

ON SUMMATION THEOREMS FOR THE ${}_3F_2(1)$ SERIES

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ABSTRACT. The intimate relation between the 3- j coefficient in Quantum Theory of Angular Momentum (QTAM) and the ${}_3F_2(1)$ hypergeometric series is exploited to derive new summation theorems, from formulas for the 3- j coefficient.

1. Introduction

In quantum mechanics, the algebra associated with angular momentum, is ubiquitous and is extensively used in atomic, molecular and nuclear physics studies. The powerful tools developed over decades, from the inception of quantum mechanics, by E. P. Wigner (1931) and G. Racah (1942-1943), have become an integral part of text books such as Rose [8], Edmonds [4] and Biedenharn and Louck [2]. The first connection between a Racah coefficient (or, angular momentum re-coupling coefficient) and the generalized hypergeometric function of unit argument, ${}_4F_3(1)$, is given in the Appendix of “Multipole Fields” by Rose [7]. “Angular Momentum in Quantum Mechanics” by Edmonds [4], along with “Elementary Theory of Angular Momentum” by Rose [8], became the first two text books in this area. One of the most comprehensive review articles about Quantum Theory of Angular Momentum (QTAM) is by Smorodinskii and Shepein [10], which reveals the many facets of QTAM.

The origin of the connection between the generalized hypergeometric series [9] of unit argument and finite groups can be traced to the article entitled, *Group theoretical basis for the terminating ${}_3F_2(1)$ series* by Srinivasa Rao et. al [12]. It has been shown [11] that from QTAM it is possible to derive new results [14] in the theory of summations and transformations of generalized hypergeometric functions of one and more variables.

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The mathematical genius Srinivasa Ramanujan [5] discovered¹, after coming to know of the definition of the Gauss hypergeometric series from Carr's [3] *Synopsis*, not only all that was known in Europe on hypergeometric series at that time, but also several new theorems, and in particular theorems on products of hypergeometric series [9], as well as several asymptotic expansions. In recent times, it has been shown [13] that new results on summation and transformations of generalized hypergeometric series in one or more variables can be derived, opening up new vistas for research. To cite an example, a recursive use of a Whipple transformation [15] for a terminating ${}_3F_2(1)$ series results in a 72-element group associated with the 18 terminating series [12], thus initiating a study of the group theory of transformations of hypergeometric functions.

The results presented here are a natural consequence of:

- (i) the intimate relation between angular momentum coupling coefficient (the Clebsch-Gordan coefficient or the 3- j coefficient) and the generalized hypergeometric function of unit argument, ${}_3F_2(1)$ and
- (ii) the special values of angular momentum coefficients presented in tabular form in an Appendix in [4].

2. The 3- j coefficient related to the ${}_3F_2(1)$

The first summation theorem in QTAM is due to Vander Monde for ${}_2F_1(a, -n; c; 1)$, which preceded the celebrated Gauss summation theorem for the ${}_2F_1(a, b; c; 1)$ and the Kummer summation theorem for ${}_2F_1(a, b; 1 + a - b; -1)$. A list of the summation theorems for ordinary hypergeometric series is given in Appendix III of [9]. In this article, new summation theorems and, as a consequence, new summation theorems for hypergeometric functions are derived from the special formulas for the 3- j coefficient.

The relation between the Clebsch-Gordan, or the 3- j angular momentum coupling coefficient, and the set of six ${}_3F_2(1)$ hypergeometric functions (c.f. [11]) is:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \delta(m_1 + m_2 + m_3, 0) \prod_{i,k=1}^3 [R_{ik}!/(J+1)!]^{1/2} \\ &\quad \times (-1)^{\sigma(pqr)} [\Gamma(1-A, 1-B, 1-C, D, E)]^{-1} \\ (1) \quad &\quad \times {}_3F_2(A, B, C; D, E; 1), \end{aligned}$$

where

$$A = -R_{2p}, \quad B = -R_{3q}, \quad C = -R_{1r}, \quad D = 1 + R_{3r} - R_{2p}, \quad E = 1 + R_{2r} - R_{3q}$$

¹It is not possible to assert when these were discovered by Ramanujan, since the 3254 Entries in the Notebooks of Ramanujan [1] are not dated.

and

$$\Gamma(x, y, \dots) = \Gamma(x)\Gamma(y)\cdots,$$

for all permutations of $(pqr) = (123)$, with

$$\sigma(pqr) = \begin{cases} R_{3p} - R_{2q}, & \text{for even permutations,} \\ R_{3p} - R_{2q} + J, & \text{for odd permutations,} \end{cases}$$

where $J = j_1 + j_2 + j_3$. Note that Eq. (1) is valid for $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$ and $-j_i \leq m_i \leq j_i$, for $i = 1, 2, 3$.

