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ON THE dTH POWER RESIDUE SYMBOL OF FUNCTION FIELDS

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Abstract. In this short notice, we prove a new result about the dth power residue symbol of function fields, by modifying the method of W. Kohnen in the paper published in Bull. Korean Math. Soc. **45** (2008), no. 2, 273–275.

W. Kohnen [1] gave a short and elementary proof of the existence of infinitely many primes p such that a given positive integer a congruent to 3 modulo 4 is a quadratic non-residue modulo p.

In this short notice, we prove a new result about the dth power residue symbol of function fields by modifying the method of Kohnen [1]. Let q be a power of an odd prime, $A = \mathbb{F}_q[t]$ be the polynomial ring over the finite field $\mathbb{F}_q$ with q elements. Let $\mathbb{F}_q^\times$ be the group of nonzero elements of $\mathbb{F}_q, g$ be a generator of $\mathbb{F}_q^\times$ and d be any divisor of $q-1$, thus $\eta = g^{\frac{q-1}{d}}$ becomes an element of order d in $\mathbb{F}_q^\times$. For any irreducible polynomial P in A, define $|P| = q^{\deg P}$. The dth power residue symbol of $\mathbb{F}_q[t]$ is defined as follows [2].

Definition 1. Let $P \in A$ be an irreducible polynomial. $a \in A$ and P does not divide a. Let $(a/P)_d$ be the unique element of $\mathbb{F}_q^\times$ such that

$$a^{\frac{|P|-1}{d}} \equiv \left(\frac{a}{P}\right)_d \pmod{P}.$$

If $P|a$ define $(a/P)_d = 0$. The symbol $\left(\frac{a}{P}\right)_d$ is called the dth power residue symbol.

Now we can state the following main result of this note.

Theorem 2. *Let a be a nonzero polynomial in $\mathbb{F}_q[t]$ and $d \nmid \deg a$. Then for any $i = 0, 1, 2, \ldots, d-1$, there exist infinitely many primes P in $\mathbb{F}_q[t]$ with $d \nmid \deg P$, such that $\left(\frac{a}{P}\right)_d = \eta^{i \deg P}$.*

When $d = 2$, we have the following result.

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Corollary 3. *Let a be a nonzero polynomial in $\mathbb{F}_q[t]$ with odd degree. Then there exist infinitely many primes in $\mathbb{F}_q[t]$ with odd degree such that $\left(\frac{a}{P}\right)_2 = -1$.*

Now we modify Kohnen's method to give a short and elementary proof of Theorem 2.

Proof. For $x \in \mathbb{R}, x \geq q$, let

$$m = -g^i \left(\prod_{|f| \leq x,\, a \not\equiv 0 \,(\mathrm{mod}\ f)} f \right)^d + a^\lambda, \tag{1}$$

where in (1) the product over all primes $|f| \leq x$ that do not divide a and λ is a positive integer with $\lambda \equiv 1 \pmod d$ such that

$$\deg a^\lambda > \deg \left(\prod_{|f| \leq x,\, a \not\equiv 0 \,(\mathrm{mod}\ f)} f \right)^d .$$

Thus $\deg m = \lambda \deg a \equiv \deg a \pmod d$ and $d \nmid \deg m$.

Let P be a prime dividing m with $d \nmid \deg P$. Then necessarily, by the definition of m, we must have $|P| > x$.

On the other hand, we find from (1)

$$a^\lambda \equiv g^i \left(\prod_{|f| \leq x,\, a \not\equiv 0 \,(\mathrm{mod}\ f)} f \right)^d \pmod P.$$

Thus

$$a^{\lambda \frac{|P|-1}{d}} \equiv g^{i \frac{|P|-1}{d}} \left(\prod_{|f| \leq x,\, a \not\equiv 0 \,(\mathrm{mod}\ f)} f \right)^{|P|-1} \pmod P.$$

From Fermat's Little Theorem (see the corollary of Proposition 1.8 in [2]), we deduce that

$$a^{\lambda \frac{|P|-1}{d}} \equiv g^{i \frac{|P|-1}{d}} \pmod P.$$

From the definition of the dth power residue symbol of $\mathbb{F}_q[t]$, we get

$$\left(\frac{a}{P}\right)_d \equiv a^{\frac{|P|-1}{d}} \equiv a^{\frac{|P|-1}{d}\lambda} \equiv g^{i \frac{|P|-1}{d}} \equiv \eta^{i \deg P} \pmod P.$$

Which conclude the proof. □

References

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