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ON SELF-RECIPROCAL POLYNOMIALS AT A POINT ON THE UNIT CIRCLE

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Abstract. Given two integral self-reciprocal polynomials having the same modulus at a point z_0 on the unit circle, we show that the minimal polynomial of z_0 is also self-reciprocal and it divides an explicit integral self-reciprocal polynomial. Moreover, for any two integral self-reciprocal polynomials, we give a sufficient condition for the existence of a point z_0 on the unit circle such that the two polynomials have the same modulus at z_0.

1. Introduction and statement of results

Throughout this paper, U denotes the unit circle and n is a positive integer. A polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is said to be a self-reciprocal polynomial of degree n if it satisfies $a_n \neq 0$ and $P(z) = z^n P(1/z)$. Thus the zeros of a self-reciprocal polynomial either lie on the unit circle or are symmetric with respect to U. There have been a number of interesting problems (for example [2]) about the distribution of zeros of self-reciprocal polynomials. Also the minimal polynomial of an algebraic number α is the unique irreducible monic polynomial $f(z)$ of smallest degree with rational coefficients such that $f(\alpha) = 0$.

In this paper, we study a generalization of an already rather general problem, that of determining the zeros of a polynomial on U. This maybe phrased as finding z with $|z| = 1$ such that $|P(z)| = 0$, where $P(z)$ is a polynomial. We propose broaden this to the problem for finding z with $|z| = 1$ such that $|P(z)| = |Q(z)|$, where $P(z)$ and $Q(z)$ are polynomials. A first priority in this fashion seems to determine the minimal polynomial $F(z)$ of an element of the set

$$\{z : |P(z)| = |Q(z)|, |z| = 1\}.$$

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Also, what can be said about the number of zeros on U of $F(z)$? We study these questions in case that the polynomials are integral and self-reciprocal.

Now we establish the first result.

Theorem 1. *Let $P(z)$ and $Q(z)$ be integral self-reciprocal polynomials with* $\deg P(z) = m \geq n = \deg Q(z)$. *Suppose that*

$$|P(z_0)| = |Q(z_0)| \neq 0$$

for some z_0 with $|z_0| = 1$ and $z_0 \neq 1$. Then the minimal polynomial of z_0 is also self-reciprocal and it divides integral self-reciprocal polynomial $P(z)^2 - z^{m-n}Q(z)^2$.

In above theorem, $z_0 \neq 1$ is required because the minimal polynomial of 1 is $z - 1$ which is not self-reciprocal. One may ask whether there always exist z_0 with $|z_0| = 1$ and $z_0 \neq 1$ such that $|P(z_0)| = |Q(z_0)|$ for any two integral self-reciprocal polynomials $P(z)$ and $Q(z)$. But an example of

$$P(z) = z^3 - 2z^2 - 2z + 1, \qquad Q(z) = z^2 - 7z + 1$$

gives the negative answer by Theorem 1. This is because $P(z)^2 - z^{m-n}Q(z)^2$ has no zeros on U. Hence it is interesting to mention the condition that the question above is true. We now give a sufficient condition for that when $P(z)$ and $Q(z)$ are even degrees of polynomials.

Theorem 2. *For even integers m and n, let*

$$P(z) = \sum_{k=0}^{m} a_k z^k, \qquad Q(z) = \sum_{k=0}^{n} b_k z^k$$

be integral self-reciprocal polynomials with $\deg P(z) = m \geq n = \deg Q(z)$. *If either*

$$\left(a_{\frac{m}{2}} - b_{\frac{n}{2}}\right)^2 < \frac{8}{4m+3}\left[\sum_{k=1}^{\frac{n}{2}} \left(a_{\frac{m}{2}-k} - b_{\frac{n}{2}-k}\right)^2 + \sum_{k=\frac{n}{2}+1}^{\frac{m}{2}} a_{\frac{m}{2}-k}^2\right]$$

or

$$\left(a_{\frac{m}{2}} + b_{\frac{n}{2}}\right)^2 < \frac{8}{4m+3}\left[\sum_{k=1}^{\frac{n}{2}} \left(a_{\frac{m}{2}-k} + b_{\frac{n}{2}-k}\right)^2 + \sum_{k=\frac{n}{2}+1}^{\frac{m}{2}} a_{\frac{m}{2}-k}^2\right],$$

then there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that

$$|P(z_0)| = |Q(z_0)|.$$

In Section 2, we provide proofs and some examples of Theorems 1 and 2.

