Bull. Korean Math. Soc. **46** (2009), No. 6, pp. 1069–1077
DOI 10.4134/BKMS.2009.46.6.1069

ON THE DECOMPOSITION OF EXTENDING LIFTING MODULES

Chaehoon Chang and Jongmoon Shin

Abstract. In 1984, Oshiro [11] has studied the decomposition of continuous lifting modules. He obtained the following: every continuous lifting module has an indecomposable decomposition. In this paper, we study extending lifting modules. We show that every extending lifting module has an indecomposable decomposition. This result is an expansion of Oshiro's result mentioned above. And we consider some application of this result.

1. Introduction

From 1958 to 1959, Matlis and Papp studied injective modules over right noetherian rings and they showed the following result: a ring R is right noetherian if and only if every injective R-module has an indecomposable decomposition. As an improved version of this result, in 1982, the following was shown by Müller-Rizvi: a ring R is right noetherian if and only if every continuous R-module has an indecomposable decomposition. Furthermore, in 1984, Okado showed the following result: a ring R is right noetherian if and only if every extending R-module has an indecomposable decomposition (cf. [8]).

On the other hand, in 1972, the result of projective modules over right perfect rings have an indecomposable decomposition was shown by Anderson-Fuller. In 1983, Oshiro showed the following result: every quasi-discrete module has an indecomposable decomposition. In addition, recently Kuratomi-Chang proved that lifting modules over right perfect rings have an indecomposable decomposition. Recently Chang showed that if every co-closed submodule of any projective module P contains $\mathrm{Rad}(P)$, then every X-lifting module over a right perfect ring has an indecomposable decomposition (cf. [1, 3, 7, 12]).

Also, quasi-injective modules, continuous modules, and projective modules over perfect rings have the exchange property. And, in 1993, Mohamed-Müller showed that continuous modules have the exchange property and, for nonsingular quasi-continuous modules, the finite exchange property implies the exchange

Received August 28, 2008.
2000 *Mathematics Subject Classification.* Primary 16D40, 16D70, 16L30.
Key words and phrases. local summand, extending module, lifting module, (internal) exchange property, local endomorphism ring.

property. Dually, in 1996, Oshiro-Rizvi proved that discrete modules have the exchange property and, for quasi-discrete modules, the finite exchange property implies the exchange property (cf. [1, 4, 8, 9]).

2. Preliminaries

Throughout this article all rings are associative and R will always denote a ring with unity. Modules are unital right R-modules unless indicated otherwise.

Let M be a module and let K be a submodule of M. K is called an *essential* submodule of M (or M is an *essential* extension of K) if $K \cap L \neq 0$ for any non-zero submodule L of M. In this case we denote $K \subseteq_e M$. Dually, a submodule K of M is called a *small* submodule (or *superfluous* submodule) of M, abbreviated $K \ll M$, in the case when, for every submodule $L \subseteq M$, $K + L = M$ implies $L = M$.

Let $N_1 \subseteq N_2 \subseteq M$. N_1 is a *co-essential* (or *cosmall*) submodule of N_2 in M, abbreviated $N_1 \subseteq_c N_2$ in M, if $N_2/N_1 \ll M/N_1$. A submodule N of M is said to be *co-closed* in M (or a *co-closed* submodule of M), if N has no proper co-essential submodule in M, i.e., $N' \subseteq_c N$ in M implies $N = N'$. Let $N_1 \subseteq N_2 \subseteq M$. N_1 is said to be a *co-closure* of N_2 in M if N_1 is a co-closed submodule of M with $N_1 \subseteq_c N_2$ in M.

A module M is said to be *extending* (or *CS*) if, any submodule A of M, there exists a direct summand A^* of M such that $A \subseteq_e A^*$ in M. Dually, a module M is said to be *lifting* if, any submodule A of M, there exists a direct summand A^* of M such that $A^* \subseteq_c A$ in M. The module M is called *continuous* if M is extending and satisfies the following condition:

(C_2) If a submodule X of M is isomorphic to a direct summand of M, then X is a direct summand of M.

The module M is called *quasi-continuous* if M is extending and satisfies the following condition:

(C_3) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

Let M be a module and let N and L be submodules of M. N is called a *supplement* of L if $M = N + L$ and $N \cap L \ll N$. Note that any supplement submodule (hence any direct summand) of a module M is co-closed in M. Following [4], a module M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A such that $P \subseteq B$. A module M is *supplemented* if every submodule of M has a supplement.

