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ON THE GAUSS MAP OF SURFACES OF REVOLUTION WITHOUT PARABOLIC POINTS

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Abstract. In this article, we study surfaces of revolution without parabolic points in a Euclidean 3-space whose Gauss map G satisfies the condition $\Delta^h G = AG, A \in \text{Mat}(3, \mathbb{R})$, where Δ^h denotes the Laplace operator of the second fundamental form h of the surface and $\text{Mat}(3, \mathbb{R})$ the set of 3×3-real matrices, and also obtain the complete classification theorem for those. In particular, we have a characterization of an ordinary sphere in terms of it.

1. Introduction

As is well known, the theory of Gauss map is always one of interesting topics in a Euclidean space and a pseudo-Euclidean space and it has been investigated from the various viewpoints by many differential geometers ([1, 2, 3, 4, 7, 10]).

Let M be a connected surface in a Euclidean 3-space $\mathbb{R}^3$, and $G : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ its Gauss map. It is well known that M has constant mean curvature if and only if $\Delta G = ||dG||^2 G$, where Δ is the Laplace operator on M corresponding to the induced metric on M from $\mathbb{R}^3$ ([14]). As a special case one can consider surfaces whose Gauss map is an eigenfunction of a Laplacian, that is, $\Delta G = \lambda G, \lambda \in \mathbb{R}$. On the generalization of this equation, F. Dillen, J. Pas, and L. Verstraelen ([9]) studied surfaces of revolution in a Euclidean 3-space $\mathbb{R}^3$ such that its Gauss map G satisfies the condition

$$\Delta G = AG, \qquad A \in \text{Mat}(3, \mathbb{R}), \tag{1.1}$$

where $\text{Mat}(3, \mathbb{R})$ denotes the set of 3×3-real matrices, and proved that such surfaces are the planes, the spheres and the circular cylinders. On the other hand, C. Baikoussis and D. E. Blair ([3]) investigated the ruled surfaces in $\mathbb{R}^3$ satisfying the condition (1.1). C. Baikoussis and L. Verstraelen ([4, 5]) studied the helicoidal surfaces and the spiral surfaces in $\mathbb{R}^3$ satisfying the condition (1.1). Also, for the Lorentz version, S. M. Choi ([7, 8]) completely classified the surfaces of revolution and the ruled surfaces with non-null base

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curve satisfying the condition (1.1) in a Minkowski 3-space $\mathbb{R}^3_1$, and L. J. Alías, A. Ferrández, P. Lucas, and M. A. Meroño ([2]) also studied the ruled surfaces with null ruling. On the extension of [8] and [2], Y. H. Kim and D. W. Yoon ([13]) investigated ruled surfaces in a Minkowski m-space $\mathbb{R}^m_1$ such that $\Delta G = AG, A \in \mathrm{Mat}(N, \mathbb{R})$, $N = \binom{m}{2}$. On the other hand, D. W. Yoon ([15]) studied the translation surface in $\mathbb{R}^3_1$ satisfying the condition (1.1).

Following the condition (1.1), an interesting geometric question is raised, the classification of all surfaces of revolution without parabolic points in a Euclidean 3-space $\mathbb{R}^3$, which satisfy the condition

$$\Delta^h G = AG, \quad A \in \mathrm{Mat}(3, \mathbb{R}), \tag{1.2}$$

where Δ^h is the Laplace operator with respect to the second fundamental form h of the surface.

Throughout this paper, we assume that all objects are smooth and all surfaces are Riemannian, unless otherwise mentioned.

2. Preliminaries

Let M be a connected surface in a Euclidean 3-space $\mathbb{R}^3$. The map $G : M \to \mathbb{S}^2(1) \subset \mathbb{R}^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of a surface M, where $\mathbb{S}^2(1)$ denotes the unit sphere of $\mathbb{R}^3$.

Now, we define a surface of revolution in $\mathbb{R}^3$. A surface of revolution is formed by revolving a plane curve about a line in $\mathbb{R}^3$.

