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AN IDEAL-BASED ZERO-DIVISOR GRAPH OF 2-PRIMAL NEAR-RINGS

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Abstract. In this paper, we give topological properties of collection of prime ideals in 2-primal near-rings. We show that Spec(N), the spectrum of prime ideals, is a compact space, and Max(N), the maximal ideals of N, forms a compact T_1-subspace. We also study the zero-divisor graph $\Gamma_I(R)$ with respect to the completely semiprime ideal I of N. We show that $\Gamma_{\mathbb{P}}(R)$, where $\mathbb{P}$ is a prime radical of N, is a connected graph with diameter less than or equal to 3. We characterize all cycles in the graph $\Gamma_{\mathbb{P}}(R)$.

1. Preliminaries

In [3], Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. In [2], Anderson and Livingston associated a graph (simple) $\Gamma(R)$ to a commutative ring R with identity with vertices $Z(R)^* = Z(R)\backslash\{0\}$, the set of nonzero zero-divisor of R, and for distinct $x, y \in Z(R)^*$, the vertices x, and y are adjacent if and only if $xy = 0$. They investigated the interplay between the ring-theoretic properties of R and the graph-theoretics properties of $\Gamma(R)$.

In [9], Redmond has generalized the notion of the zero-divisor graph. For a given ideal I of R, he defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R\backslash I : xy \in I$ for some $y \in R\backslash I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

In this paper, we study the undirected graph $\Gamma_I(N)$ of near-rings for any completely semiprime ideal I of N. We extend the results obtained by K. Samei [11] for reduced rings to 2-primal near-rings. Clearly, reduced rings are 2-primal near-rings.

Let N be a near-ring with identity. Let J be a completely semiprime ideal of N. The zero-divisor graph of N with respect to the ideal J, denoted by $\Gamma_J(N)$, is the graph whose vertices are the set $\{x \in N\backslash J : xy \in J$ for some $y \in N\backslash J\}$ with distinct vertices x and y are adjacent if and only if $xy \in J$. If $J = 0$, then

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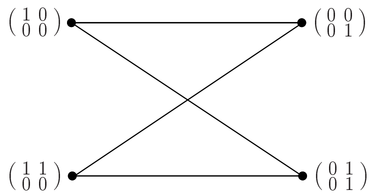
$\Gamma_J(N) = \Gamma(N)$, and J is a non-zero completely prime ideal of N if and only if $\Gamma_J(N) = \phi$.

Example 1.1. Let $N = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where $F = \{0, 1\}$ is the field under addition and multiplication modulo 2. Then its prime radical $P = \left\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right\}$ is a completely reflexive ideal of the near-ring N and its ideal based zero-divisor graph $\widehat{\Gamma_P(N)}$ is:



Remark 1.2. In the above example, N is a 2-primal near-ring, but neither reduced nor commutative.

Throughout this paper N is a zero symmetric near-ring with identity unless otherwise stated, and its prime radical is not a prime ideal of N.

Let $\mathbb{P}$ denote the prime radical, and let $N(N)$ denote the set of nilpotent elements of N. For any vertices x, y in a graph G, if x and y are adjacent, we denote it as $x \approx y$. A near-ring N is called a 2-primal if $\mathbb{P} = N(N)$. A near-ring N is said to be reduced if $N(N) = 0$. Clearly, reduced near-rings are 2-primal, but the converse need not be true (See Example 1.3 of [5]). A near-ring N is called pm if each prime ideal in N is contained in a unique maximal ideal of N.

We use $\text{Spec}(N)$, $\text{Max}(N)$, and $\text{Min}(N)$ for the spectrum of prime ideals, maximal ideal and minimal prime ideals of N, respectively.

For any ideal J of N and $a \in N$, we define $V(a) = \{P \in \text{Spec}(N) \ : \ a \in P\}$ and $D(J) = \text{Spec}(N) \backslash V(J)$. Let $V(J) = \cap_{a \in J} V(a)$. Then $F = \{V(J) \ : \ J$ is an ideal of $N\}$ is closed under finite union and arbitrary intersections, so that there is a topology on $\text{Spec}(N)$ for which F is the family of closed sets. This is called the Zariski topology. Note that $V(A) = (\langle J \rangle)$ for any subset A of N. Let $\mathcal{B} = \{D(a) \ : \ a \in N\}$. Then $\mathcal{B}$ is a basis for a topology on $\text{Spec}(N)$.

The operations *cl* and *int* denote the closure and the interior in $\text{Spec}(N)$. We also set $V^{'}(a) = V(a) \cap \text{Min}(N)$; $D^{'}(a) = D(a) \cap \text{Min}(N)$.

