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SKEW LAURENT POLYNOMIAL EXTENSIONS OF BAER AND P.P.-RINGS

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Abstract. Let R be a ring and α a monomorphism of R. We study the skew Laurent polynomial rings $R[x, x^{-1}; \alpha]$ over an α-skew Armendariz ring R. We show that, if R is an α-skew Armendariz ring, then R is a Baer (resp. p.p.-)ring if and only if $R[x, x^{-1}; \alpha]$ is a Baer (resp. p.p.-) ring. Consequently, if R is an Armendariz ring, then R is a Baer (resp. p.p.-)ring if and only if $R[x, x^{-1}]$ is a Baer (resp. p.p.-)ring.

1. Introduction

Throughout this paper R denotes an associative ring with unity and $\alpha : R \to R$ is an endomorphism, which is not assumed to be surjective. We denote $R[x; \alpha]$ the Ore extension whose elements are the polynomials $\Sigma_{i=0}^{n} r_i x^i$, $r_i \in R$, where the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. The set $\{x^j\}_{j \geq 0}$ is easily seen to be a left Ore subset of $R[x\ ; \alpha]$, so that one can localize $R[x; \alpha]$ and form the skew Laurent polynomial ring $R[x, x^{-1}\ ; \alpha]$. Elements of $R[x, x^{-1}\ ; \alpha]$ are finite sums of elements of the form $x^{-j} r x^i$, where $r \in R$ and i and j are nonnegative integers.

A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j. The term Armendariz was introduced by Rege and Chhawchharia [20]. This nomenclature was used by them since it was Armendariz [2, Lemma 1] who initially showed that a reduced ring (i.e., a ring without nonzero nilpotent elements) always satisfies this condition.

According to Krempa [16], an endomorphism α of a ring R is called to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called α-rigid if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by Hong et al. [10].

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Properties of α-rigid rings have been studied in Krempa [16], Hong et al. [10], and Hirano [9].

A generalization of α-rigid rings and Armendariz rings is introduced and well studied by C. Y. Hong, N. K. Kim, and T. Kwak in [11].

By Hong et al. [11], a ring R is called *α-skew Armendariz* if, for polynomials $f(x) = a_0+a_1x+\cdots+a_nx^n$, $g(x) = b_0+b_1x+\cdots+b_mx^m$ in the skew polynomial ring $R[x;\alpha]$, $f(x)g(x) = 0$ implies that $a_i\alpha^i(b_j) = 0$ for each i, j. By [10] every α-rigid ring is reduced and α-skew Armendariz; and by [18] reduced α-skew Armendariz rings are α-rigid.

Hong et al. in [11, Theorems 21 and 22] proved that:

If α is an automorphism of a ring R with $\alpha(e) = e$ for any $e^2 = e \in R$, and R is an α-skew Armendariz ring, then R is a Baer (resp. p.p.-)ring if and only if $R[x;\alpha]$ is a Baer (resp. p.p.-)ring.

Following Hong et al.'s results [10 and 11], in this paper we study on the skew Laurent polynomial rings $R[x, x^{-1}\ ;\alpha]$ when R is an α-skew Armendariz ring. We first give a short and simple proof of [18] and prove that, for an endomorphism α of a ring R, R is an α-rigid ring if and only if α is injective, R is reduced and α-skew Armendariz. We then show that:

If α is a monomorphism of a ring R and R is an α-skew Armendariz ring, then R is a Baer (resp. p.p.-)ring if and only if the skew Laurent polynomial ring $R[x, x^{-1};\alpha]$ is a Baer (resp. p.p.-)ring. Consequently, we deduce that:

If R is an Armendariz ring, then R is a Baer (resp. p.p.-)ring if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is a Baer (resp. p.p.-)ring.

Finally we construct some new examples of non reduced α-skew Armendariz rings.