For a module M, we use $\mathrm{End}_R(M)$, $K <_\oplus M$, and $\mathrm{Rad}(M)$ to denote the endomorphism ring, direct summand, and Jacobson radical of M, respectively.

For a module M and an index set I, we denote by $M^{(I)}$ the direct sum of I copies of M.

For undefined terms, the reader is referred to [1, 4, 8].

Remark 2.1 (cf. [4, 8]). It is well-known that the following implications hold for a module:

(a) injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ extending.

(b) projective $\Rightarrow$ quasi-projective $\not\Rightarrow$ discrete $\Rightarrow$ quasi-discrete $\Rightarrow$ lifting $\Rightarrow$ amply supplemented $\Rightarrow$ supplemented.

Lemma 2.2 (cf. [4, 20.22])**.** *Every factor module of a* (*amply*) *supplemented module is* (*amply*) *supplemented.*

$\sum \oplus_{\lambda\in\Lambda} X_\lambda \subseteq X$ is called a *local summand* of X if $\sum \oplus_{\lambda\in F} X_\lambda <_\oplus X$ for every finite subset $F \subseteq \Lambda$.

The following lemma is useful.

Lemma 2.3 (cf. [11, Lemma 2.4])**.** *If every local summand of M is a direct summand, then M has an indecomposable decomposition.*

Lemma 2.4. *Let M be an amply supplemented module and let $f : M \to N$ be an isomorphism. If K is co-closed in M, then $f(K)$ is co-closed in N.*

Proof. Since M is amply supplemented, there exists a supplement submodule L of K in M. Since $N = f(M)$ is amply supplemented, there is a co-closure T of $f(K)$ in N. Then there exists a submodule $K^{'}$ of K such that $f(K^{'}) = T$. This implies that $N = f(M) = f(L) + f(K) = f(L) + T = f(L) + f(K^{'})$. Thus $M = L + K^{'} + \ker f$. By [7, Lemma 1.5], $K^{'} \subseteq_c K$ in M. Since K is co-closed in M, $K = K^{'}$. Hence $f(K) = f(K^{'}) = T$ is co-closed in N. □

A module M is said to have the (*finite*) *exchange property* if, for any (finite) index set I, whenever $M \oplus N = \oplus_I A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_I B_i)$ for submodules $B_i \subseteq A_i$. A module M has the (*finite*) *internal exchange property* if, for any (finite) direct sum decomposition $M = \oplus_I M_i$ and any direct summand X of M, there exist submodules $\overline{M_i} \subseteq M_i$ such that $M = X \oplus (\oplus_I \overline{M_i})$. A decomposition $M = \oplus_A M_\alpha$ of a module M as a direct sum of non-zero submodules $(M_\alpha)_{\alpha\in A}$ is said to *complement direct summands* if, for every direct summand K of M, there is a subset $B \subseteq A$ with $M = (\oplus_B M_\beta) \oplus K$.

Lemma 2.5. *Let M be a module and let $M = \oplus_\Lambda M_\lambda$ be an indecomposable decomposition. Then the following are equivalent*:

(i) *The decomposition $M = \oplus_\Lambda M_\lambda$ complement direct summands*;

(ii) *$M = \oplus_\Lambda M_\lambda$ has the internal exchange property.*

Proof. (ii) $\Longrightarrow$ (i) is obvious. (i) $\Longrightarrow$ (ii) Let $M = \oplus_I N_i$ and let X be a direct summand of M. For $N_i <_\oplus M$, by assumption, there is a direct sum decomposition $M = N_i \oplus (\oplus_{\Lambda'} M_\lambda)$, where $\Lambda^{'} \subseteq \Lambda$. Thus $N_i \simeq \oplus_{\Lambda-\Lambda'} M_\lambda$. Put $\Lambda_i = \Lambda - \Lambda^{'}$. Then $M = \oplus_I N_i \overset{f}{\simeq} \oplus_I(\oplus_{\Lambda_i} M_\alpha)$. Hence $M = \oplus_I(\oplus_{\Lambda_i} f^{-1}(M_\alpha))$, where $\oplus_{\Lambda_i} f^{-1}(M_\alpha) = N_i$. By [1, Corollary 12.5], $M = \oplus_I(\oplus_{\Lambda_i} f^{-1}(M_\alpha))$ complement direct summands. Therefore $M = X \oplus [\oplus_I(\oplus_{\Lambda_i{}'} f^{-1}(M_\alpha))]$, where $\Lambda_i{}^{'} \subseteq \Lambda_i$. Put $\overline{N_i} = \oplus_{\Lambda_i{}'} f^{-1}(M_\alpha)$. Then $M = X \oplus (\oplus_I \overline{N_i})$. Hence M has the internal exchange property. □