For any subset S of N, we define $\mathbb{P}_S = \{n \in N : nS \subseteq \mathbb{P}\}$. We set $\text{Supp}(a) = \cap_{x \in \mathbb{P}_a} V(x)$.

For distinct vertices x and y of $\Gamma_{\mathbb{P}}(N)$, let $d(x, \ y)$ be the length of the shortest path from x to y. The diameter of a connected graph is the supremum of the distances between vertices. The associated number $e(a)$ for a vertex a in $\Gamma_{\mathbb{P}}(R)$ is defined by $e(a) = \max\{d(a, \ b) \ : \ a \neq b\}$.

A graph G is called triangulated (hyper-triangulated) if each vertex (edge) of G is a vertex (edge) of a triangle.

A point P of $\mathrm{Spec}(N)$ is said to be quasi-isolated if P is a minimal prime ideal and P is not contained in the union of all minimal prime ideals of N different from P.

If a and b are the two vertices in $\Gamma_{\mathbb{P}}(N)$, by $c(a,\ b)$ we mean the length of the smallest cycle containing a and b. For every two vertices a and b, all possible cases for $c(a,\ b)$ are given in Theorem 3.9. In this paper the notations of graph theory are from [4], the notations of near-ring are from [8], and the notations of topology are from [6] and [7].

2. Topological space of Spec(N)

In this section, we associate the near-ring properties of N and the topological properties of $\mathrm{Spec}(N)$. We start this section with the following useful lemma.

Lemma 2.1. *Let N be a near-ring. If A is a subset of* $\mathrm{Spec}(N)$, *then there exists an ideal $J = \cap A$ of N with $cl(A) = V(J)$. In particular, if A is a closed subset of* $\mathrm{Spec}(N)$, *then $A = V(J)$ for some ideal J of N.*

Proof. Let $P_1 \in V(J)$ and let $D(x)$ be any arbitrary element in $\mathcal{B}$ such that $P_1 \in D(x)$. Suppose that $D(x) \cap A = \phi$. Then $x \in J$, and so $P_1 \in V(x)$, a contradiction. Thus $D(x) \cap A \neq \phi$, and hence, the result follows from Theorem 17.5 of [7]. □

In view of above lemma, we have the following remarks.

Remark 2.2. Let N be a near-ring.

(i) The closure of $P \in \mathrm{Spec}(N)$ is $V(P)$.

(ii) A point $P \in \mathrm{Spec}(N)$ is closed if and only if $P \in \mathrm{Max}(N)$.

(iii) If $P, Q \in \mathrm{Spec}(N)$ with $cl(P) = cl(Q)$, then $P = Q$.

With the help of Lemma 2.1, we have the following some important characterizations of $\mathrm{Spec}(N)$.

Theorem 2.3. *Let N be a near-ring.*

(i) *If $F \subseteq \mathrm{Spec}(N)$ is a closed set and $D(K)$ is an open set in* $\mathrm{Spec}(N)$ *satisfying $F \cap \mathrm{Max}(N) \subseteq D(K)$, then $F \subseteq D(K)$.*

(ii) $\mathrm{Spec}(N)$ *is a compact space.*

(iii) $\mathrm{Max}(N)$ *is a compact T_1 subspace.*

(iv) *If* $\mathrm{Spec}(N)$ *is normal, then* $\mathrm{Max}(N)$ *is a Hausdorff space.*

(v) *If $\mathbb{P} = \cap \mathrm{Max}(N)$ and* $\mathrm{Max}(N)$ *is a Hausdorff space, then* $\mathrm{Spec}(N)$ *is normal.*

Proof. (i) Suppose that there is $P \in F$ with $P \notin D(K)$. Then $K + L \subseteq P$ since $F = V(L)$ for some ideal L of N. Hence, each maximal ideal M containing P is also in F. Then $M \in F \cap \mathrm{Max}(N)$, and so $M \in D(K)$, a contradiction.