Let Π be a plane in $\mathbb{R}^3$, let l and C be a line and a point set of a plane curve which does not intersect l in Π, respectively. When C is revolved in $\mathbb{R}^3$ about l, the resulting point set M is called the *surface of revolution* generated by C. In this case, C is called the *profile curve* of M and the line l is called the *axis of revolution* of M. For convenience we choose Π to be the xz-plane and l to be the z-axis. We shall assume that the point set C has a parametrization $\gamma : I = (a, b) \to C$ defined by $u \mapsto (f(u), 0, g(u))$, which is differentiable. Without loss of generality, we can assume that $f(u)$ is a positive function, and g is a function on I. On the other hand, a subgroup of the rotation group which fixes the vector $(0, 0, 1)$ is generated by

$$\begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $v \in \mathbb{R}$. Hence the surface M of revolution can be written as

$$\begin{aligned} x(u, v) &= \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix} \\ &= (f(u) \cos v, f(u) \sin v, g(u)). \end{aligned} \tag{2.1}$$

Without loss of generality, we may assume that γ has the arc-length parametrization, i.e., it satisfies

$$(f'(u))^2 + (g'(u))^2 = 1. \tag{2.2}$$

Then, using the natural frame $\{x_u, x_v\}$ of M defined by

$$x_u = (f'(u)\cos v, f'(u)\sin v, g'(u))$$

and

$$x_v = (-f(u)\sin v, f(u)\cos v, 0),$$

the components of the first fundamental form of the surface are obtained as

$$g_{11} = \langle x_u, x_u\rangle = 1, \quad g_{12} = \langle x_u, x_v\rangle = 0, \quad g_{22} = \langle x_v, x_v\rangle = f^2(u).$$

Then, the Gauss map G is computed by $\frac{1}{f}(x_u \times x_v)$ or, equivalently,

$$G = (-g'(u)\cos v, -g'(u)\sin v, f'(u)). \tag{2.3}$$

Accordingly G can be regarded as a map of M into the 2-dimensional unit sphere $\mathbb{S}^2$. By using (2.2) the components of the second fundamental form h are

$$h_{11} = -f''(u)g'(u) + f'(u)g''(u), \quad h_{12} = 0, \quad h_{22} = f(u)g'(u). \tag{2.4}$$

Therefore, using the data described above, the mean curvature H is given by

$$H = \frac{1}{2}\left(\frac{g'}{f} - f''g' + f'g''\right). \tag{2.5}$$

If a surface M in $\mathbb{R}^3$ has no parabolic points, then we have $h_{11}h_{22} - h_{12}^2 \neq 0$. Thus, the second fundamental form h is regarded as a new (pseudo-)Riemannian metric.

Let $\{x_1, x_2\}$ be a local coordinate system of M. For the components $h_{ij}(i, j = 1, 2)$ of the second fundamental form h on M we denote by (h^{ij}) (resp. $\mathcal{H}$) the inverse matrix (resp. the determinant) of the matrix (h_{ij}). The Laplace operator Δ^h of the second fundamental form h on M is formally defined by

$$\Delta^h = -\frac{1}{\sqrt{|\mathcal{H}|}}\sum_{i,j=1}^{2}\frac{\partial}{\partial x^i}\Big(\sqrt{|\mathcal{H}|}\; h^{ij}\frac{\partial}{\partial x^j}\Big). \tag{2.6}$$

3. Main theorems

In this section, we will classify the surfaces of revolution in $\mathbb{R}^3$ satisfying the condition (1.2).

Theorem 3.1. *The only surfaces of revolution in a Euclidean 3-space whose Gauss map G satisfies*

$$\Delta^h G = AG, \quad A \in \mathrm{Mat}(3, \mathbb{R}) \tag{3.1}$$

are locally the catenoid and the sphere.

Proof. Let M be a surface of revolution in $\mathbb{R}^3$ defined by (2.1). We may assume that the profile curve γ is of unit speed; thus

$$(f'(u))^2 + (g'(u))^2 = 1. \tag{3.2}$$

We may put

$$f'(u) = \cos t, \quad g'(u) = \sin t \tag{3.3}$$

for the smooth function $t = t(u)$. Since the surface has no parabolic points, the functions $f(u)$, t' and $\sin t$ are non-vanishing everywhere. Furthermore, the mean curvature H given by (2.5) becomes

$$H = \frac{1}{2}\left(t' + \frac{\sin t}{f}\right). \tag{3.4}$$

By a straightforward computation, the Laplacian Δ^h of the second fundamental form h on M with the help of (2.4), (2.6) and (3.3) turns out to be

$$\Delta^h = -\frac{1}{t'}\frac{\partial^2}{\partial u^2} - \frac{1}{f\sin t}\frac{\partial^2}{\partial v^2} + \left(\frac{t''}{2t'^2} - \frac{\cos t}{2ft'} - \frac{\cos t}{2\sin t}\right)\frac{\partial}{\partial u}. \tag{3.5}$$