Proposition 2.6 (cf. [4, Corollary 11.14]). *Let M be a module. Then the following are equivalent*:

(i) *M has the finite exchange property*;

(ii) *For any finite subset $\{f_1, \ldots, f_n\} \subseteq \mathrm{End}_R(M)$ with $\sum_{i=1}^{n} f_i = 1$, there exist orthogonal idempotents $e_1, \ldots, e_n \in \mathrm{End}_R(M)$ with $e_i \in f_i\mathrm{End}_R(M)$, for each i and $\sum_{i=1}^{n} e_i = 1$*;

(iii) *For any $f \in \mathrm{End}_R(M)$, there exists an idempotent $e \in f\mathrm{End}_R(M)$ with $1 - e \in (1 - f)\mathrm{End}_R(M)$.*

Let A and B be modules. A is said to be *generalized B-injective* (or *B-ojective*) if, for any submodule X of B and any homomorphism $f : X \to A$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : B_1 \to A_1$ and a monomorphism $h_2 : A_2 \to B_2$, and for $x = b_1 + b_2$ and $f(x) = a_1 + a_2$ one has $a_1 = h_1(b_1)$ and $b_2 = h_2(a_2)$. As the dual notion of generalized relative injective modules, A is said to be *generalized B-projective* (or *B-dual ojective*) if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$, and an epimorphism $h_2 : B_2 \to A_2$ such that $g \circ h_1 = f\,|_{A_1}$ and $f \circ h_2 = g\,|_{B_2}$ (cf. [4] or [6]).

Proposition 2.7 (cf. [4, 4.42]). *Let $M = A \oplus B$. Then A is generalized B-projective if and only if whenever $M = X + B$, we have $M = X^* \oplus A^* \oplus B^* = X^* + B$ with $X^* \subseteq X$, $A^* \subseteq A$ and $B^* \subseteq B$.*

Proposition 2.8 (cf. [4, 4.43]). *Let B^* be a direct summand of B. If A is generalized B-projective, then A is generalized B^*-projective.*

We recall that a non-zero module U is called *uniform* if it is indecomposable extending. Dually, a module H is called *hollow* if it is indecomposable lifting.

Lemma 2.9 (cf. [6, Theorem 3.7], [10, Corollary 21]). *Let H be a hollow module and let U be a uniform module. Then the following hold*:

(i) *$H \oplus H$ is lifting with the (finite) internal exchange property if and only if H is generalized H-projective*;

(ii) *$U \oplus U$ is extending with the (finite) internal exchange property if and only if U is generalized U-injective.*

3. Results

In 1984, Oshiro [11, Lemma 2.5] showed the following: every continuous lifting module has an indecomposable decomposition. On the other hand, in 2007, Kuratomi-Chang [7, Theorem 3.10] obtained the following:

(a) every lifting module over a right perfect ring has an indecomposable decomposition.

(b) for a lifting module over a right perfect ring, the finite exchange property implies the exchange property.

In this section, we consider the following problems:

Problem 1. Does any extending lifting module have an indecomposable decomposition?

Problem 2. Does any extending lifting module have the exchange property?

Since every quasi-discrete module has an indecomposable decomposition, every discrete module has an indecomposable decomposition. Now we give a direct proof of this result.

Proposition 3.1. *Every discrete module can be expressed as a direct sum of indecomposable modules.*

Proof. Let M be a discrete module and let $N = \sum \oplus_I M_i$ be a local summand of M. Since M is lifting, there exists a direct sum decomposition $M = A \oplus B$ such that $A \subseteq_c N$ in M. Then $N = A \oplus (N \cap B)$ and $N \cap B \ll M$. As $N/(N \cap B) \simeq A$ and M is discrete, $N \cap B <_\oplus M$. Hence $N \cap B = 0$, as required. □

We show the following theorem.