2. Proofs and examples

Proof of Theorem 1. The first part of the theorem follows from the well known fact that the integral minimal polynomial $f(z)$ of degree d of z_0 with $|z_0| = 1$ is self-reciprocal. This is because

$$z_0^d f\left(z_0^{-1}\right) = z_0^d f(\overline{z_0}) = 0,$$

and z_0 is a zero of the polynomial $z^n f\left(z^{-1}\right)$ which has degree d. Since the minimal is unique, we have $f(z) = z^d f\left(z^{-1}\right)$. We now prove the second part of the theorem. Suppose that $P(z)$ and $Q(z)$ are integral self-reciprocal polynomials with $\deg P(z) = m \geq n = \deg Q(z)$. Consider $2m$ degree polynomial

$$F(z) = P(z)^2 - z^{m-n} Q(z)^2.$$

Then $F(z)$ is an integral self-reciprocal polynomial since

$$\begin{aligned} z^{2m} F(z^{-1}) &= z^{2m}(P(z^{-1})^2 - z^{-m+n} Q(z^{-1})^2) \\ &= z^{2m}(z^{-2m} P(z)^2 - z^{-m+n} z^{-2n} Q(z)^2) \\ &= P(z)^2 - z^{m-n} Q(z)^2 = F(z). \end{aligned}$$

Suppose that $|P(z_0)|^2 = |Q(z_0)|^2$ for some z_0 with $|z_0| = 1$ and $z_0 \neq 1$. Using $\overline{z_0} = 1/z_0$ and $P(z)$, $Q(z)$ self-reciprocal, we have

$$\begin{aligned} 0 &= P(z_0)\overline{P(z_0)} - Q(z_0)\overline{Q(z_0)} = P(z_0)P(z_0^{-1}) - Q(z_0)Q(z_0^{-1}) \\ &= z_0^{-m} P(z_0)^2 - z_0^{-n} Q(z_0)^2 = z_0^{-m}(P(z_0)^2 - z_0^{m-n} Q(z_0)^2) \\ &= z_0^{-m} F(z_0), \end{aligned}$$

which completes the proof. □

Example 3. Let $P(z) = z^4 + 1$ and $Q(z) = z^2 + 1$. For $z_0 = \frac{1 \pm i\sqrt{3}}{2}$ and $z_1 = \frac{-1 \pm i\sqrt{3}}{2}$, we may compute that

$$|P(z_0)| = |Q(z_0)| = |P(z_1)| = |Q(z_1)| = 1.$$

Also the minimal polynomials of z_0 and z_1 are

$$z^2 - z + 1$$

and

$$z^2 + z + 1,$$

respectively. Now we can confirm that the two polynomials above, $z^2 \pm z + 1$, are factors of

$$(z^4 + 1)^2 - z^2(z^2 + 1)^2 = (z - 1)^2(z + 1)^2(z^2 + z + 1)^2(z^2 - z + 1)^2.$$

Example 4. Consider the self-reciprocal polynomials

$$z^3 + 1 \quad \text{and} \quad z^2 + z + 1$$

having all their zeros on U. By Theorem 1, a complex number z_0 on U with $|z_0^3+1| = |z_0^2+z_0+1|$ must have the minimal polynomial

$$F(z) = z^6 - z^5 - 2z^4 - z^3 - 2z^2 - z + 1$$

since

$$(z^3+1)^2 - z(z^2+z+1) = F(z),$$

and $F(z)$ is irreducible.

The minimal polynomials of z_0 and z_1 in Example 3 have all their zeros on U. However we may verify that $F(z)$ in Example 4 has two zeros not on U. Hence it is natural to ask which self-reciprocal polynomials $P(z)$ and $Q(z)$ in Theorem 1 give the minimal polynomial of z_0 having all its zeros on U. We now provide two examples of such pairs of polynomials:

(1) $P(z) = z^{n+k}+1$, $Q(z) = z^n + 1$.