The defining relations for the numerator and denominator parameters, R_{ik} 's, are the elements of the 3×3 square symbol in Regge [6]:

$$(2) \quad \|R_{ik}\| = \begin{vmatrix} -j_1 + j_2 + j_3 & j_1 - j_2 + j_3 & j_1 + j_2 - j_3 \\ j_1 - m_1 & j_2 - m_2 & j_3 - m_3 \\ j_1 + m_1 & j_2 + m_2 & j_3 + m_3 \end{vmatrix}.$$

3. Summation theorems

A special formula for the 3- j coefficient, Eq. (3.7.11) in [4], is:

$$(3) \quad \begin{pmatrix} j_1 & j_2 & j_1 + j_2 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} = (-1)^{j_1 - j_2 + m_1 + m_2} \times \left[\frac{(2j_1)!(2j_2)!(j_1 + j_2 + m_1 + m_2)!(j_1 + j_2 - m_1 - m_2)!}{(2j_1 + 2j_2 + 1)!(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!} \right]^{1/2}.$$

The above relation is the one which when rearranged into the form of a ${}_3F_2(1)$, yields the following new summation theorem, the first Master formula, for the ${}_3F_2(1)$:

$$(4) \quad {}_3F_2 \left[\begin{matrix} -(j_1 + m_1 + j_2 + m_2), & -(j_1 + m_1), & -2j_1 \\ 1 - j_1 - m_1, & 1 + j_2 - j_1 - m_1 - m_2 \end{matrix} ; 1 \right] = (-1)^{j_1 + m_1} \times \frac{\Gamma(1 + j_1 + j_2 + m_1 + m_2, 1 + 2j_1, 1 - j_1 - m_1, 1 + j_2 - j_1 - m_1 - m_2)}{\Gamma(1 + j_1 - m_1, 1 + j_2 - m_2, 1 + j_2 + m_2)}.$$

which is valid for $-j_i \leq m_i \leq j_i$, for $i = 1$ and 2 .

Another special relation for the 3- j coefficient, corresponding to the Eq. (3.7.10) of [4] is:

$$(5) \quad \begin{pmatrix} j_1 & j_2 & j_3 \\ j_1 & -j_1 - m_3 & m_3 \end{pmatrix} = (-1)^{-j_1 + j_2 + m_3} \times \left[\frac{(2j_1)!(-j_1 + j_2 + j_3)!(j_1 + j_2 + m_3)!(j_3 - m_3)!}{(j_1 + j_2 + j_3 + 1)!(j_1 - j_2 + j_3)!(j_1 + j_2 - j_3)!(-j_1 + j_2 - m_3)!(j_3 + m_3)!} \right]^{1/2}.$$

When this is rearranged into the form of a hypergeometric function, the resultant summation formula, the second Master formula, reads:

$$\begin{aligned}
 & {}_3F_2 \left[\begin{matrix} m_3 - j_3, -2j_1, j_2 - j_3 - j_1 \\ 1 + j_2 - j_3 - j_1, 1 + j_2 - j_1 + m_3 \end{matrix} ; 1 \right] \\
 (6) \quad & = (-1)^{j_2 - j_3 - j_1} \\
 & \quad \times \frac{\Gamma(1 + j_3 - m_3, 1 + 2j_1, 1 + j_2 - j_3 - j_1, 1 + j_2 - j_1 + m_3)}{\Gamma(1 + j_3 + m_3, 1 + j_1 + j_2 - j_3, 1 + j_2 - j_1 - m_3)}.
 \end{aligned}$$

which is valid for $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$ and $\max(-j_3, -j_1 - j_2) \leq m_3 \leq \min(j_3, j_2 - j_1)$.

For the ‘parity’ Clebsch-Gordan (3- j) coefficient, the projection quantum numbers are zero, i.e., $m_1 = m_2 = m_3 = 0$. From the recurrence relation Eq. (3.7.13), given in [4], for the 3- j coefficient, it has been shown that the formula for parity 3- j coefficient is, for j_1, j_2, j_3 integers and $J = j_1 + j_2 + j_3$, an even integer:

$$\begin{aligned}
 & \begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} \\
 (7) \quad & = (-1)^{\frac{J}{2}} \left[\frac{(J - 2j_1)!(J - 2j_2)!(J - 2j_3)!}{(J + 1)!} \right]^{1/2} \\
 & \quad \times \frac{(J/2)!}{(J/2 - j_1)!(J/2 - j_2)!(J/2 - j_3)!}.
 \end{aligned}$$

Similarly, when this is rearranged into the form of a hypergeometric function, the resultant summation formula, the third Master formula, reads:

$$\begin{aligned}
 & {}_3F_2 \left[\begin{matrix} -j_1, -j_2, j_3 - j_1 - j_2 \\ 1 + j_3 - j_1, 1 + j_3 - j_2 \end{matrix} ; 1 \right] \\
 (8a) \quad & = (-1)^{J/2} \times \frac{\Gamma(1 + J/2, 1 + j_1 + j_2 - j_3, 1 + j_3 - j_1, 1 + j_3 - j_2)}{\Gamma(1 + j_3, 1 + J/2 - j_1, 1 + J/2 - j_2, 1 + J/2 - j_3)},
 \end{aligned}$$

where j_1, j_2, j_3 are positive integers and $J = j_1 + j_2 + j_3$, is even and $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$.