(ii) Let $\mathcal{B} = \{D(s_i)\ :\ s_i \in J\}$ be the basis of N, for any subset J of N, and suppose that $\mathrm{Spec}(N) = \cup_{j\in J} D(s_j)$. Then $\phi = \cap_{j\in J}(\mathrm{Spec}(N)\backslash D(s_j)) = \cap_{j\in J} V(s_j) = V(\langle s_j; j \in J\rangle) = V(\sum_{j\in J}\langle s_j\rangle)$ which gives $\sum_{j\in J}\langle s_j\rangle = N$. Then there exists $K \subset J$ finite with $1 = \sum_{k \in K} s_k^{'}$, where $s_k^{'} \in \langle s_k \rangle$ which implies $\mathrm{Spec}(N) = \cup_{k \in K} D(s_k^{'})$. Indeed, clearly $\cup_{k \in K} D(s_k^{'}) \subseteq \mathrm{Spec}(N)$ and suppose $P \in \mathrm{Spec}(N)$ with $P \notin \cup_{k \in K} D(s_k^{'})$. Then $s_k^{'} \in P$ for all $k \in K$ which implies $1 \in P$, a contradiction. Hence $\mathrm{Spec}(N)$ is a compact space.

(iii) Let $\mathcal{B} = \{D(s_i) \ : \ s_i \in J\}$ be the basis of N, for any subset J of N, and suppose that $\mathrm{Max}(N) = (\cup_{i \in J} D(s_i)) \cap \mathrm{Max}(N)$. Then

$$\begin{aligned} \phi = \cap_{i \in J}(\mathrm{Max}(N) \backslash D(s_i)) &= (\cap_{i \in J} V(s_i)) \cap \mathrm{Max}(N) \\ &= V(\sum_{i \in I} \langle s_i \rangle) \cap \mathrm{Max}(N) \end{aligned}$$

which imply $\sum_{i \in J} \langle s_i \rangle = N$. Then there exists $J_1 \subset J$ finite with $1 = \sum_{j \in J_1} s_j$, and so $\mathrm{Max}(N) = \cup_{j \in J_1} D(s_j)$.

Let M_1 and M_2 be two distinct elements in $\mathrm{Max}(N)$. Then $M_1 \in D(M_2)$ and $M_2 \in D(M_1)$, and so $\mathrm{Max}(N)$ is a T_1 space.

(iv) Let M_1 and M_2 be distinct elements in $\mathrm{Max}(N)$. Then $\{M_1\}$ and $\{M_2\}$ are closed subsets in both $\mathrm{Spec}(N)$ and $\mathrm{Max}(N)$. If $\mathrm{Spec}(N)$ is normal, then there exist disjoint open sets $D(I)$ and $D(J)$ such that $\{M_1\} \subseteq D(I)$ and $\{M_2\} \subseteq D(J)$ for some ideals I and J of N, respectively. So, $M_1 \in D(I) \cap \mathrm{Max}(N)$, and $M_2 \in D(J) \cap \mathrm{Max}(N)$, which imply $\mathrm{Max}(N)$ is a Hausdorff space.

(v) Let F_1 and F_2 be two disjoint closed subsets of $\mathrm{Spec}(N)$. Then $F_1 \cap \mathrm{Max}(N)$ and $F_2 \cap \mathrm{Max}(N)$ are also disjoint subsets of $\mathrm{Max}(N)$. By Theorem 32.3 in [7], $\mathrm{Max}(N)$ is normal. So, there are open subsets $D(J)$ and $D(J_1)$ of $\mathrm{Spec}(N)$ such that $F_1 \cap \mathrm{Max}(N) \subseteq A$, $F_2 \cap \mathrm{Max}(N) \subseteq B$ and $A \cap B = \phi$, where $A = D(J) \cap \mathrm{Max}(N)$ and $B = D(J_1) \cap \mathrm{Max}(N)$.

Assume $\mathbb{P} = \cap \mathrm{Max}(N)$. Then $JJ_1 \subseteq \cap \mathrm{Max}(N) = \mathbb{P}$ since $D(J) \cap D(J_1) = D(JJ_1)$, and so $D(J) \cap D(J_1) = \phi$. By (i), we have $F_1 \subseteq D(J)$ and $F_2 \subseteq D(J_1)$. □

Theorem 2.4. *Let N be a 2-primal near-ring. Then $\mathbb{P}_S = \cap V(\mathbb{P}_S)$ for any subset S of N.*

Proof. Clearly, $\mathbb{P}_S \subseteq \cap V(\mathbb{P}_S)$. Let $a \in N \backslash \mathbb{P}_S$. Then $as \notin P$ for some $P \in \mathrm{Spec}(N)$ and $s \in S$ which implies $\mathbb{P}_S \subseteq P$. Thus, $a \notin P \in V(\mathbb{P}_S)$, and hence, $\cap V(\mathbb{P}_S) \subseteq \mathbb{P}_S$. □

Lemma 2.5. *Let N be a 2-primal near-ring and let $a, b \in N$. Then int $V(a) \subseteq$ int $V(b)$ if and only if $\mathbb{P}_a \subseteq \mathbb{P}_b$.*

Proof. Let *int* $V(a) \subseteq$ *int* $V(b)$ for any $a, b \in N$ and let $x \in \mathbb{P}_a$. Then $\mathrm{Spec}(N) \backslash V(x) \subseteq$ *int* $V(a) \subseteq$ *int* $V(b) \subseteq V(b)$, which gives $bx \in \mathbb{P}$, so $x \in \mathbb{P}_b$.