2. α-skew Armendariz rings

In this section we provide a simple proof of Matzuk's main result [18]. Some equivalent characterizations of α-skew Armendariz rings is given and some properties of the skew Laurent polynomial ring $R[x, x^{-1};\alpha]$, over an α-skew Armendariz ring, is studied.

We start by observing that for an endomorphism α of a ring R, R is an α-skew Armendariz ring, if for elements $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + \cdots + b_mx^m \in R[x;\alpha]$, $f(x)g(x) = 0$ implies $a_0b_j = 0$ for all integers $0 \le j \le m$. If we take $\alpha = id_R$, we deduce the following equivalent condition for a ring to be Armendariz:

A ring R is Armendariz if and only if for every polynomials $f(x) = a_0 + a_1x+\cdots+a_nx^n$ and $g(x) = b_0+b_1x+\cdots+b_mx^m$ in $R[x]$, $f(x)g(x) = 0$ implies $a_0b_j = 0$ for each $0 \le j \le m$.

Since the skew Laurent polynomial ring $R[x, x^{-1}\ \alpha]$ is a localization of $R[x;\alpha]$ with respect to the set of powers of x, we prove an equivalent condition for a ring to be α-skew Armendariz, related to the skew Laurent polynomial ring $R[x, x^{-1}\ \alpha]$:

Proposition 1. *Let R be a ring and α a monomorphism of R. Then R is an α-skew Armendariz ring if and only if for elements $f(x) = x^r a_r + x^{r+1}a_{r+1} + \cdots + a_0 + a_1x + \cdots + a_n x^n$ and $g(x) = b_0 + \cdots + b_m x^m \in R[x, x^{-1}\ ; \alpha]$, where r is a negative integer, $f(x)g(x) = 0$ implies $a_0 b_j = 0$ for all integers $0 \leq j \leq m$.*

Proof. Suppose that R is an α-skew Armendariz ring and $f(x)g(x) = 0$ for elements $f(x) = x^r a_r + x^{r+1}a_{r+1} + \cdots + x^{-1}a_{-1} + a_0 + a_1x + \cdots + a_n x^n$ and $g(x) = b_0 + \cdots + b_m x^m \in R[x, x^{-1}\ ; \alpha]$, where r is a negative integer. We show that this implies that $a_0 b_j = 0$ for all integers $0 \leq j \leq m$. Multiply $f(x)g(x) = 0$ by x^{-r} from left yields

$$(a_r + xa_{r+1} + \cdots + x^{-r-1}a_{-1} + x^{-r}a_0 + x^{-r}a_1x + \cdots + x^{-r}a_n x^n) \cdot (b_0 + \cdots + b_m x^m) = 0.$$

Hence $a_r b_j = 0$ for each $0 \leq j \leq m$, since R is α-skew Armendariz. Repeating the argument for

$$(a_{r+1} + xa_{r+2} + \cdots + x^{-r}a_{-1} + x^{-r-1}a_0 + x^{-r-1}a_1x + \cdots + x^{-r-1}a_n x^n) \cdot (b_0 + \cdots + b_m x^m) = 0,$$

yields $a_{r+1}b_j = 0$ for each $0 \leq j \leq m$. Continuing in this way we get $(a_0 + a_1x + \cdots + a_n x^n)(b_0 + \cdots + b_m x^m) = 0$, and α-skew Armendariz condition implies that $a_0 b_j = 0$ for each $0 \leq j \leq m$. □

Theorem 2. *Let α be an endomorphism of a ring R. Then R is an α-rigid ring if and only if α is injective, R is reduced and α-skew Armendariz.*

Proof. Suppose that R is a reduced α-skew Armendariz ring and $a\alpha(a) = 0$ for $a \in R$. Now, consider $h(x) = \alpha(a) - \alpha(a)x$ and $k(x) = a + \alpha(a)x \in R[x; \alpha]$. Then $h(x)k(x) = 0$. Since R is α-skew Armendariz, we have $\alpha(a)\alpha(a) = 0$. But R is reduced and α is a monomorphism, therefore $a = 0$. The converse follows by [10, Proposition 6]. □

