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EXTENSIONS OF GENERALIZED STABLE RINGS

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Abstract. In this paper, we investigate the extensions of generalized stable rings. It is shown that a ring R is a generalized stable ring if and only if R has a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents such that $e_1Re_1, \ldots, e_nRe_n$ are generalized stable rings. Also, we prove that a ring R is a generalized stable ring if and only if $R[[X]]$ is a generalized stable ring if and only if $T(R, M)$ is a generalized stable ring.

1. Introduction

Let R be an associative ring with identity. A ring R is said to have stable range one provided that $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by \in U(R)$, where $U(R)$ is the set of all units in R. This definition is left-right symmetric. Moreover, we know that a right R-module M can be cancelled from direct sums if and only if $\mathrm{End}_R(M)$ has stable range one. Many authors have studied stable range one conditions and its generalizations from different viewpoints such as [1, 3–12, 14].

As a generalization of rings with stable range one, in [9], Chen introduced the generalized stable rings. A ring R is called a generalized stable ring provided that $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by \in K(R)$, where $K(R) = \{x \in R \mid \text{there exist } s, t \in R \text{ such that } sxt = 1\}$. Obviously, if a ring R has stable range one, then it is a generalized stable ring, but the converse is not true, see [9, Example 3]. Thus generalized stable rings are nontrivial generalizations of rings with stable range one. For more details and examples of generalized stable rings, see [5, 6, 9].

In this paper, we investigate the extensions of generalized stable rings. It is shown that a ring R is a generalized stable ring if and only if R has a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents such that $e_1Re_1, \ldots, e_nRe_n$ are generalized stable rings. Also, we prove that a ring R is a generalized stable ring if and only if $R[[X]]$ is a generalized stable ring if and only if $T(R, M)$ is a generalized stable ring.

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2. Main results

The following result appeared in [9, Theorem 9].

Lemma 2.1. *Let A be a right R-module, $E = \mathrm{End}_R(A)$ be its endomorphism ring. Then the following are equivalent*:

(1) *E is a generalized stable ring.*
(2) *Given any decompositions $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A \cong A_2$, there exist $C, D, E \leq M$ such that $M = C \oplus D \oplus B_1 = C \oplus E \oplus B_2$ with $C \cong A$.*

Lemma 2.2. *Let $M_1, M_2, \ldots, M_n$ be right R-modules. If $\mathrm{End}_R(M_1), \mathrm{End}_R(M_2), \ldots, \mathrm{End}_R(M_n)$ are generalized stable rings, then so is $\mathrm{End}_R(M_1 \oplus M_2 \oplus \cdots \oplus M_n)$.*

Proof. Given right R-module decompositions $M = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n \cong A_2$, then we have $A_1 = A_{11} \oplus A_{12} \oplus \cdots \oplus A_{1n}$ and $A_2 = A_{21} \oplus A_{22} \oplus \cdots \oplus A_{2n}$ with $A_{1i} \cong M_i \cong A_{2i}$ for $1 \leq i \leq n$. So $M = A_{11} \oplus A_{12} \oplus \cdots \oplus A_{1n} \oplus B = A_{21} \oplus A_{22} \oplus \cdots \oplus A_{2n} \oplus C$. Since $\mathrm{End}_R(M_1)$ is a generalized stable ring and $A_{11} \cong M_1 \cong A_{21}$, by Lemma 2.1, we can find submodules $C_1, D_1, E_1 \leq M$ such that $M = C_1 \oplus D_1 \oplus A_{12} \oplus \cdots \oplus A_{1n} \oplus B = C_1 \oplus E_1 \oplus A_{22} \oplus \cdots \oplus A_{2n} \oplus C$ with $C_1 \cong M_1$. Likewise, we have submodules $C_2, \ldots, C_n, D_2, \ldots, D_n, E_2, \ldots, E_n$ such that $M = C_1 \oplus \cdots \oplus C_n \oplus D_1 \oplus \cdots \oplus D_n \oplus B = C_1 \oplus \cdots \oplus C_n \oplus E_1 \oplus \cdots \oplus E_n \oplus C$ with $C_i \cong M_i$ for $1 \leq i \leq n$. Clearly, $C_1 \oplus \cdots \oplus C_n \cong M_1 \oplus \cdots \oplus M_n$. By Lemma 2.1, we conclude that $\mathrm{End}_R(M_1 \oplus M_2 \oplus \cdots \oplus M_n)$ is a generalized stable ring. □