Conversely, let $\mathbb{P}_a \subseteq \mathbb{P}_b$ and let $P \in$ *int* $V(a)$. Suppose $P \notin V(b)$. By Lemma 2.1, if $P \notin \mathrm{Spec}(N) \backslash$*int* $V(a)$, then there is $0 \neq c \in N$ with $\mathrm{Spec}(N) \backslash$*int* $V(a) \subseteq V(c)$ and $c \notin P$. Clearly $ac \in \mathbb{P}$ and $bc \notin \mathbb{P}$. Then $c \in \mathbb{P}_a$ and $c \notin \mathbb{P}_b$, a contradiction. □

Lemma 2.6. *Let N be a 2-primal near-ring. Then for every $a \in N$, $cl(D(a)) = V(\mathbb{P}_a) = \mathrm{Supp}(a) = \mathrm{Spec}(N) \backslash int\ V(a)$.*

Proof. Let $a \in N$, $P \in V(\mathbb{P}_a)$, and let $D(x)$ be any arbitrary basis element in $\mathcal{B}$ such that $P \in D(x)$. Let $P \notin D(a)$ and suppose $D(a) \cap D(x) = \phi$. Then $D(xa) \subseteq D(x) \cap D(a) = \phi$, and so $xa \in \mathbb{P}$ which implies $x \in P$, a contradiction. Thus, $D(a) \cap D(x) \neq \phi$, and hence, $V(\mathbb{P}_a) = cl(D(a))$.

Let $P \in cl(D(a))$ and suppose that $P \in int\ V(a)$. Then there exists an open set U of $\mathrm{Spec}(N)$ with $P \in U \subseteq V(a)$, and so $P \notin \mathrm{Spec}(N) \backslash U$, a contradiction. Let $P \in \mathrm{Spec}(N) \backslash int\ V(a)$ and let $D(x)$ be any arbitrary element in $\mathcal{B}$ with $P \in D(x)$. Suppose that $D(x) \cap D(a) = \phi$. Then $P \in D(\mathbb{P}_a) \subseteq V(a)$, a contradiction. □

The following result gives the condition under which a subset of $\mathrm{Spec}(N)$ of 2-primal near-ring to be clopen, which will be used in our main result in Section 3.

Lemma 2.7. *Let N be a 2-primal near-ring. Then A is a clopen subset of* $\mathrm{Spec}(N)$ *if and only if there exists an element $a \in N$ with $a \in P$ or $-1+a \in P$ for all $P \in \mathrm{Spec}(N)$ and $A = V(a)$.*

Proof. Suppose that A is a clopen subset of $\mathrm{Spec}(N)$. Let $J = \cap A$ and $J_1 = \cap A^c$. Then by Lemma 2.1 $A = cl(A) = V(J)$ and $A^c = V(J_1)$. So, $V(J) \cap V(J_1) = \phi$, which gives $J + J_1 = N$. Then there exists $a \in J$ and $a^{'} \in J_1$ such that $a + a^{'} = 1$. Therefore $a(-1+a) \in \mathbb{P}$. Thus, for every prime ideal P, we have $a \in P$ or $-1 + a \in P$. Consequently, $A = V(J) = V(a)$. The converse is trivial. □

Theorem 2.8. *Let N be a 2-primal and pm near-ring. Then* $\mathrm{Max}(N)$ *is a compact Hausdorff space.*

Proof. By Lemma 2.3(iii), $\mathrm{Max}(N)$ is a compact space. Let $M_1, M_2 \in \mathrm{Max}(N)$ and consider the multiplicative subset

$$S = \{a_1 b_1 \cdots a_{n-1} b_{n-1} a_n b_n \ :\ a_i \notin M_1,\ b_i \notin M_2,\ n,\ i \in \{1, 2, \ldots, n\}\}.$$

Suppose that $0 \notin S$. Then there is a prime ideal P of N with $P \cap S = \phi$ and hence $P \subseteq M_1 \cap M_2$, a contradiction. So, there exist $a_i \notin M_1$ and $b_i \notin M_2$ such that $a_1 b_1 \cdots a_n b_n = 0$. We now have elements $x_1 \notin M_1$ and $x_2 \notin M_2$ with $x_1 x_2 \in \mathbb{P}$, which imply $D(x_1)$ and $D(x_2)$ are disjoint with $M_1 \in D(x_1)$ and $M_2 \in D(x_2)$. □

The following is an immediate corollary of Theorem 2.8.