Accordingly, we get

$$\Delta^h G = \begin{pmatrix} \left(-t'\sin t - \frac{1}{2f} - \frac{\sin^2 t}{2f} + \frac{t''\cos t}{2t'} + \frac{t'\cos^2 t}{2\sin t}\right)\cos v \\ \left(-t'\sin t - \frac{1}{2f} - \frac{\sin^2 t}{2f} + \frac{t''\cos t}{2t'} + \frac{t'\cos^2 t}{2\sin t}\right)\sin v \\ \frac{3}{2}t'\cos t + \left(\frac{\cos t}{2f} + \frac{t''}{2t'}\right)\sin t \end{pmatrix}. \tag{3.6}$$

By the assumption (3.1) and the above equation we get the following system of differential equations:

$$\begin{cases} \left(-t'\sin t - \frac{1}{2f} - \frac{\sin^2 t}{2f} + \frac{t''\cos t}{2t'} + \frac{t'\cos^2 t}{2\sin t} + a_{11}\sin t\right)\cos v \\ \qquad +a_{12}\sin t\sin v - a_{13}\cos t = 0, \\ \left(-t'\sin t - \frac{1}{2f} - \frac{\sin^2 t}{2f} + \frac{t''\cos t}{2t'} + \frac{t'\cos^2 t}{2\sin t} + a_{22}\sin t\right)\sin v \\ \qquad +a_{21}\sin t\cos v - a_{23}\cos t = 0, \\ \frac{3}{2}t'\cos t + \left(\frac{\cos t}{2f} + \frac{t''}{2t'}\right)\sin t + a_{31}\sin t\cos v + a_{32}\sin t\sin v - a_{33}\cos t = 0, \end{cases} \tag{3.7}$$

where $a_{ij}(i,j = 1,2,3)$ denote the components of the matrix A given by (3.1).

In order to prove the theorem we have to solve the above system of ordinary differential equations. From (3.7) we easily deduce that $a_{12} = a_{21} = a_{13} = a_{23} = a_{31} = a_{32} = 0$ and $a_{11} = a_{22}$, i.e., the matrix A is diagonal. We put $a_{11} = a_{22} = \lambda$ and $a_{33} = \mu, \lambda, \mu \in \mathbb{R}$. Then, the system (3.7) reduces now to the following equations

$$-t'\sin t - \frac{1}{2f} - \frac{\sin^2 t}{2f} + \frac{t''\cos t}{2t'} + \frac{t'\cos^2 t}{2\sin t} = -\lambda \sin t, \tag{3.8}$$

$$\frac{3}{2}t'\cos t + \left(\frac{\cos t}{2f} + \frac{t''}{2t'}\right)\sin t = \mu \cos t. \tag{3.9}$$

We discuss five cases according to the constants λ and μ.

Case I. Let $\lambda = \mu = 0$.

If we multiply (3.8) by $\sin t$ and (3.9) by $-\cos t$, and add the resulting equations, we easily get

$$t' + \frac{\sin t}{f} = 0,$$

which implies the mean curvature H vanishes identically because of (3.4). Therefore, the surface is minimal, that is, it is a catenoid. Furthermore, a catenoid satisfies the condition (3.1).

Case II. Let $\lambda = \mu \neq 0$.