Theorem 3.2. *Every local summand of an extending lifting module is a direct summand.*

Proof. Let M be an extending lifting module and let $N = \sum \oplus_I M_i$ be a local summand of M. Since M is lifting, there exists a direct sum decomposition $M = A \oplus B$ such that $A \subseteq_c N$ in M. Then we see

$$N = A \oplus (N \cap B), \quad N \cap B \ll M. \tag{3.1}$$

Assume $0 \neq x \in N \cap B$. Then there exists a finite subset K of I such that $x \in \sum \oplus_K M_k$. By hypothesis, $\sum \oplus_K M_k$ is a direct summand of M. Since M is extending, $\sum \oplus_K M_k$ is extending. Hence there exists a direct sum decomposition $\sum \oplus_K M_k = T \oplus T^*$ such that $xR \subseteq_e T$ in $\sum \oplus_K M_k$. Since $xR \cap A = 0$, we see $T \cap A = 0$. Thus $N \cap B$ contains a submodule C isomorphic to T. Since $C \subseteq N \cap B \ll M$, $C \ll M$. As $C \simeq T$ and T is co-closed in M, by Lemma 2.4, C is co-closed in M. Since M is amply supplemented, $C <_\oplus M$, which contradicts to $C \ll M$. Therefore $N \cap B = 0$, as required. □

By Lemma 2.3 and Theorem 3.2, we obtain the first main theorem.

Theorem 3.3. *Every extending lifting module has an indecomposable decomposition.*

Example 3.4. Let D be a division ring. Consider the upper triangular matrix ring

$$R = \begin{pmatrix} D & D & \dots & D \\ 0 & D & \dots & D \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & D \end{pmatrix}$$

of $n \times n$ matrices over D. Then R is artinian and R can be expressed as $R = e_{11}R \oplus \cdots \oplus e_{nn}R$, where $\{e_{ii}\}_{i=1}^{n}$ is a complete set of orthogonal primitive idempotents of R and

$$e_{11} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}, \dots, e_{nn} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Furthermore,

$$J(R) = \begin{pmatrix} 0 & D & \dots & D \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & D \\ 0 & \dots & 0 & 0 \end{pmatrix},\ e_{ii}J(R) = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & \\ 0 & & D & \dots & D \\ \vdots & & & & \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

and each $e_{ii}J(R)$ has a unique composition series. Hence R is a generalized uniserial ring. Let M be an extending (or lifting) module. First we assume that M is extending. Then M is lifting by [4, 29.7]. Hence M has an indecomposable decomposition. Next we assume that M is lifting. Since R is generalized uniserial, R is right perfect. Then there exists a direct sum decomposition $M = \oplus_I M_i$, where M_i is hollow by [7, Theorem 3.4]. Furthermore, M is extending.

Recall that a module H is called *local* if it is hollow and $\mathrm{Rad}(H)$ is small in H.

By Remark 2.1(b) and Theorem 3.3, we get the following two corollaries:

Corollary 3.5. *Every finitely generated extending lifting module can be expressed as a direct sum of local modules.*

Corollary 3.6 (cf. [11, Lemma 2.5])**.** *Every* (*quasi-*)*continuous lifting module has an indecomposable decomposition.*

Lemma 3.7 (cf. [13, Proposition 1])**.** *Let H be a uniform* (*or hollow*) *module. If $H \oplus H$ has the internal exchange property, then H has a local endomorphism ring.*

Proof. First assume that H is a uniform module. Suppose that $\mathrm{End}_R(H)$ is not local. Then there exist non-units $f, g \in \mathrm{End}_R(H)$ such that $1 = f - g$. Let $\pi_i : M = H_1 \oplus H_2 \to H_i$ be a projection, where $H_i = H$, $(i = 1, 2)$. Define a map $(f, g) : H \to H_1 \oplus H_2$ by $x \rightsquigarrow (f(x), g(x))$. Then (f, g) is an R-homomorphism. Put $\mathrm{Im}\ f = H^{'}$. Define a map $(1_H, 1_H) : H \to H_1 \oplus H_2$ by $x \rightsquigarrow (x, x)$. Put $\mathrm{Im}\ (1_H, 1_H) = K$. Then $M = H^{'} \oplus K$. Since M has the internal exchange property, we see

$$M = H^{'} \oplus \overline{H_1} \oplus \overline{H_2},\ \ \overline{H_i} \subseteq H_i,\ \ (i = 1, 2). \tag{3.2}$$

Since H_i is uniform, either $M = H^{'} \oplus H_1$ or $M = H^{'} \oplus H_2$. If $M = H^{'} \oplus H_1 = H_2 \oplus H_1$, then $H^{'} \overset{\pi_2|H^{'}}{\simeq} H_2$. Moreover, $(f, g) : H \to H^{'}$ is an isomorphism. Hence $H \overset{(f,g)}{\simeq} H^{'} \overset{\pi_2|H^{'}}{\simeq} H_2$. Therefore g is an automorphism, which contradicts to g is not-unit. If $M = H^{'} \oplus H_2$, then by the same argument, f is an automorphism, which contradicts to f is not-unit. Thus H has a local endomorphism ring.