Corollary 2.9 ([12], Lemma 2.1)**.** *If R is a 2-primal and pm ring, then* $\mathrm{Max}(R)$ *is a compact Hausdorff space.*

3. Distance and cycles in $\Gamma_{\mathbb{P}}(N)$

In this section, we associate the near-ring properties of N and the graph properties of $\Gamma_{\mathbb{P}}(N)$.

Theorem 3.1. *Let N be a 2-primal near-ring. Then $\Gamma_{\mathbb{P}}(N)$ is connected and diam $\Gamma_{\mathbb{P}}(N) \leq 3$.*

Proof. Let $x, y \in \Gamma_{\mathbb{P}}(N)$ be distinct. If $xy \in \mathbb{P}$, then $d(x, y) = 1$. Otherwise, there are $a, b \in N \backslash (\mathbb{P} \cup \{x, y\})$ such that $ax, by \in \mathbb{P}$.

If $a = b$, then $x \approx a \approx y$ is a path of length 2. Thus, we assume that $a \neq b$. If $ab \in \mathbb{P}$, then $x \approx a \approx b \approx y$ is a path of length 3; and hence $d(x, y) \leq 3$. Otherwise, $x \approx ab \approx y$ is a path of length 2; thus, $d(x, y) = 2$. Hence, $d(x, y) \leq 3$. □

Lemma 3.2. *Let N be a 2-primal near-ring and let $a, b \in \Gamma_{\mathbb{P}}(N)$. Then*

(i) $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$ *if and only if* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \subseteq V(c)$ *for some $c \in \Gamma_{\mathbb{P}}(N)$.*

(ii) *$D(a) \cap D(b) \neq \phi$ if and only if there exists $c \in \Gamma_{\mathbb{P}}(N)$ such that $\phi \neq D(a) \cap D(b) \subseteq V(c)$.*

Proof. (i) Suppose $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$. Then there exists an element $P \in \mathrm{Spec}(N)$ with $x, y \notin P$ for some $x \in \mathbb{P}_a$ and $y \in \mathbb{P}_b$. So, $xy \notin \mathbb{P}$. It is easy to see that $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \subseteq V(xy)$.

Conversely, suppose that $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$. Then $c \in \mathbb{P}$, a contradiction. Hence, $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$.

(ii) Straightforward. □

Now by Theorem 3.1, and Lemma 3.2, we have the following characterizations of the diameter of $\Gamma_{\mathbb{P}}(N)$.

Theorem 3.3. *Let N be a 2-primal near-ring and let $a, b \in \Gamma_{\mathbb{P}}(N)$ be distinct elements. Then*

(i) *For any $c \in \Gamma_{\mathbb{P}}(N)$, we have c is adjacent to both a and b if and only if* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \subseteq V(c)$.

(ii) *$d(a, b) = 1$ if and only if $D(a) \cap D(b) = \phi$.*

(iii) *$d(a, b) = 2$ if and only if $D(a) \cap D(b) \neq \phi$ and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$.

(iv) *$d(a, b) = 3$ if and only if $D(a) \cap D(b) \neq \phi$ and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$.

Proof. (i) Let $c \in \Gamma_{\mathbb{P}}(N)$. Then c is adjacent to both a and b if and only if $D(a) \cap D(c) = D(b) \cap D(c) = \phi$ if and only if $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \subseteq V(c)$.

(ii) Trivial.

(iii) Let $a, b \in \Gamma_{\mathbb{P}}(N)$. Then $d(a, b) = 2$ if and only if $ab \notin \mathbb{P}$ and there exists $c \in \Gamma_{\mathbb{P}}(N)$ such that c is adjacent to both a and b if and only if $D(a) \cap D(b) \neq \phi$ and $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \subseteq V(c)$ if and only $D(a) \cap D(b) \neq \phi$ and $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$ by Lemma 3.2.

(iv) By Theorem 3.1, $d(a,b) = 3$ if and only if $d(a,b) \neq 1, 2$ if and only if $D(a) \cap D(b) \neq \phi$ and $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$ by (i) and (ii). □

Since the reduced commutative ring is also a 2-primal near-ring, the following corollary is immediate.