Now we consider D. A. Jordan's construction of the ring $A(R, \alpha)$ (See [13], for more details). Let $A(R, \alpha)$ be the subset $\{x^{-i}rx^i \mid r \in R\ ,\ i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. For each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with $x^{-i}rx^i + x^{-j}rx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}$ for $r, s \in R$ and $i, j \geq 0$. Note that α is actually an automorphism of $A(R, \alpha)$. We have $R[x, x^{-1}; \alpha] \simeq A(R, \alpha)[x, x^{-1}; \alpha]$, by way of an isomorphism which maps $x^{-i}rx^j$ to $\alpha^{-i}(r)x^{j-i}$.

Theorem 3. *A ring R is α-rigid if and only if $R[x, x^{-1}; \alpha]$ is a reduced ring.*

Proof. If $R[x, x^{-1}; \alpha]$ is a reduced ring and for $a \in R$, $a\alpha(a) = 0$ then $axax = 0$ and hence $ax = 0$. So R is α-rigid. Conversely assume that R is an α-rigid ring. We first show that the Jordan extension $A(R, \alpha)$ is α-rigid. Let $(x^{-i}rx^i)\alpha(x^{-i}rx^i) = 0$, where $i \geq 0$ and $r \in R$. Then $r\alpha(r) = 0$, so $r = 0$, since R is α-rigid. Therefore $A(R, \alpha)$ is α-rigid. Since by [13], $R[x, x^{-1}; \alpha] \simeq$

$A(R,\alpha)[x,x^{-1};\alpha]$, we will assume that α is an automorphism of R and R is an α-rigid ring. Assume that $f^2=0$, with $f(x)=a_mx^m+a_{m+1}x^{m+1}+\cdots+a_nx^n\in R[x,x^{-1};\alpha]$, and integers m,n. Then we have $(a_nx^n)(a_nx^n)=a_n\alpha^n(a_n)x^{2n}=0$. Since R is α-rigid, $a_n=0$. Hence we can deduce that $f=0$ and the result follows. □

The following proposition partially extends [10, Proposition 5] and hence [8, Lemma 3] and [16, Theorem 3.3].

Proposition 4. *Let R be an α-skew Armendariz ring. Then for each idempotent element $e\in R$, we have $\alpha(e)=e$.*

Proof. Consider $f(x)=1-e+(1-e)\alpha(e)x$ and $g(x)=e+(e-1)\alpha(e)x$. Then $f(x)g(x)=0$. Since R is α-skew Armendariz, $(1-e)(e-1)\alpha(e)=0$ and hence $\alpha(e)=e\alpha(e)$. Now suppose that $h(x)=e+e(1-\alpha(e))x$ and $k(x)=1-e-e(1-\alpha(e))x$. Then $h(x)k(x)=0$. Hence $e(e(1-\alpha(e))=0$ and so $e=e\alpha(e)=\alpha(e)$. □

Theorem 5. *Every α-skew Armendariz ring is abelian.*

Proof. Let $r\in R$ and $e^2=e\in R$. Consider $h(x)=e-er(1-e)x$ and $k(x)=(1-e)+er(1-e)x\in R[x;\alpha]$. We have $h(x)k(x)=0$. Since R is α-skew Armendariz, $eer(1-e)=0$. Thus $er=ere$. Now take $f=(1-e)-(1-e)rex$ and $g=e+(1-e)rex$. Then $fg=0$. Since R is α-skew Armendariz, $(1-e)(1-e)re=0$. So $re=ere$. Therefore $re=ere=er$, and that R is abelian. □

Corollary 6. *Every Armendariz ring is abelian.*

3. Skew Laurent polynomial extensions of Baer and p.p.-rings

Now we turn our attention to the relationship between the Baerness and p.p.-property of a ring R and these of the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$ in case R is an α-skew Armendariz ring.