Next assume that H is a hollow module. By the same argument as above, we see that H has a local endomorphism ring. □

By [8, Theorem 2.25], Theorems 3.2, 3.3, and Lemma 3.7, we obtain the following corollary.

Corollary 3.8. *Let M be an extending lifting module. Then M has the exchange property if and only if every uniform hollow summand of M has a local endomorphism ring.*

Proof. Assume that M has the exchange property. Let H be a unform hollow submodule of M such that $H <_{\oplus} M$. Then $H \oplus H$ has the exchange property. By Lemma 3.7, H has a local endomorphism ring. Conversely, assume that every uniform hollow summand of M has a local endomorphism ring. By hypothesis and Theorem 3.3, there exists a direct sum decomposition $M = \oplus_I M_i$, where each M_i has a local endomorphism ring. By Theorem 3.2 and [8, Theorem 2.25], M has the exchange property. □

By Corollary 3.8, Lemmas 2.5, 2.9, and 3.7, we obtain the second main theorem.

Theorem 3.9. *Let M be an extending lifting module and let $M = \oplus_I M_i$ be a decomposition with each M_i is a uniform hollow module satisfying one of the following*:

(i) *M is generalized M-projective and $M \oplus M$ is amply supplemented*;
(ii) *M is generalized M-injective*;
(iii) *M has the finite exchange property*;
(iv) *$M \oplus M$ has the finite internal exchange property*;
(v) *The decomposition $M = \oplus_I M_i$ complement direct summands.*

Then M has the exchange property.

Proof. (ii), (iv), and (v) follow from Lemmas 2.5, 2.9, 3.7, and Corollary 3.8. We may show only (i) and (iii).

(i) Let $M = M_i \oplus (\oplus_{I-\{i\}} M_i)$. Now we put $N = M_i \oplus (\oplus_I M_i)$. First we show that M_i is generalized $\oplus_I M_i$-projective.

Assume that $N = X + (\oplus_I M_i)$. By Lemma 2.2, N is amply supplemented. Thus $\oplus_I M_i$ has a supplement Y in N with $Y \subseteq X$. Then we see

$$M \oplus M = [(\oplus_{I-\{i\}} M_i) \oplus Y] + (\oplus_I M_i). \tag{3.3}$$

It is easy to see that $[(\oplus_{I-\{i\}}M_i)+Y]\cap(\oplus_I M_i)=Y\cap(\oplus_I M_i)$. Therefore $(\oplus_{I-\{i\}}M_i)\oplus Y$ is a supplement of $\oplus_I M_i$. Since M is generalized M-projective and $(\oplus_{I-\{i\}}M_i)\oplus Y$ is a supplement of $\oplus_I M_i$, by Proposition 2.7, we have

$$\begin{aligned} M\oplus M &= [(\oplus_{I-\{i\}}M_i)\oplus Y]\oplus \overline{M}\oplus \overline{\overline{M}} \\ &= [(\oplus_{I-\{i\}}M_i)\oplus Y]+(\oplus_I M_i),\ \ \overline{M},\overline{\overline{M}}\subseteq M. \end{aligned} \tag{3.4}$$

Consider the projection $\pi_{M_i}:\oplus_I M_i\to M_i$. Then $(\oplus_{I-\{i\}}M_i)\oplus\overline{M}=(\oplus_{I-\{i\}}M_i)\oplus\pi_{M_i}(\overline{M})$. This implies that

$$M\oplus M=(\oplus_{I-\{i\}}M_i)\oplus(Y\oplus\pi_{M_i}(\overline{M})\oplus\overline{\overline{M}}). \tag{3.5}$$

Thus $N=Y\oplus\pi_{M_i}(\overline{M})\oplus\overline{\overline{M}}$ with $Y\subseteq X$, $\pi_{M_i}(\overline{M})\subseteq\overline{M}$ and $\overline{\overline{M}}\subseteq M$. Hence M_i is generalized $\oplus_I M_i$-projective. By Proposition 2.8, M_i is generalized M_i-projective. By Lemmas 2.9(i), 3.7, and Corollary 3.8, M has the exchange property.

