$\nu^M_{h,\ge k+1}(z) = \begin{cases} 0, & \text{if} \quad \nu_h(z) < k; \\ \nu^M_h(z), & \text{if} \quad \nu_h(z) \ge k + 1. \end{cases}$$

Similarly, we can get the corresponding definitions of $N_{\varphi,\ge k+1}(r)$, $\overline{N}_{\varphi,\ge k+1}(r)$, etc.

Let $\{a_j\}_{j=0}^q$ be meromorphic functions on $\mathbb{C}$ with reduced representations $a_j = (a_{j0} : a_{j1})$ $(0 \le j \le q)$. For each $0 \le j \le q$, we fix an index $k_j \in \{0, 1\}$ such that $a_{jk_j} \not\equiv 0$ and set $a^*_j := (a_{j1} : -a_{j0})$, $\tilde{a}_j := \left(\frac{a_{j0}}{a_{jk_j}} : \frac{a_{j1}}{a_{jk_j}}\right)$, $\tilde{a}^*_j := \left(\frac{a_{j1}}{a_{jk_j}} : -\frac{a_{j0}}{a_{jk_j}}\right)$.

Let f be a meromorphic function on $\mathbb{C}$ with reduced representation $f = (f_0 : f_1)$. For each $0 \le j \le q$, we set $(f, \tilde{a}_j) = \frac{a_{j0}f_0 + a_{j1}f_1}{a_{jk_j}}$, $(f, \tilde{a}^*_j) = \frac{a_{j1}f_0 - a_{j0}f_1}{a_{jk_j}}$.

For a meromorphic function f on $\mathbb{C}$, we define the proximity function of f by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \ge 0$. Then

$$T_f(r) = N_f(r) + m(r, f) + O(1).$$

Let $a \in \mathcal{R}(f)$. We denote the deficiency of a with respect to f by

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{T_f(r)}=1-\limsup_{r\to\infty}\frac{N_{(f,a)}(r)}{T_f(r)},$$

and denote the reduced deficiency by

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}_{(f,a)}(r)}{T_f(r)}.$$

As usual, by the notation "$\|P$" we mean the assertion P holds for all $r\in[0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\int_E dr<\infty$.

Theorem C ([5], The second main theorem for small functions)**.** *Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let a_1, a_2, ..., a_q be distinct meromorphic functions on $\mathbb{C}$. Assume that a_i are small functions with respect to f for all $1\le i\le q$. Then for each $\varepsilon>0$, the following holds*

$$\left\|(q-2-\varepsilon)T_f(r)\le\sum_{i=1}^{q}\overline{N}_{(f,a_i)}(r)+o\left(T_f(r)\right).\right.$$

Definition 2.1. Let $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$, and let $a(z)$ be any element in $\mathcal{R}(f)$. If $f(z)-a(z)$ has no zero points, then $a(z)$ is called a Picard exceptional small function of $f(z)$.

From Theorem C, one can see that any transcendental meromorphic function on $\mathbb{C}$ has at most two Picard exceptional small functions.

Lemma 2.1 ([4])**.** *Let f be a nonconstant meromorphic function on $\mathbb{C}$ and a_1,a_2 be two distinct small functions with respect to f. Then*

$$T_{\frac{(f,\tilde{a}_1)}{(f,\tilde{a}_2)}}(r)=T_f(r)+o\left(T_f(r)\right).$$

Lemma 2.2 ([1])**.** *Let f be a nonconstant meromorphic function on $\mathbb{C}$, a be a small function with respect to f, and k be a positive integer or infinity. Then*

$$\overline{N}_{(f,a)}(r)\quad\le\quad\frac{k}{k+1}\overline{N}_{(f,a),\le k}(r)+\frac{1}{k+1}N_{(f,a)}(r);$$

and

$$\overline{N}_{(f,a)}(r)\quad\le\quad\frac{k}{k+1}\overline{N}_{(f,a),\le k}(r)+\frac{1}{k+1}T_f(r)+o\left(T_f(r)\right).$$

3. Unicity theorems of meromorphic functions

We now show our main theorems. The first one is:

Theorem 3.1. *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$. Suppose that a_j $(j=1,2,\ldots,q)$ and b are $q+1$ distinct meromorphic functions in $\mathcal{R}(f_1)\cap\mathcal{R}(f_2)$ such that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b \not\equiv 0$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. *Let* k_j $(j = 1, 2, \ldots, q)$ *be positive integers or* ∞ *satisfying*

$$k_1 \geq k_2 \geq \cdots \geq k_q \tag{1}$$

and

$$\nu^1_{(f_1,S_j),\leq k_j} = \nu^1_{(f_2,S_j),\leq k_j}, \quad (j = 1, 2, \ldots, q). \tag{2}$$