A finite orthogonal set of idempotents $e_1, \ldots, e_n$ in a ring R is said to be complete in case $e_1 + \cdots + e_n = 1 \in R$.

Theorem 2.3. *The following conditions are equivalent*:

(1) *R is a generalized stable ring.*
(2) *There exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents such that $e_1Re_1, \ldots, e_nRe_n$ are generalized stable rings.*

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (1) It follows by $e_iRe_i \cong \mathrm{End}_R(e_iR)$ and Lemma 2.2 that $\mathrm{End}_R(e_1R \oplus \cdots \oplus e_nR)$ is a generalized stable ring. One easily checks that $R = e_1R \oplus \cdots \oplus e_nR$. Hence $R \cong \mathrm{End}_R(R) = \mathrm{End}_R(e_1R \oplus \cdots \oplus e_nR)$ is a generalized stable ring. □

Let $e_1, \ldots, e_n$ be idempotents of a ring R. Clearly, with the usual matrix operation, $\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix} = \left\{ \begin{pmatrix} e_1r_{11}e_1 & \cdots & e_1r_{1n}e_n \\ \vdots & \ddots & \vdots \\ e_nr_{n1}e_1 & \cdots & e_nr_{nn}e_n \end{pmatrix} \middle| r_{ij} \in R (1 \leq i, j \leq n) \right\}$ forms a ring with the identity $\mathrm{diag}(e_1, \ldots, e_n)$.

Proposition 2.4. *Let $e_1, \ldots, e_n$ be idempotents of a ring R. If $e_1Re_1, \ldots, e_nRe_n$ are all generalized stable rings, then so is the ring*

$$\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}.$$

Proof. Set

$$T = \begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}.$$

Choose $f_1 = \mathrm{diag}(e_1, 0, \ldots, 0)$, $f_2 = \mathrm{diag}(0, e_2, \ldots, 0), \ldots, f_n = \mathrm{diag}(0, 0, \ldots, e_n) \in T$. Then we have a complete orthogonal set $\{f_1, f_2, \ldots, f_n\}$ of idempotents such that $f_iTf_i \cong e_iRe_i$ are generalized stable rings. By Theorem 2.3, T is a generalized stable ring. □

Let $M_n(R)$ denotes the n by n matrix ring over R, from Proposition 2.4, we have the following result in [9].

Corollary 2.5. *If R is a generalized stable ring, then so is $M_n(R)$ for all positive integer n.*

A Morita context denoted by (A, B, M, N, ψ, ϕ) consists of two rings A, B, two bimodules ${}_AN_B$, ${}_BM_A$ and a pair of bimodule homomorphisms (called pairings) $\psi : N \otimes_B M \to A$ and $\phi : M \otimes_A N \to B$ which satisfy the following associativity: $\psi(n \otimes m)n' = n\phi(m \otimes n')$, $\phi(m \otimes n)m' = m\psi(n \otimes m')$ for any $m, m' \in M, n, n' \in N$. These conditions ensure that the set T of generalized matrices $\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right)$, $a \in A, b \in B, m \in M, n \in N$ forms a ring, called the ring of the Morita context. Particularly, if $N = 0$, then the Morita context is the formal triangular matrix ring.