(iii) By assumption, each M_i has the finite exchange property. Assume that $\mathrm{End}_R(M_i)$ is not local. Then there exists an $f\in\mathrm{End}_R(M_i)$ such that both f and $1-f$ are non-isomorphisms of M_i. Since M_i has the finite exchange property, by Proposition 2.6, for any $f\in\mathrm{End}_R(M_i)$, there is an idempotent $g\in\mathrm{End}_R(M_i)$ such that $g\in f\mathrm{End}_R(M_i)$ and $1-g\in(1-f)\mathrm{End}_R(M_i)$. As g is either 0_{M_i} or 1_{M_i}, f or $1-f$ is right invertible. Hence f or $1-f$ is a unit. This is a contradiction. □

The following corollary is a direct consequence of Theorem 3.9.

Corollary 3.10 (cf. [4]). *Let M be a quasi-continuous lifting module. Suppose that M has the finite exchange property. Then M has the exchange property.*

Recently Er has studied infinite direct sums of lifting modules. He showed the following:

Let M be a continuous module such that $M^{(\mathbb{N})}$ is a lifting module. Then M is a direct sum of local modules and for any index set I, $M^{(I)}$ is lifting with the exchange property (cf. [5, Proposition 4]).

Motivated by Er's result mentioned above, we consider the following problem:

Problem 3. Let M be an extending module such that $M^{(\mathbb{N})}$ is a lifting module. When does the lifting property on $M^{(\mathbb{N})}$ imply the same on $M^{(I)}$ for arbitrary index set I? Moreover, does M have an indecomposable decomposition such that each indecomposable summand is local?

We show the following proposition.

Proposition 3.11. *Let M be an extending module having the internal exchange property such that $M^{(\mathbb{N})}$ is a lifting module. Then M is a direct sum of local modules and for any index set I, $M^{(I)}$ is a lifting module with the exchange property.*

Proof. By assumption and Theorem 3.3, there exists an indecomposable decomposition $M = \oplus_I M_i$, where each M_i is uniform hollow. Since M has the internal exchange property, each M_i has a local endomorphism ring by Lemma 3.7. Furthermore, M_i is local by [2, Lemma 4]. Hence M is a direct sum of local modules. By the proof of [5, Proposition 4], $M^{(I)}$ is a lifting module with the exchange property. □

References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1992.

[2] Y. Baba and M. Harada, *On almost M-projectives and almost M-injectives*, Tsukuba J. Math. **14** (1990), no. 1, 53–69.

[3] C. Chang, *X-lifting modules over right perfect rings*, Bull. Korean Math. Soc. **45** (2008), no. 1, 59–66.

[4] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules*, Supplements and projectivity in module theory. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[5] N. Er, *Infinite direct sums of lifting modules*, Comm. Algebra **34** (2006), no. 5, 1909–1920.

[6] Y. Kuratomi, *On direct sums of lifting modules and internal exchange property*, Comm. Algebra **33** (2005), no. 6, 1795–1804.

[7] Y. Kuratomi and C. Chang, *Lifting modules over right perfect rings*, Comm. Algebra **35** (2007), no. 10, 3103–3109.

[8] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, Cambridge University Press, Cambridge, 1990.

[9] ______, *The exchange property for quasi-continuous modules*, Ring theory (Granville, OH, 1992), 242–247, World Sci. Publ., River Edge, NJ, 1993.

[10] ______, *Ojective modules*, Comm. Algebra **30** (2002), no. 4, 1817–1827.

[11] K. Oshiro, *Lifting modules, extending modules and their applications to QF-rings*, Hokkaido Math. J. **13** (1984), no. 3, 310–338.

[12] ______, *Semiperfect modules and quasisemiperfect modules*, Osaka J. Math. **20** (1983), no. 2, 337–372.

[13] R. B. Warfield, *A Krull-Schmidt theorem for infinite sums of modules*, Proc. Amer. Math. Soc. **22** (1969), 460–465.

Chaehoon Chang
Information Technology Manpower Development Program
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yamaguchi21@hanmail.net

Jongmoon Shin
Department of Mathematics Education
Dongguk University
Gyeongju 780-714, Korea
E-mail address: shinjm@dongguk.ac.kr