Furthermore, let

$$\Theta(f_i) = \sum_a \Theta(0, f_i - a) - \sum_{j=1}^{q} \sum_{s=0}^{l-1} \Theta(0, f_i - (a_j + sb)), \ (i = 1, 2),$$

and

$$\begin{aligned} A_1 \;=\;& \frac{\sum_{j=1}^{m-1} \sum_{s=0}^{l-1} \delta(0, f_1 - (a_j + sb))}{k_m + 1} + \sum_{j=m}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_1 - (a_j + sb))}{k_{j+1}} \\ &+ \frac{(lm - 3l + 1)k_m}{k_m + 1} - \frac{(2l - 1)k_n}{k_n + 1} + \Theta(f_1) - 2, \end{aligned}$$

$$\begin{aligned} A_2 \;=\;& \frac{\sum_{j=1}^{n-1} \sum_{s=0}^{l-1} \delta(0, f_2 - (a_j + sb))}{k_n + 1} + \sum_{j=n}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_2 - (a_j + sb))}{k_{j+1}} \\ &+ \frac{(ln - 3l + 1)k_n}{k_n + 1} - \frac{(2l - 1)k_m}{k_m + 1} + \Theta(f_2) - 2, \end{aligned}$$

where m *and* n *are positive integers in* $\{1, 2, \ldots, q\}$ *and* a *is an arbitrary element in* $\mathcal{R}(f_1) \cap \mathcal{R}(f_2)$. *If*

$$\min\{A_1, A_2\} \geq 0 \quad \textit{and} \quad \max\{A_1, A_2\} > 0, \tag{3}$$

then $f_1(z) \equiv f_2(z)$.

Proof. Suppose that $f_1(z) \not\equiv f_2(z)$. Without loss of generality, we assume that there exist infinitely many small functions d with respect to f such that $\Theta(0, f_1 - d) > 0$ and $d \notin \{a_j + sb : j = 1, 2, \ldots, q, \ s = 0, 1, \ldots, l - 1\}$. We denote them by d_k $(k = 1, 2, \ldots, \infty)$. Obviously, $\Theta(f_1) = \sum_{k=1}^{\infty} \Theta(0, f_1 - d_k)$. Thus there exits a p such that $\sum_{k=1}^{p} \Theta(0, f_1 - d_k) > \Theta(f_1) - \varepsilon$ holds for any given ε (> 0). From Theorem C (The second main theorem for small functions) we have

$$\begin{aligned} &\|(ql + p - 2 - \varepsilon)T_{f_1}(r) \\ \leq\; & \sum_{j=1}^{q} \sum_{s=0}^{n-1} \overline{N}_{(f_1, a_j + sb)}(r) + \sum_{k=1}^{p} \overline{N}_{(f_1, d_k)}(r) + o(T_{f_1}(r)). \end{aligned}$$

By the definition of reduced deficiency, we have

$$\overline{N}_{(f_1, d_k)}(r) < (1 - \Theta(0, f_1 - d_k))\, T_{f_1}(r) + o(T_{f_1}(r)).$$

From Lemma 2.2 and the definition of deficiency, we get for $s \in \{0, 1, \ldots, l-1\}$

$$\begin{aligned}\overline{N}_{(f_1,a_j+sb)}(r) &\le \frac{k_j}{k_j+1}\overline{N}_{(f_1,a_j+sb),\le k_j}(r) + \frac{1}{k_j+1}N_{(f_1,a_j+sb)}(r)\\ &< \frac{k_j}{k_j+1}\overline{N}_{(f_1,a_j+sb),\le k_j}(r)\\ &\quad + \frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)T_{f_1}(r)+o(T_{f_1}(r)).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\|(ql+p-2-\varepsilon)T_{f_1}(r)\\ &< \left\{\sum_{k=1}^{p}(1-\Theta(0,f_1-d_k))\right\}T_{f_1}(r)+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}\overline{N}_{(f_1,a_j+sb),\le k_j}(r)\\ &\quad+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T_{f_1}(r)+o(T_{f_1}(r)).\end{aligned}$$