Corollary 3.4 ([11], Proposition 2.2)**.** *Let R be a commutative reduced ring and let $a, b, c \in \Gamma(R)$ be distinct elements. Then*

(i) *c is adjacent to both a and b if and only if* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \subseteq V(c)$.
(ii) *$d(a,b) = 1$ if and only if $D(a) \cap D(b) = \phi$.*
(iii) *$d(a,b) = 2$ if and only if $D(a) \cap D(b) \neq \phi$ and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(R)$.
(iv) *$d(a,b) = 3$ if and only if $D(a) \cap D(b) \neq \phi$ and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(R)$.

The following theorem shows that every minimal prime ideal of 2-primal near-ring that doesn't contain both a and $\mathbb{P}_a$ for any $a \in N$.

Theorem 3.5. *Let N be a 2-primal near-ring and let $a \in N$. Then $V^{'}(a) = D^{'}(\mathbb{P}_a)$ and $D^{'}(a) = V^{'}(\mathbb{P}_a)$. In particular, $V^{'}(a)$ and $V^{'}(\mathbb{P}_a)$ are disjoint clopen subsets of* $\mathrm{Spec}(N)$*. Also,* $\mathrm{Min}(N)$ *is a Hausdorff space.*

Proof. Let $P \in V^{'}(a)$ and suppose $P \notin D^{'}(\mathbb{P}_a)$. Let $M = \{a, a^2, \ldots\}$ be multiplicative closed system and let $S = \{I \trianglelefteq N \ : \ I \subseteq P \text{ and } I \cap M = \phi\}$. Since $\mathbb{P}_a \in S$, $S \neq \phi$. Then by Zorn's Lemma, there exists a maximal ideal $\overline{P}$ in S with $\overline{P} \subseteq P$ and $\overline{P} \cap M = \phi$. Let J and J_1 be ideals of N such that $\overline{P} \subset J$ and $\overline{P} \subset J_1$.

Case (i): If $P \subset J$ and $P \subset J_1$, then $JJ_1 \nsubseteq P$. So $JJ_1 \nsubseteq \overline{P}$.

Case (ii): If $J \subseteq P$ and $J_1 \subseteq P$, then $J \cap M \neq \phi$ and $J_1 \cap M \neq \phi$. Then there exist $j \in J \cap M$ and $j_1 \in J_1 \cap M$ with $j^{'} j_1^{'} \in M$ for some $j^{'} \in J$ and $j_1^{'} \in J_1$, which gives $JJ_1 \cap M \neq \phi$. So, $JJ_1 \nsubseteq \overline{P}$.

Case (iii): If $J \subseteq P$ and $P \subset J_1$, then by Case (ii), we have $JP \nsubseteq \overline{P}$. So $JJ_1 \nsubseteq \overline{P}$.

Thus, $\overline{P}$ is a prime ideal with $\overline{P} \subset P$, contradicting the minimality of P. Hence, $V^{'}(a) = D^{'}(\mathbb{P}_a)$. Similarly, we have $D^{'}(a) = V^{'}(\mathbb{P}_a)$.

Let $P \neq P^{'} \in \mathrm{Min}(N)$ and $a \in P \backslash P^{'}$. Then $V^{'}(a)$ and $V^{'}(\mathbb{P}_a)$ are disjoint open sets containing P and $P^{'}$, respectively. □

Lemma 3.6. *Let N be a 2-primal near-ring and let $a \in \Gamma_{\mathbb{P}}(N)$. If $e(a) = 1$, then $\mathbb{P}_a$ is a completely prime ideal of N.*

Proof. Straightforward. □

Theorem 3.7. *Let N be a 2-primal near-ring and $2 \notin \mathbb{P}$. Then*

(i) *$\Gamma_{\mathbb{P}}(N)$ is a triangulated graph if and only if* $\mathrm{Spec}(N)$ *has no quasi-isolated points.*

(ii) $\Gamma_{\mathbb{P}}(N)$ *is a hyper-triangulated graph if and only if* $\mathrm{Spec}(N)$ *is connected space and for any* $a, b \in \Gamma_{\mathbb{P}}(N)$, *we have that* $ab \in \mathbb{P}$ *and* $D(a) \cup D(b) \neq \mathrm{Spec}(N)$ *imply* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$.