Corollary 2.6. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ). If A and B are generalized stable rings, then so is T.*

Proof. Set $e_1 = \left(\begin{smallmatrix} 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right), e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_B \end{smallmatrix}\right)$. Then $\{e_1, e_2\}$ is a complete orthogonal set of idempotents such that $e_1Te_1 \cong A$, $e_2Te_2 \cong B$ are generalized stable rings. Then we obtain the result by Theorem 2.3. □

Corollary 2.7. *Let $T = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$ be the formal triangular matrix ring. If A and B are generalized stable rings, then so is T.*

Theorem 2.8. *Let $e_1, \ldots, e_n$ be idempotents of a ring R. If $e_1Re_1, \ldots, e_nRe_n$ are all generalized stable rings, then so is the ring*

$$\begin{pmatrix} e_1Re_1 & 0 & \cdots & 0 \\ e_2Re_1 & e_2Re_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_nRe_1 & e_nRe_2 & \cdots & e_nRe_n \end{pmatrix}.$$

Proof. Clearly, the result holds for $n = 1$. Assume now that the result holds for $n = k \geqslant 1$. Let $n = k + 1$. Set

$$B = \begin{pmatrix} e_2Re_2 & 0 & \cdots & 0 \\ e_3Re_2 & e_3Re_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{k+1}Re_2 & e_{k+1}Re_3 & \cdots & e_{k+1}Re_{k+1} \end{pmatrix}, \quad M = \begin{pmatrix} e_2Re_1 \\ e_3Re_1 \\ \vdots \\ e_{k+1}Re_1 \end{pmatrix}.$$

Then $T = \left(\begin{smallmatrix} e_1Re_1 & 0 \\ M & B \end{smallmatrix}\right)$. By hypothesis, B is a generalized stable ring. Let $f_1 = \mathrm{diag}(e_1, 0), f_2 = \mathrm{diag}(0, \mathrm{diag}(e_2, \ldots, e_{k+1}))$. Then $\{f_1, f_2\}$ is a complete orthogonal set of idempotents such that $f_1Tf_1 \cong e_1Re_1, f_2Tf_2 \cong B$ are generalized stable rings. Then we obtain the result by Theorem 2.3. $\square$

Corollary 2.9. *If R is a generalized stable ring, then so is the ring of $n \times n$ lower triangular matrices over R.*

Analogously, we deduce that if R is a generalized stable ring, then so is the ring of $n \times n$ upper triangular matrices over R.

In [9, Proposition 5], the author proved that a ring R is a generalized stable ring if and only if $R/J(R)$ is a generalized stable ring. Use the same method in [9, Proposition 5], we can prove the following result. We omit its proofs.

Lemma 2.10. *Let R be a ring, I an ideal of R such that $I \subseteq J(R)$. The following conditions are equivalent*:

(1) *R is a generalized stable ring.*

(2) *R/I is a generalized stable ring.*

Let $R[[X]]$ denotes the power series ring over R. From Lemma 2.10, we have the following result.

Theorem 2.11. *A ring R is a generalized stable ring if and only if $R[[X]]$ is a generalized stable ring.*

Proof. We construct a map $\phi : R[[X]] \to R$ such that $\phi(\Sigma_{i=0}^{\infty} a_i x^i) \to a_0$ for any $\Sigma_{i=0}^{\infty} a_i x^i \in R[[X]]$. It is easy to verify that $\mathrm{Ker}\phi = \{\Sigma_{i=0}^{\infty} a_i x^i | a_0 = 0\}$. Since for any $\Sigma_{i=0}^{\infty} a_i x^i \in \mathrm{Ker}\phi$ and $y \in R[[X]]$, $1 - y(\Sigma_{i=0}^{\infty} a_i x^i) = 1 + c_1x + c_2x^2 + \cdots$, so $1 - y(\Sigma_{i=0}^{\infty} a_i x^i) \in U(R[[X]])$. Hence, we have $\Sigma_{i=0}^{\infty} a_i x^i \in J(R[[X]])$, so $\mathrm{Ker}\phi \subseteq J(R[[X]])$. Since $R \cong R[[X]]/\mathrm{Ker}\phi$, we obtain the result by Lemma 2.10. $\square$