For $k \geq 1$,

$$\begin{aligned} &(z^{n+k}+1)^2 - z^k(z^n+1)^2 \\ &= (z^k-1)(z^{2n+k}-1) \\ &= (z-1)^2(z^{k-1}+z^{k-2}+\cdots+1)(z^{2n+k-1}+z^{2n+k-2}+\cdots+1). \end{aligned}$$

(2) $P(z) = \frac{z^m-1}{z-1}$, $Q(z) = \frac{z^n-1}{z-1}$.

For $m \geq n$,

$$\begin{aligned} &\left(\frac{z^m-1}{z-1}\right)^2 - z^{m-n}\left(\frac{z^n-1}{z-1}\right)^2 \\ &= \frac{(z^{m-n}-1)(z^{m+n}-1)}{(z-1)^2} \\ &= (z^{m-n-1}+z^{m-n-2}+\cdots+1)(z^{m+n-1}+z^{m+n-2}+\cdots+1). \end{aligned}$$

For the proof of Theorem 2, we need the following lemma which is the Nikolskii-type inequality (see Theorem 2.6 of [1]) for the class of real trigonometric polynomials of degree at most n.

Let $\mathbf{K} := \mathbb{R} \pmod{2\pi}$. For $f \in C(\mathbf{K})$, let

$$\|f\|_p := \left(\int_0^{2\pi} |f(\theta)|^p \, d\theta\right)^{1/p}, \qquad 0 < p < \infty.$$

Lemma 5. *Let T_n be a real trigonometric polynomial of degree at most n, and $0 < q \leq p \leq \infty$. Then we have*

$$\|T_n\|_p \leq \left(\frac{2rn+1}{2\pi}\right)^{\frac{1}{q}-\frac{1}{p}} \|T_n\|_q,$$

where $r := r(q)$ is the smallest integer not less than $q/2$.

Proof of Theorem 2. For even integers m and n, let

$$P(z) = \sum_{k=0}^{m} a_k z^k, \qquad Q(z) = \sum_{k=0}^{n} b_k z^k$$

be integral self-reciprocal polynomials with $\deg P(z) = m \geq n = \deg Q(z)$. Suppose that

$$|P(z)| \neq |Q(z)|$$

for all $z \in \mathbb{C}$ with $|z| = 1$. Write $F(z) = F_1(z)F_2(z)$, where

$$F_1(z) = P(z) - z^{\frac{m-n}{2}} Q(z), \qquad F_2(z) = P(z) + z^{\frac{m-n}{2}} Q(z).$$

Then both $F_1(z)$ and $F_2(z)$ have no zeros on U and $\deg F_1(z) = \deg F_2(z) = m$. Now we have

$$\frac{F_1(z)}{z^{\frac{m}{2}}} = \frac{P(z)}{z^{\frac{m}{2}}} - \frac{Q(z)}{z^{\frac{n}{2}}}.$$

Since, for $z = e^{i\theta}$, we have

$$\begin{aligned}
\frac{P(z)}{z^{\frac{m}{2}}} &= a_{\frac{m}{2}} + a_{\frac{m}{2}-1}\left(z + \frac{1}{z}\right) + a_{\frac{m}{2}-2}\left(z^2 + \frac{1}{z^2}\right) + \cdots + a_0\left(z^{\frac{m}{2}} + \frac{1}{z^{\frac{m}{2}}}\right) \\
&= a_{\frac{m}{2}} + 2\left(a_{\frac{m}{2}-1} Re\, z + \cdots + a_0 Re\, z^{\frac{m}{2}}\right) \\
&= a_{\frac{m}{2}} + 2\left(a_{\frac{m}{2}-1}\cos(\theta) + \cdots + a_0 \cos\left(\frac{m}{2}\theta\right)\right)
\end{aligned}$$

and similarly

$$\frac{Q(z)}{z^{\frac{n}{2}}} = b_{\frac{n}{2}} + 2\left(b_{\frac{n}{2}-1}\cos(\theta) + \cdots + b_0 \cos\left(\frac{n}{2}\theta\right)\right).$$