(iii) *If* $2 \notin \Gamma_{\mathbb{P}}(N)$, *then every vertex of* $\Gamma_{\mathbb{P}}(N)$ *is a 4-cycle vertex.*

Proof. (i) Let $\Gamma_{\mathbb{P}}(N)$ be a triangulated graph and suppose $\mathrm{Spec}(N)$ has a quasi-isolated point P. Then $D^{'}(\mathbb{P}_a) = V^{'}(a) = \{P\}$ for some $a \in P$. Clearly, $a \in \Gamma_{\mathbb{P}}(N)$, and since $\Gamma_{\mathbb{P}}(N)$ is a triangulated graph, there are $b, c \in \Gamma_{\mathbb{P}}(N)$ such that $ab, ac, bc \in \mathbb{P}$. Thus, $D^{'}(a) \subseteq V^{'}(b)$, and $\phi \neq D^{'}(c) \subseteq V^{'}(a) \cap V^{'}(b) = \{P\}$, which gives $V^{'}(b) = \mathrm{Min}(N)$, a contradiction. Hence, $\mathrm{Spec}(N)$ does not contain quasi-isolated points.

Conversely, suppose that $\mathrm{Spec}(N)$ does not contain quasi-isolated points and take $a \in \Gamma_{\mathbb{P}}(N)$. Then there are two different points $P, P^{'} \in V^{'}(a) = D^{'}(\mathbb{P}_a)$. Since $\mathbb{P}_a \nsubseteq P^{'}$, there exists $z \in \mathbb{P}_a$ such that $z \notin P^{'}$. Also, there exists $y \in P$ with $y \notin P^{'}$. Clearly, $zy \notin \mathbb{P}$ and $P \in V^{'}(zy) = D^{'}(\mathbb{P}_{zy})$, which imply $P \notin \mathrm{Supp}(zy)$. Thus $\mathrm{Supp}(a) \cup \mathrm{Supp}(zy) \neq \mathrm{Spec}(N)$. Then by Lemma 3.2, there exists $c \in \Gamma_{\mathbb{P}}(N)$ such that $\mathrm{Supp}(a) \cup \mathrm{Supp}(zy) \subseteq V(c)$, so by Theorem 3.3 (i), c is adjacent to both a and zy.

(ii) Let $\Gamma_{\mathbb{P}}(N)$ be a hyper-triangulated graph. If $\mathrm{Spec}(N)$ is not connected, then by Lemma 2.7, there exists an element $a \in \Gamma_{\mathbb{P}}(N)$. Since $\mathrm{Supp}(a) \cup$ *supp*$(-1+a) = \mathrm{Spec}(N)$, by Theorem 3.3, there is no vertex adjacent to both a and $-1+a$, a contradiction. The second part follows from Lemma 3.2 and Theorem 3.3.

Conversely, let $a \approx b$ be an edge in $\Gamma_{\mathbb{P}}(N)$. Since $D(a) \cap D(b) = \phi$ and $\mathrm{Spec}(N)$ is connected, $D(a) \cup D(b) \neq \mathrm{Spec}(N)$. Thus by hypothesis, $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$. Therefore, by Lemma 3.2 and Theorem 3.3, there exists a vertex adjacent to both a and b.

(iii) Let $a \in \Gamma_{\mathbb{P}}(N)$. Then there exists $b \in N \backslash \mathbb{P}$ such that $ab \in \mathbb{P}$. Since $2 \notin \Gamma_{\mathbb{P}}(N)$, we have $2a \neq b$ and $a \neq 2b$. So $a, b, 2a$ and $2b$ are all distinct. Also, $ab, (2a)b, (2a)(2b)$ and $a(2b)$ belong to $\mathbb{P}$. Hence $a, b, 2a$ and $2b$ is a cycle with length 4 containing a. □

As an immediate application of Theorem 3.7, we have the following corollary.

Corollary 3.8 ([11], Theorem 3.1). *Let R be a commutative reduced ring. Then*

(i) $\Gamma(R)$ *is a triangulated graph if and only if* $\mathrm{Spec}(R)$ *has no quasi-isolated points.*

(ii) $\Gamma(R)$ *is a hyper-triangulated graph if and only if* $\mathrm{Spec}(R)$ *is connected space and for any* $a, b \in \Gamma(R)$, *we have that* $ab \in \mathbb{P}$ *and* $D(a) \cup D(b) \neq \mathrm{Spec}(R)$ *imply* $\mathrm{Spec}(R) \cup \mathrm{Spec}(R) \neq \mathrm{Spec}(R)$.

(iii) *If* $2 \notin Z(R)$, *then every vertex of* $\Gamma(R)$ *is a 4-cycle vertex.*

The next theorem will help to characterize all possible cycles in the ideal-based zero-divisor graph.