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(a_1, x_1)(a_2, x_2) = (a_1a_2, a_1x_2 + x_1a_2).$$

This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Theorem 2.12. *A ring R is a generalized stable ring if and only if $T(R, M)$ is a generalized stable ring.*

Proof. From [13], we have $J(T(R,M)) = T(J(R),M)$. We construct a map $\phi : R \longrightarrow T(R,M)/J(T(R,M))$ such that $\phi(x) = \overline{(x,0)}$ for any $x \in R$. Suppose that $\overline{(x,m)} \in T(R,M)/J(T(R,M))$, then $\overline{(x,m)} = \overline{(x,0)} + \overline{(0,m)} = \phi(x)$. Hence ϕ is a epimorphism and $\mathrm{Ker}\phi = \{x \in R | (x,m) \in J(T(R,M))\} = J(R)$. So $T(R,M)/J(T(R,M)) \cong R/J(R)$. So we complete the proof by Lemma 2.10. □

Now we consider the following subring of the triangular matrix ring. Let R be a ring and let

$$R_n = \left\{ \left. \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \right| a, a_{ij} \in R \right\}$$

be a subset of the triangular matrix ring. Then R_n is a ring with addition point-wise and usual matrix multiplication.

Proposition 2.13. *A ring R is a generalized stable ring if and only if R_n is a generalized stable ring.*

Proof. It is clear that

$$J(R_n) = \left\{ \left. \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \right| a \in J(R), a_{ij} \in R \right\}.$$

Since $R_n/J(R_n) \cong R/J(R)$, we obtain the result by Lemma 2.10. □

Let R be a ring and let

$$T_n(R) = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \right| a_i \in R (1 \le i \le n) \right\}$$

be a subset of the triangular matrix ring. Then $T_n(R)$ is a ring with addition point-wise and usual matrix multiplication.

Proposition 2.14. *A ring R is a generalized stable ring if and only if $T_n(R)$ is a generalized stable ring.*

Proof. It is clear that

$$J(T_n(R)) = \left\{ \left. \begin{pmatrix} a & a_2 & a_3 & \cdots & a_n \\ 0 & a & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \right| a \in J(R), a_i \in R (2 \leq i \leq n) \right\}.$$

Since $T_n(R)/J(T_n(R)) \cong R/J(R)$, we obtain the result by Lemma 2.10. □

Corollary 2.15. *Let R be a ring and n any positive integer. Then R is a generalized stable ring if and only if $R[x]/(x^n)$ is a generalized stable ring, where (x^n) is the ideal generated by x^n.*

Proof. Observe that $T_n(R) \cong R[x]/(x^n)$ for any positive integer n. Thus the result follows from Proposition 2.14. □

Recall from [2], an idempotent $e \in R$ is left (resp. right) semicentral in R if $ere = re$ (resp. $ere = er$) for all $r \in R$. It is well known that if a ring R has (weakly) stable range one, then eRe has (weakly) stable range one for any $e^2 = e \in R$. But we don't know the result whether or not true for generalized stable rings. For semicentral idempotents, we have the following result.

Theorem 2.16. *Let R be a generalized stable ring. Then eRe is also a generalized stable ring for any left (right) semicentral idempotent $e \in R$.*

Proof. Given $ax + b = e$ with $a, x, b \in eRe$, then we have $(a + 1 - e)(x + 1 - e) + b = 1$. Since R is a generalized stable ring, there exists $z \in R$ such that $a + 1 - e + bz = u \in K(R)$. Assume $sut = 1$ for some $s, t \in R$, that is, $s(a+1-e+bz)t = 1$, thus $es(a+1-e+bz)te = e$. Since e is a left semicentral idempotent, we have $(1 - e)te = 0$, hence $es(a + bz)te = e$. In addition, we know that $a, b \in eRe$, thus $ese(a+b(eze))ete = e$, that is, $a+b(eze) \in K(eRe)$, as desired. □

Similarly, we can prove eRe is also a generalized stable ring for any right semicentral idempotent $e \in R$.

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