Following the same procedure as in Case I, we can obtain

$$t' = \lambda - \frac{\sin t}{f}. \tag{3.10}$$

Differentiating (3.10) with respect to u we have

$$t'' = -\frac{\cos t}{f^2}(\lambda f - 2\sin t). \tag{3.11}$$

Substituting (3.10) and (3.11) in (3.9) we get

$$\lambda^2 f^2 - 4\lambda \sin t f + 4\sin^2 t = 0, \tag{3.12}$$

from which,

$$f(u) = \frac{2}{\lambda}\sin t. \tag{3.13}$$

Furthermore, (3.13) together with the equation (3.10) becomes $t' = \frac{\lambda}{2}$, that is,

$$t(u) = \frac{\lambda}{2}u + k, \quad k \in \mathbb{R}. \tag{3.14}$$

On the other hand, by (3.3) and (3.14) we have

$$g'(u) = \sin\left(\frac{\lambda}{2}u + k\right),$$

from which,

$$g(u) = -\frac{2}{\lambda}\cos\left(\frac{\lambda}{2}u + k\right) + c, \quad c \in \mathbb{R}. \tag{3.15}$$

Consequently, from (3.13)-(3.15) we get

$$\langle x(u,v)-\mathbf{C}, x(u,v)-\mathbf{C}\rangle = f(u)^2+(g(u)-c)^2 = \frac{4}{\lambda^2} > 0, \quad \mathbf{C} = (0,0,c), \tag{3.16}$$

which means that the surface M is contained in the sphere $\mathbb{S}^2$ centered at $\mathbf{C}$ with radius $\frac{2}{|\lambda|}$. Also, a sphere satisfies the condition (3.1).

Case III. Let $\lambda \neq 0, \mu = 0$.

In this case (3.8) and (3.9) are given respectively by

$$-t' \sin t - \frac{1}{2f} - \frac{\sin^2 t}{2f} + \frac{t'' \cos t}{2t'} + \frac{t' \cos^2 t}{2 \sin t} = -\lambda \sin t, \tag{3.17}$$

$$\frac{3}{2} t' \cos t + \left(\frac{\cos t}{2f} + \frac{t''}{2t'}\right) \sin t = 0. \tag{3.18}$$

From this system of ODEs we have

$$t' = -\frac{\sin t}{f} + \lambda \sin^2 t, \tag{3.19}$$

and thus

$$t'' = -\frac{1}{f^2}(t' \cos t f - \sin t \cos t) + 2\lambda t' \sin t \cos t. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18) we get

$$5\lambda^2 \sin^2 t f^2 - 8\lambda \sin t f + 4 = 0. \tag{3.21}$$

Differentiating the above equation gives

$$5\lambda \sin t f - 4 = 0. \tag{3.22}$$

If we take the differentiation of the equation once again, we get

$$\lambda \cos t \sin^2 t f = 0.$$

Since f is a positive function and $\lambda \neq 0$, $\cos t \sin^2 t = 0$ for every t, which implies M is a part of a plane whose points are parabolic. Thus, there are no surfaces of revolution satisfying this case.

Case IV. Let $\lambda = 0, \mu \neq 0$.

The system of equations (3.8) and (3.9), in this case, takes the form

$$-t' \sin t - \frac{1}{2f} - \frac{\sin^2 t}{2f} + \frac{t'' \cos t}{2t'} + \frac{t' \cos^2 t}{2 \sin t} = 0, \tag{3.23}$$

$$\frac{3}{2} t' \cos t + \left(\frac{\cos t}{2f} + \frac{t''}{2t'}\right) \sin t = \mu \cos t. \tag{3.24}$$

Applying the same algebraic method as above, we also obtain

$$\begin{aligned} t' &= -\frac{\sin t}{f} + \mu\cos^2 t, \\ t'' &= -\frac{1}{f^2}(t'\cos t f - \sin t\cos t) - 2\mu t'\sin t\cos t. \end{aligned} \tag{3.25}$$

Furthermore, by (3.24) and (3.25) we get

$$\alpha_1 f^2 + \alpha_2 f + \alpha_3 = 0, \tag{3.26}$$

where we put

$$\begin{aligned} \alpha_1 &= 5\mu^2\sin^4 t - 6\mu^2\sin^2 t + \mu^2, \\ \alpha_2 &= 8\mu\sin^3 t - 4\mu\sin t, \\ \alpha_3 &= 4\sin^2 t. \end{aligned}$$

Differentiating the equation (3.26) and using (3.25) we find

$$\beta_1 f^2 + \beta_2 f + \beta_3 = 0, \tag{3.27}$$

where

$$\begin{aligned} \beta_1 &= \mu^2(-10\sin^8 t + 8\sin^6 t + 12\sin^4 t - 8\sin^2 t - 2), \\ \beta_2 &= \mu(-40\sin^7 t + 56\sin^5 t - 24\sin^3 t + 8\sin^2 t), \\ \beta_3 &= -40\sin^6 t + 48\sin^4 t - 8\sin^2 t. \end{aligned}$$