For $a = j_3 - j_1 - j_2, b = -j_2, c = -j_1$, Eq. (8a) becomes

$$\begin{aligned}
 & {}_3F_2 \left[\begin{matrix} a, b, c \\ 1 + a - b, 1 + a - c \end{matrix} ; 1 \right] \\
 (8b) \quad & = \frac{\Gamma(1 + \frac{a}{2} - b - c, 1 - a, 1 + a - b, 1 + a - c)}{\Gamma(1 + a - b - c, 1 + \frac{a}{2} - b, 1 + \frac{a}{2} - c, 1 - \frac{a}{2})},
 \end{aligned}$$

where a, b, c are negative integers and a is even and $|b - c| + b + c \leq a \leq 0$.

A minor noteworthy point occurs when $a = -2, b = -5, c = -3$, which when substituted in Eq. (8b), gives directly the value $-7/4$ for both the LHS and RHS. However, for the same values of a, b, c , Dixon’s theorem yields the desired result only after one resorts to a limiting process. Thus, Eq. (8a) and

(8b) are both special cases of Dixon’s theorem. What is novel is that we have derived a special case of Dixon’s theorem from an entirely different approach — from the ${}_3F_2(1)$ form of (7) for the parity Clebsch-Gordan $(3 - j)$ coefficient.

The first entry in Appendix 2, Table 2 of [4] corresponds to the case of the parity $3 - j$ coefficient discussed above. In addition, there are eighteen, independent, special formulas for the $3 - j$ coefficient in Table 2 of [4]. In this article, it is shown that by substituting, in Eq. (5), special values for j_1, j_2, j_3, m_3 , four of the tabulated results of Edmonds follow. These then are rearranged to obtain four summation theorems for the ${}_3F_2(1)$.

The four substitutions to be made, for the parameters, in Eq. (5) for the $3 - j$ coefficient and also in the second Master formula, Eq. (6), to get the corresponding summation formulas for the ${}_3F_2(1)$, are given in the table below:

j_1	j_2	j_3	m_3	j_1	j_2	j_3	m_3
$3/2$	$J + 1/2$	J	$-M - 3/2$	2	J	$J + 1$	M
1	J	J	M	2	J	J	M

These four substitutions yield four $3 - j$ formulas given in Appendix 2, Table 2 of [4]. The key for the identification of a formula with an entry in Edmonds is with the help of the $3 - j$ coefficient only.

Likewise, eight summation theorems can be derived by suitable substitutions in Eq. (3), to obtain the formulas given in Appendix 2, Table 2 of [4] and in turn, these substitutions in Eq. (4) give rise to eight different summation theorems for the ${}_3F_2(1)$ s.

Similarly, the substitutions to be made, for the parameters, in the Eq. (3) for the $3 - j$ coefficient and in the first Master formula, Eq. (4), to get the corresponding eight summation formula for the ${}_3F_2(1)$, are given in the table below:

j_1	m_1	j_2	m_2	j_1	m_1	j_2	m_2
J	$-M - 1/2$	$1/2$	$1/2$	J	$-M$	1	0
J	$-M - 3/2$	$3/2$	$3/2$	J	$-M - 2$	2	2
J	$-M - 1/2$	$3/2$	$1/2$	J	$-M - 1$	2	1
J	$-M - 1$	1	1	J	$-M$	2	0

Besides these twelve formulas, there are six more formulas given in Appendix 2, Table 2 of [4]. These are independent of the three Master formulas, (4), (6), and (8a), cited above. Listed below are the six formulas for the $3 - j$ coefficient:

$$\begin{aligned}
 (9) \quad & \begin{pmatrix} J + \frac{1}{2} & J & \frac{3}{2} \\ M & -M - \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
 & = (-1)^{J-M-\frac{1}{2}} \left(J + 3M + \frac{3}{2} \right) \times \left[\frac{(J - M + \frac{1}{2})}{(2J + 3)(2J + 2)(2J + 1)2J} \right]^{1/2},
 \end{aligned}$$

$$(10) \quad \begin{pmatrix} J & J & 1 \\ M & -M & 0 \end{pmatrix} = (-1)^{J-M} \left[\frac{M}{[(2J+1)(J+1)J]^{1/2}} \right],$$