Noting that

$$1 \ge \frac{k_1}{k_1+1} \ge \frac{k_2}{k_2+1} \ge \cdots \ge \frac{k_q}{k_q+1} \ge \frac{1}{2},$$

we can deduce that

$$\begin{aligned}&\|(ql+p-2-\varepsilon)T_{f_1}(r)\\ &< \left(p-\Theta(f_1)+\varepsilon\right)T_{f_1}(r)+\frac{k_m}{k_m+1}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{(f_1,a_j+sb),\le k_j}(r)\\ &\quad+\left\{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\left(\frac{k_j}{k_j+1}-\frac{k_m}{k_m+1}\right)\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T_{f_1}(r)\\ &\quad+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1-\delta(0,f_1-(a_j+sb))}{k_j+1}\right\}T_{f_1}(r)+o(T_{f_1}(r)),\end{aligned}$$

namely,

$$\begin{aligned}&\left\|\left(\frac{l(m-1)k_m}{k_m+1}+B_1-2\varepsilon\right)T_{f_1}(r)\right.\\ &< \sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{(f_1,a_j+sb),\le k_j}(r)+o(T_{f_1}(r)),\end{aligned}$$

where

$$B_1 = \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0, f_1-(a_j+sb))}{k_m+1} + \sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0, f_1-(a_j+sb))}{k_{j+1}} + \Theta(f_1) - 2.$$

By similar discussion, we have

$$\left\|\left(\frac{l(n-1)k_n}{k_n+1} + B_2 - 2\varepsilon\right)T_{f_2}(r) < \sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{(f_2,a_j+sb),\le k_j}(r) + o(T_{f_2}(r)),\right.$$

where

$$B_2 = \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0, f_2-(a_j+sb))}{k_n+1} + \sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0, f_2-(a_j+sb))}{k_{j+1}} + \Theta(f_2) - 2.$$

Hence

$$\left\|\left(\frac{l(m-1)k_m}{k_m+1} + B_1 - 2\varepsilon\right)T_{f_1}(r) + \left(\frac{l(n-1)k_n}{k_n+1} + B_2 - 2\varepsilon\right)T_{f_2}(r)\right.$$
$$< \sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{(f_1,a_j+sb),\le k_j}(r) + \sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{(f_2,a_j+sb),\le k_j}(r) + o(T_{f_1}(r)+T_{f_2}(r)).$$

Let a_0 be a nonzero meromorphic function on $\mathbb{C}$ such that

$$a_0 \in (\mathcal{R}(f_1)\cap\mathcal{R}(g_2)) \setminus \{a_j+sb : j=1,2,\dots,q,\ s=0,1,\dots,l-1\}.$$

Since $f_1 \not\equiv f_2$, there exists one of $\{a_j+sb : j=1,2,\dots,q,\ s=0,1,\dots,l-1\}$, assume that a_1, such that $\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)} \not\equiv \frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}$. Obviously, we have $f(z)-g(z)\not\equiv sb$ for each $s\in\{1,2,\dots,l-1\}$. Otherwise, a_j $(j=1,2,\dots,q)$ are the Picard exceptional small functions of f_1 and $a_j+(l-1)b$ $(j=1,2,\dots,q)$ are the Picard exceptional small functions of f_2. This is impossible. Thus, there exists one of $\{a_j : j=1,2,\dots,q\}$, assume also that a_1, such that $\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)} \not\equiv \frac{(f_2+sb,\tilde{a}_1)}{(f_2+sb,\tilde{a}_0)}$. Similarly, we have $f_2-f_1\not\equiv sb$ for each $s\in\{1,2,\dots,l-1\}$. Thus, there exists one of $\{a_j : j=1,2,\dots,q\}$, assume also that a_1, such that $\frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)} \not\equiv \frac{(f_1+sb,\tilde{a}_1)}{(f_1+sb,\tilde{a}_0)}$. Since $\nu^1_{(f_1,S_j),\le k_j} = \nu^1_{(f_2,S_j),\le k_j}$ for $j=1,2,\dots,q$, we have $f_1=f_2$, or $f_1 = f_2+sb$ or $f_2=f_1+sb$ on $\bigcup_{j=1}^{q}\{z : \nu_{(f_1,S_j),\le k_j} > 0\}$, where $s=1,2\dots,l-1$. It is easy to see that $(\alpha^*,\beta)=0$ on $\{z : (f_1,\alpha)(z)=0$ and $(f_1,\alpha)(z)=0\}$, where α and β are distinct elements in $\{a_j+sb : j=1,2,\dots,q$ and $s=0,1,\dots,l-1\}$.