Since $F_1(z)$ has no zeros on U,

$$\begin{aligned}
T(\theta) := \frac{F_1(z)}{z^{\frac{m}{2}}} &= \frac{P(z)}{z^{\frac{m}{2}}} - \frac{Q(z)}{z^{\frac{n}{2}}} \\
&= \left(a_{\frac{m}{2}} + 2\left(a_{\frac{m}{2}-1}\cos(\theta) + \cdots + a_0 \cos\left(\frac{m}{2}\theta\right)\right)\right) \\
&\quad - \left(b_{\frac{n}{2}} + 2\left(b_{\frac{n}{2}-1}\cos(\theta) + \cdots + b_0 \cos\left(\frac{n}{2}\theta\right)\right)\right) \\
&= a_{\frac{m}{2}} - b_{\frac{n}{2}} + 2\sum_{k=1}^{\frac{n}{2}} \left(a_{\frac{m}{2}-k} - b_{\frac{n}{2}-k}\right)\cos(k\theta) \\
&\quad + 2\sum_{k=\frac{n}{2}+1}^{\frac{m}{2}} a_{\frac{m}{2}-k}\cos(k\theta)
\end{aligned}$$

has no any real zeros. Without loss of generality we may assume that T is positive on the real line. Then we have

$$\|T\|_1 = \int_0^{2\pi} T(\theta)\, d\theta = 2\pi\left(a_{\frac{m}{2}} - b_{\frac{n}{2}}\right).$$

Using the Parseval formula, we also have

$$\|T\|_2^2 = \int_0^{2\pi} T(\theta)^2\, d\theta = \frac{\pi}{2}\left(a_{\frac{m}{2}} - b_{\frac{n}{2}}\right)^2 + 4\pi\left[\sum_{k=1}^{\frac{n}{2}}\left(a_{\frac{m}{2}-k} - b_{\frac{n}{2}-k}\right)^2 + \sum_{k=\frac{n}{2}+1}^{\frac{m}{2}} a_{\frac{m}{2}-k}^2\right].$$

By Lemma 5,

$$\|T\|_2^2 \le \left(\frac{m+1}{2\pi}\right)\|T\|_1^2$$

and so

$$\frac{1}{2}\left(a_{\frac{m}{2}} - b_{\frac{n}{2}}\right)^2 + 4\left[\sum_{k=1}^{\frac{n}{2}}\left(a_{\frac{m}{2}-k} - b_{\frac{n}{2}-k}\right)^2 + \sum_{k=\frac{n}{2}+1}^{\frac{m}{2}} a_{\frac{m}{2}-k}^2\right] \le \left(\frac{m+1}{2\pi}\right)4\pi\left(a_{\frac{m}{2}} - b_{\frac{n}{2}}\right)^2 = 2(m+1)\left(a_{\frac{m}{2}} - b_{\frac{n}{2}}\right)^2,$$

i.e.,

$$\left(a_{\frac{m}{2}} - b_{\frac{n}{2}}\right)^2 \ge \frac{8}{4m+3}\left[\sum_{k=1}^{\frac{n}{2}}\left(a_{\frac{m}{2}-k} - b_{\frac{n}{2}-k}\right)^2 + \sum_{k=\frac{n}{2}+1}^{\frac{m}{2}} a_{\frac{m}{2}-k}^2\right].$$

Using $F_2(z)$ having no zeros on U, we follow above method to get

$$\left(a_{\frac{m}{2}} + b_{\frac{n}{2}}\right)^2 \ge \frac{8}{4m+3}\left[\sum_{k=1}^{\frac{n}{2}}\left(a_{\frac{m}{2}-k} + b_{\frac{n}{2}-k}\right)^2 + \sum_{k=\frac{n}{2}+1}^{\frac{m}{2}} a_{\frac{m}{2}-k}^2\right],$$

which completes the proof. □

References

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