$$(11) \quad \begin{pmatrix} J+1 & J & 2 \\ M & -M-1 & 1 \end{pmatrix} \\ = (-1)^{J-M+1} \times \left[\frac{4(J-M+1)(J-M)(J+2M+2)^2}{(2J+4)(2J+3)(2J+2)(2J+1)2J} \right]^{1/2},$$

$$(12) \quad \begin{pmatrix} J+1 & J & 2 \\ M & -M & 0 \end{pmatrix} \\ = (-1)^{J-M+1} 2M \times \left[\frac{6(J+M+1)(J-M+1)}{(2J+4)(2J+3)(2J+2)(2J+1)2J} \right]^{1/2},$$

$$(13) \quad \begin{pmatrix} J & J & 2 \\ M & -M-1 & 1 \end{pmatrix} \\ = (-1)^{J-M} (1+2M) \times \left[\frac{6(J+M+1)(J-M)}{(2J+3)(2J+2)(2J+1)(2J)(2J-1)} \right]^{1/2},$$

$$(14) \quad \begin{pmatrix} J & J & 2 \\ M & -M & 0 \end{pmatrix} \\ = (-1)^{J-M} \frac{2[3M^2 - J(J+1)]}{[(2J+3)(2J+2)(2J+1)(2J)(2J-1)]^{1/2}}.$$

Corresponding to the six $3-j$ formulas are the following six ${}_3F_2(1)$ summation theorems:

$$(15) \quad {}_3F_2 \left[\begin{matrix} M-J-\frac{1}{2}, M-J+\frac{1}{2}, 1-2J \\ M-J+\frac{5}{2}, M-J+\frac{5}{2} \end{matrix} ; 1 \right] \\ = (-1)^{J-M-\frac{1}{2}} \\ \times \frac{[J+3M+\frac{3}{2}] \Gamma(J-M+\frac{3}{2}, 2J, M-J+\frac{5}{2}, M-J+\frac{5}{2})}{2 \Gamma(M+J+\frac{3}{2})},$$

$$(16) \quad {}_3F_2 \left[\begin{matrix} M-J, M-J, 1-2J \\ M-J+2, M-J+2 \end{matrix} ; 1 \right] \\ = (-1)^{J-M} \times \frac{[2M] \Gamma(2J, M-J+2, M-J+2, J-M+1)}{\Gamma(M+J+1)},$$

$$\begin{aligned}
 (17) \quad & {}_3F_2 \left[\begin{matrix} M - J - 1, M - J + 1, 1 - 2J \\ M - J + 3, M - J + 3 \end{matrix} ; 1 \right] \\
 &= (-1)^{J - M + 1} \\
 &\quad \times \frac{[J + 2M + 2] \Gamma(2J, M - J + 3, M - J + 3, J - M + 2)}{3 \Gamma(M + J + 2)},
 \end{aligned}$$

$$\begin{aligned}
 (18) \quad & {}_3F_2 \left[\begin{matrix} M - J - 1, M - J, 1 - 2J \\ M - J + 2, M - J + 3 \end{matrix} ; 1 \right] \\
 &= (-1)^{J - M} \times \frac{[M] \Gamma(2J, M - J + 2, M - J + 3, J - M + 2)}{\Gamma(M + J + 1)},
 \end{aligned}$$

$$\begin{aligned}
 (19) \quad & {}_3F_2 \left[\begin{matrix} M - J, M - J + 1, 2 - 2J \\ M - J + 4, M - J + 3 \end{matrix} ; 1 \right] \\
 &= (-1)^{J - M + 1} \\
 &\quad \times \frac{[1 + 2M] \Gamma(2J - 1, M - J + 4, M - J + 3, J - M + 1)}{2 \Gamma(M + J + 1)},
 \end{aligned}$$

$$\begin{aligned}
 (20) \quad & {}_3F_2 \left[\begin{matrix} M - J, M - J, 2 - 2J \\ M - J + 3, M - J + 3 \end{matrix} ; 1 \right] \\
 &= (-1)^{J - M} [3M^2 - J(J + 1)] \\
 &\quad \times \frac{\Gamma(2J - 1, M - J + 3, M - J + 3, J - M + 1)}{2 \Gamma(M + J + 1)}.
 \end{aligned}$$

4. Concluding remarks

In fine, it is shown that there are terminating summation theorems for the ${}_3F_2(1)$ s which are a direct consequence of the special formulas for the Clebsch-Gordan or 3- j coefficient. The restrictions that pertain to any 3- j coefficient formula is to be taken note of and the same applies to the corresponding summation formula derived from it. Needless to say, the nine independent summation formulas – including the two Master formulas, Eq. (4) and Eq. (6) – and the twelve derived from these Master formulas are subject to the quantum mechanical nature (i.e., for half-integral and integral values) of angular momentum.

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