Hence, by Lemma 2.1 we have

$$
\begin{aligned}
&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{(f_1,a_j+sb),\leq k_j}(r)\\
&\leq N_{\frac{(f_1,a_1)}{(f_1,a_0)}-\frac{(f_2,a_1)}{(f_2,a_0)}}(r)+\sum_{s=1}^{l-1}N_{\frac{(f_1,a_1)}{(f_1,a_0)}-\frac{(f_2+sb,a_1)}{(f_2+sb,a_0)}}(r)+\sum_{s=1}^{l-1}N_{\frac{(f_2,a_1)}{(f_2,a_0)}-\frac{(f_1+sb,a_1)}{(f_1+sb,a_0)}}(r)\\
&\quad+\sum_{\alpha,\beta\in\{a_j+sb:j=1,2,\ldots,q,\ s=0,1,\ldots,l-1\}}N_{((\alpha^*,\beta))}(r)\\
&=N_{\left(\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)}-\frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}\right)\cdot\frac{a_{1k_1}}{a_{0k_0}}}(r)+\sum_{s=1}^{l-1}N_{\left(\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)}-\frac{(f_2+sb,\tilde{a}_1)}{(f_2+sb,\tilde{a}_0)}\right)\cdot\frac{a_{1k_1}}{a_{0k_0}}}(r)\\
&\quad+\sum_{s=1}^{l-1}N_{\left(\frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}-\frac{(f_1+sb,\tilde{a}_1)}{(f_1+sb,\tilde{a}_0)}\right)\cdot\frac{a_{1k_1}}{a_{0k_0}}}(r)\\
&\quad+\sum_{\alpha,\beta\in\{a_j+sb:j=1,2,\ldots,q,\ s=0,1,\ldots,l-1\}}N_{((\alpha^*,\beta))}(r)\\
&\leq(2l-1)(T_{f_1}(r)+T_{f_2}(r))+o(T_{f_1}(r)+T_{f_2}(r)).
\end{aligned}
$$

Similarly, we get

$$\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{(f_2,a_j+sb),\leq k_j}(r)\leq(2l-1)(T_{f_1}(r)+T_{f_2}(r))+o(T_{f_1}(r)+T_{f_2}(r)).$$

Therefore, from the above discussion we obtain

$$
\begin{aligned}
&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T_{f_1}(r)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T_{f_2}(r)\\
&<(2l-1)\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T_{f_1}(r)+T_{f_2}(r))+o(T_{f_1}(r)+T_{f_2}(r)),
\end{aligned}
$$

namely,

$$(A_1-\varepsilon)\,T_{f_1}(r)+(A_2-\varepsilon)\,T_{f_2}(r)<o(T_{f_1}(r)+T_{f_2}(r)).$$

Noting that ε is arbitrary, the above inequality contradicts to (3). we complete the proof of Theorem 3.1. □

Obviously, Theorem 1 in [1] is the special case for $l=0$. Furthermore, from Theorem 3.1, we obtain the following corollaries.

Corollary 3.1. *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$. Suppose that a_j $(j=1,2,\ldots,q)$ and b are $q+1$ distinct meromorphic functions in $\mathcal{R}(f_1)\cap\mathcal{R}(f_2)$ such that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b \not\equiv 0$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. *Let* k_j $(j = 1, 2, \ldots, q)$ *be positive integers or* ∞ *satisfying*

$$k_1 \geq k_2 \geq \cdots \geq k_q.$$

and

$$\nu^1_{(f_1,S_j),\leq k_j} = \nu^1_{(f_2,S_j),\leq k_j}, \quad (j = 1, 2, \ldots, q).$$

If

$$\sum_{j=3}^{q} \sum_{s=0}^{l-1} \frac{k_j}{k_{j+1}} + \frac{(2-2l)k_3}{k_3+1} > 2,$$

then $f_1(z) \equiv f_2(z)$.

Proof. Let $m = n = 3$. Noting that $\Theta(f_i) \geq 0$ and $\delta(0, f_i - (a_j + sb)) \geq 0$ for $j = 1, 2, \ldots, q$ and $i = 1, 2$, one can deduce from Theorem 3.1 that Corollary 3.1 follows. □

The following corollary is an analog of Theorem A due to H.-X. Yi for small functions for small functions.