Theorem 3.9. *Let N be a 2-primal near-ring, $a, b \in \Gamma_{\mathbb{P}}(N)$ and $2 \notin \mathbb{P}$. If $2 \notin \Gamma_{\mathbb{P}}(N)$, then*

(i) $c(a, b) = 3$ *if and only if* $D(a) \cap D(b) = \phi$ *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$.

(ii) $c(a, b) = 4$ *if and only if either* $D(a) \cap D(b) \neq \phi$ *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$, *or* $D(a) \cap D(b) = \phi$, *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$.

(iii) $c(a, b) = 6$ *if and only if* $D(a) \cap D(b) \neq \phi$ *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$.

Proof. (i) Follows from Lemma 3.2 and Theorem 3.3.

(ii) If $D(a) \cap D(b) = \phi$ and $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$, there exists a path with vertices $a, b, 2a$ and $2b$, i.e., $c(a, b) \leq 4$. Now (i) implies that $c(a, b) = 4$. If $D(a) \cap D(b) \neq \phi$ and $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(N)$, then by Theorem 3.3, there exists $c \in \Gamma_{\mathbb{P}}(N)$ such that c is adjacent to both a and b. Thus, the path with vertices a, c, b and $2c$ is a cycle with length 4.

(iii) If $c(a, b) = 6$, then parts (i) and (ii) imply that $D(a) \cap D(b) \neq \phi$ and $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$. Conversely, let $D(a) \cap D(b) \neq \phi$ and $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(N)$. Then by Theorem 3.3, $d(a, b) = 3$. Also, (i) and (ii) implies that $c(a, b) > 4$. Hence, there are vertices c and d such that $ac, cd, bd \in \mathbb{P}$. Now, if some vertex e is adjacent to b, then $be \in \mathbb{P}$. Therefore, $\mathrm{Spec}(N) = \mathrm{Supp}(a) \cup \mathrm{Supp}(b) \subseteq V(c) \cup V(e)$. However, $d(a, b) = 3$ implies that a is not adjacent to e, i.e., $c(a, b) = 6$. If we consider the vertices $2c$ and $2d$, then we have a cycle with vertices $a, c, b, 2d$ and $2c$, i.e., $c(a, b) = 6$. □

From Theorem 3.9, we have the following corollary.

Corollary 3.10 ([11], Theorem 3.4)**.** *Let R be a commutative reduced ring, $a, b \in \Gamma(R)$, and $2 \notin \Gamma(R)$. Then*

(i) $c(a, b) = 3$ *if and only if* $D(a) \cap D(b) = \phi$ *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(R)$.

(ii) $c(a, b) = 4$ *if and only if either* $D(a) \cap D(b) \neq \phi$ *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(R)$ *or* $D(a) \cap D(b) = \phi$ *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(R)$.

(iii) $c(a, b) = 6$ *if and only if* $D(a) \cap D(b) \neq \phi$ *and* $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(R)$.

As an immediate application of Theorem 3.9 or Corollary 3.10, we have the following corollary.

Corollary 3.11 ([11], Corollary 3.5)**.** *Let R be a commutative reduced ring and $2 \notin \Gamma(R)$. Then every edge of a cycle with length* 3 *or* 4.

Proof. Let $a \approx b$ be an edge in a cycle. Then $ab \in \mathbb{P}$ and $D(a) \cap D(b) = \phi$. If $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \neq \mathrm{Spec}(R)$, then by Corollary 3.10, we have $c(a, b) = 3$. Otherwise, $\mathrm{Supp}(a) \cup \mathrm{Supp}(b) = \mathrm{Spec}(R)$. Then by Corollary 3.10, we have $c(a, b) = 4$. □

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References

[1] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), no. 2, 500–514.

[2] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.

[3] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[5] P. Dheena and B. Elavarasan, *On strong IFP near-ring*, Submitted.

[6] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[7] J. R. Munkres, *Topology*, Prentice-Hall of Indin, New Delhi, 2005.

[8] G. Pilz, *Near-Rings*, North-Holland, Amsterdam, 1983.

[9] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), no. 9, 4425–4443.

[10] K. Samei, *On summand ideals in commutative reduced rings*, Comm. Algebra **32** (2004), no. 3, 1061–1068.

[11] ———, *The zero-divisor graph of a reduced ring*, J. Pure Appl. Algebra **209** (2007), no. 3, 813–821.

[12] S. H. Sun, *Noncommutative rings in which every prime ideal is contained in a unique maximal ideal*, J. Pure Appl. Algebra **76** (1991), no. 2, 179–192.

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