Combining (3.26) and (3.27) we show that

$$(\alpha_2\beta_1 - \alpha_1\beta_2)f + \alpha_3\beta_1 - \alpha_1\beta_3 = 0. \tag{3.28}$$

Differentiating once again this equation and using the same algebraic techniques above we find the following trigonometric polynomial in $\sin t$ satisfying

$$\sum_{i=1}^{12} c_i\mu^2\sin^{2i-1} t = 0, \tag{3.29}$$

where $c_i (i = 1, 2, \ldots, 12)$ denote the coefficients as non-zero constant of the function $\sin^{2i-1} t$. Since this polynomial is equal to zero for every t, all its coefficients must be zero. Thus, we have $\mu = 0$. So we get a contradiction and therefore, in this case there are no surfaces of revolution.

Case V. Let $\lambda \neq 0, \mu \neq 0$ and $\lambda \neq \mu$.
From (3.8) and (3.9) we have:

$$t' = \lambda\sin^2 t + \mu\cos^2 t - \frac{\sin t}{f}, \tag{3.30}$$

from which, the equation (3.9) is written as

$$\phi_1 f^2 + \phi_2 f + \phi_3 = 0, \tag{3.31}$$

where

$$\begin{aligned}\phi_1 &= 5\alpha^2 \sin^4 t + 6\mu\alpha \sin^2 t + \mu,\\ \phi_2 &= -8\alpha \sin^3 t - 4\mu \sin t,\\ \phi_3 &= 4 \sin^2 t,\\ \alpha &= \lambda - \mu.\end{aligned}$$

Differentiating the equation (3.31) and using (3.30), we find

$$\delta_1 f^2 + \delta_2 f + \delta_3 = 0, \tag{3.32}$$

where

$$\begin{aligned}\delta_1 &= -10\alpha^4 \sin^8 t - 8\alpha^3\mu \sin^6 t + (2\alpha^2\mu^2 + 10\alpha^2\mu) \sin^4 t\\ &\quad + (-4\alpha\mu^3 + 12\alpha\mu^2) \sin^2 t + 2\mu^3 - 4\mu^4,\\ \delta_2 &= 40\alpha^3 \sin^7 t + 56\alpha^2\mu \sin^5 t + 24\alpha\mu^2 \sin^3 t + 8\mu^3 \sin t,\\ \delta_3 &= -8 \sin^2 t.\end{aligned}$$

Combining (3.31) and (3.32), we have

$$(\phi_2\delta_1 - \delta_2\phi_1)f + \phi_3\delta_1 - \delta_3\phi_1 = 0.$$

Hence, by this procedure the equation (3.31) is reduced to a linear one with respect to the function f. Therefore if we repeat this method one more time, we can find the following polynomial:

$$\begin{aligned}&192000\alpha^{14} \sin^{32} t + \sum_{i=1}^{13} \alpha^i p_i(\lambda, \mu) \sin^{2i+4} t\\ &+ (1024\mu^{13} - 2560\mu^{12} + 1536\mu^{11} + 512\mu^{10} - 512\mu^9) \sin^4 t = 0,\end{aligned} \tag{3.33}$$

where $p_i(\lambda, \mu)(i = 1, \ldots, 13)$ are the known polynomials in λ and μ. Since this polynomial is equal to zero for every t, all its coefficients must be zero. Therefore, we conclude that $\alpha = 0$, equivalently $\lambda = \mu$ and $\mu = 0$ or $\mu = 1$ or $\mu = -\frac{1}{2}$, which is a contradiction. Consequently, there are no surfaces of revolution in this case. This completes the proof. □

Combining the results of [9] and our Theorem 3.1, we have

Theorem 3.2 (Characterization)**.** *Let M be a surface of revolution without parabolic points in a Euclidean 3-space $\mathbb{R}^3$. Then for some non-singular matrices $A, A_h \in \mathrm{Mat}(3, \mathbb{R})$ the following are equivalent*:

(1) $\Delta G = AG$.
(2) $\Delta^h G = A_h G$.
(3) M is an open part of a sphere.

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