Corollary 3.2. *Let* f_1 *and* f_2 *be two nonconstant meromorphic functions on* $\mathbb{C}$. *Suppose that* a_j $(j = 1, 2, \ldots, q)$ *and* b *are* $q+1$ *distinct meromorphic functions in* $\mathcal{R}(f_1) \cap \mathcal{R}(f_2)$ *such that*

$$S_j = \{a_j, a_j + b, \ldots, a_j + (l-1)b\}, \quad j = 1, 2, \ldots, q,$$

with $b \not\equiv 0$, $q > 4$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. *If*

$$\nu^1_{(f_1,S_j),\leq k_j} = \nu^1_{(f_2,S_j),\leq k_j}, \quad (j = 1, 2, \ldots, q),$$

then $f_1(z) \equiv f_2(z)$.

Proof. Let $k_1 = k_2 = \cdots = k_q = \infty$. One can deduce from Corollary 3.1 that Corollary 3.2 follows immediately. □

Here we show another main theorem.

Theorem 3.2. *Let* f_1 *and* f_2 *be two nonconstant meromorphic functions on* $\mathbb{C}$. *Suppose that* a_j $(j = 1, 2, \ldots, q)$, c *and* w *are* $q+2$ *distinct meromorphic functions in* $\mathcal{R}(f_1) \cap \mathcal{R}(f_2)$ *such that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

with $a_j \not\equiv 0$, $(j = 1, 2, \ldots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. *Let* k_j $(j = 1, 2, \ldots, q)$ *be positive integers or* ∞ *satisfying*

$$k_1 \geq k_2 \geq \cdots \geq k_q \tag{4}$$

and

$$\nu^1_{(f_1,S_j),\leq k_j} = \nu^1_{(f_2,S_j),\leq k_j}, \quad (j = 1, 2, \ldots, q). \tag{5}$$

Furthermore, let

$$\Theta(f_i)=\sum_{a}\Theta(0,f_i-a)-\sum_{j=1}^{q}\sum_{s=0}^{l-1}\Theta(0,f_i-(c+a_jw^s)),\ (i=1,2),$$

and

$$A_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(c+a_jw^s))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(c+a_jw^s))}{k_{j+1}}$$
$$+\frac{l(m-2)k_m}{k_m+1}-\frac{lk_n}{k_n+1}+\Theta(f_1)-2,$$

$$A_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(c+a_jw^s))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(c+a_jw^s))}{k_{j+1}}$$
$$+\frac{l(n-2)k_n}{k_n+1}-\frac{lk_m}{k_m+1}+\Theta(f_2)-2,$$

where m *and* n *are positive integers in* $\{1,2,\dots,q\}$ *and* a *is an arbitrary element in* $\mathcal{R}(f_1)\cap\mathcal{R}(f_2)$. *If*

$$\min\{A_1,A_2\}\geq 0\quad \textit{and}\quad \max\{A_1,A_2\}>0, \tag{6}$$

then $(f_1(z)-c)^l\equiv(f_1(z)-c)^l$.

Proof. We assume that $(f_1(z)-c)^l\not\equiv(f_2(z)-c)^l$. Without loss of generality, we assume that there exist infinitely many small functions d with respect to f_1 such that $\Theta(0,f_1-d)>0$ and $d\notin\{c+a_jw^s: j=1,2,\dots,q,\ s=0,1,\dots,l-1\}$. We denote them by d_k $(k=1,2,\dots,\infty)$. Obviously, $\Theta(f_1)=\sum_{k=1}^{\infty}\Theta(0,f_1-d_k)$. Thus there exits a p such that $\sum_{k=1}^{p}\Theta(0,f_1-d_k)>\Theta(f_1)-\varepsilon$ holds for any given ε (>0).

Using similar discussion as in the proof of Theorem 3.1, we obtain

$$\left(\frac{l(m-1)k_m}{k_m+1}+B_1-2\varepsilon\right)T_{f_1}(r)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-2\varepsilon\right)T_{f_2}(r)$$
$$<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{(f_1,c+a_jw^s),\leq k_j}(r)$$
$$+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{(f_2,c+a_jw^s),\leq k_j}(r)+o(T_{f_1}(r)+T_{f_2}(r)),$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(c+a_jw^s))}{k_m+1}$$
$$+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(c+a_jw^s))}{k_{j+1}}+\Theta(f_1)-2,$$

$$B_2 = \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0, f_2-(c+a_jw^s))}{k_n+1} + \sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0, f_2-(c+a_jw^s))}{k_{j+1}} + \Theta(f_2) - 2.$$

Let a_0 be a nonzero meromorphic function on $\mathbb{C}$ such that

$$a_0 \in (\mathcal{R}(f_1)\cap\mathcal{R}(g_2)) \setminus \{c+a_jw^s : j=1,2,\ldots,q,\ s=0,1,\ldots,l-1\}.$$

Since $(f_1(z)-c)^l \not\equiv (f_2(z)-c)^l$, there exists one of $\{c+a_jw^s\ :\ j=1,2,\ldots,q,$ $s=0,1,\ldots,l-1\}$, assume that $\delta_1 := c+a_1$, such that $\frac{(f_1,\tilde{\delta}_1)}{(f_1,\tilde{a}_0)} \not\equiv \frac{(f_2,\tilde{\delta}_1)}{(f_2,\tilde{a}_0)}$. Since $\nu^1_{(f_1,S_j),\le k_j} = \nu^1_{(f_2,S_j),\le k_j}$ for $j=1,2,\ldots,q$, we have $(f_1(z)-c)^l = (f_2(z)-c)^l$ on $\bigcup_{j=1}^{q}\{z : \nu_{(f_1,S_j),\le k_j} > 0\}$. One can see that $(\alpha^*,\beta)=0$ on $\{z : (f_i,\alpha)(z)=0$ and $(f_i,\alpha)(z)=0\}$, where $i=1,2$, α and β are distinct elements in $\{c+a_jw^s :$ $j=1,2,\ldots,q,\ s=0,1,\ldots,l-1\}$.

Using similar discussion as in the proof of Theorem 3.1, we deduce by Lemma 2.1 that

$$\begin{aligned}
&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{(f_1,c+a_jw^s),\le k_j}(r)\\
&< N_{\frac{((f_1-c)^l,\delta_1)}{((f_1-c)^l,a_0)}-\frac{((f_2-c)^l,\delta_1)}{((f_2-c)^l,a_0)}}(r) + \sum_{\alpha,\beta\in\{c+a_jw^s : j=1,2,\ldots,q,\ s=0,1,\ldots,l-1\}} N_{((\alpha^*,\beta))}(r)\\
&\le l(T_{f_1}(r)+T_{f_2}(r)) + o(T_{f_1}(r)+T_{f_2}(r)).
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{(f_2,c+a_jw^s),\le k_j}(r)\\
&< N_{\frac{((f_2-c)^l,\delta_1)}{((f_2-c)^l,a_0)}-\frac{((f_1-c)^l,\delta_1)}{((f_1-c)^l,a_0)}}(r) + \sum_{\alpha,\beta\in\{c+a_jw^s : j=1,2,\ldots,q,\ s=0,1,\ldots,l-1\}} N_{((\alpha^*,\beta))}(r)\\
&\le l(T_{f_1}(r)+T_{f_2}(r)) + o(T_{f_1}(r)+T_{f_2}(r)).
\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}
&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-2\varepsilon\right)T_{f_1}(r) + \left(\frac{l(n-1)k_n}{k_n+1}+B_2-2\varepsilon\right)T_{f_2}(r)\\
&< l\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T_{f_1}(r)+T_{f_2}(r)) + o(T_{f_1}(r)+T_{f_2}(r)),
\end{aligned}$$

namely,

$$(A_1-2\varepsilon)\,T_{f_1}(r) + (A_2-2\varepsilon)\,T_{f_2}(r) < o(T_{f_1}(r)+T_{f_2}(r)).$$

Noting that ε is arbitrary, the above inequality contradicts to (3). Therefore, we complete the proof of Theorem 3.2. □

We have an analog of Theorem B due to H.-X. Yi for small functions.

Corollary 3.3. *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$. Suppose that a_j $(j = 1, 2, \ldots, q)$, c and w are $q + 2$ distinct meromorphic functions in $\mathcal{R}(f_1) \cap \mathcal{R}(f_2)$ such that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

with $a_j \not\equiv 0$, $q > 2 + \frac{2}{l}$, $(j = 1, 2, \ldots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If

$$\nu^1_{(f_1,S_j),\leq k_j} = \nu^1_{(f_2,S_j),\leq k_j}, \quad (j = 1, 2, \ldots, q), \tag{7}$$

then $(f_1(z) - c)^l \equiv (f_2(z) - c)^l$.

Proof. Let $m = n = 1$ and $k_1 = k_2 = \cdots = \infty$. Note that $\Theta(f_i) \geq 0$ and $\delta(0, f_i - (a_j + sb)) \geq 0$ for $j = 1, 2, \ldots, q$ and $i = 1, 2$, Corollary 3.3 follows immediately from Theorem 3.2. □

One can see that Nevanlinna's five-value theorem for small functions is the special case where $l = 1$.

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