Theorem 7. *Let R be a ring and α a monomorphism of R. If R is α-skew Armendariz and $e^2=e\in R[x,x^{-1};\alpha]$, then $e\in R$.*

Proof. Let $e=x^{-i_1}e_1x^{j_1}+\cdots+x^{-i_n}e_nx^{j_n}$, with $e_i\in R$ and nonnegative integers $i_1,\ldots,i_n,j_1,\ldots,j_n$. Let $i=\max\{i_1,\ldots,i_n\}$. Then

$$\begin{aligned}e&=x^{-i}(x^{i-i_1}e_1x^{j_1}+\cdots+x^{i-i_n}e_nx^{j_n})\\&=x^{-i}(\alpha^{i-i_1}(e_1)x^{i-i_1+j_1}+\cdots+\alpha^{i-i_n}(e_n)x^{i-i_n+j_n}).\end{aligned}$$

Since $e(1-e)=(1-e)e=0$, we have

$$(1-e)x^{-i}(\alpha^{i-i_1}(e_1)x^{i-i_1+j_1}+\cdots+\alpha^{i-i_n}(e_n)x^{i-i_n+j_n})=0.$$

Thus

$$(1-x^iex^{-i})(\alpha^{i-i_1}(e_1)x^{i-i_1+j_1}+\cdots+\alpha^{i-i_n}(e_n)x^{i-i_n+j_n})=0.$$

But $e = x^{-i_1}e_1x^{j_1} + \cdots + x^{-i_n}e_nx^{j_n}$, so $x^iex^{-i} = \alpha^{i-i_1}(e_1)x^{j_1-i_1} + \cdots + \alpha^{i-i_n}(e_n)x^{j_n-i_n}$. Thus

$$(1 - \alpha^{i-i_1}(e_1)x^{j_1-i_1} - \cdots - \alpha^{i-i_n}(e_n)x^{j_n-i_n}) \cdot (\alpha^{i-i_1}(e_1)x^{i-i_1+j_1} + \cdots + \alpha^{i-i_n}(e_n)x^{i-i_n+j_n}) = 0.$$

Now, if for all $1 \le t \le n$, $j_t \ne i_t$, then by Proposition 4, we have $\alpha^{i-i_t}(e_t) = 0$ for all $1 \le t \le n$, and hence $e_1 = e_2 = \cdots = e_n = 0$ and that $e = 0$, so the result follows. Otherwise for some $1 \le t \le n$, $j_t = i_t$. In this case it is enough to assume that for only one index t with $1 \le t \le n$, $i_t = j_t$. This is because, if $i_t = j_t$ and $i_k = j_k$, with $1 \le k < t \le n$, then we have,

$$\begin{aligned} x^{-i_t}e_tx^{j_t} + x^{-i_k}e_kx^{j_k} &= x^{-i_t-i_k}\alpha^{i_k}(e_t)x^{j_t+j_k} + x^{-i_t-i_k}\alpha^{i_t}(e_k)x^{j_t+j_k} \\ &= x^{-i_s}[\alpha^{i_k}(e_t) + \alpha^{i_t}(e_k)]x^{j_s}. \end{aligned}$$

Therefore we assume that for only one index t with $1 \le t \le n$, $i_t = j_t$. In this case we have $(1 - \alpha^{i-i_t}(e_t))(\alpha^{i-i_l}(e_l)) = 0$ for all $1 \le l \le n$. Thus $\alpha^{i-i_t}(e_t) = \alpha^{i-i_t}(e_t)\alpha^{i-i_t}(e_t)$. Since α is a monomorphism, $e_t = e_t^2$. Also for each $k \ne t$, $\alpha^{i-i_k}(e_k) = \alpha^{i-i_t}(e_t)\alpha^{i-i_k}(e_k)$.

(1) On the other hand, $e(1-e) = 0$ implies that $x^{-i}(\alpha^{i-i_1}(e_1)x^{i-i_1+j_1} + \cdots + \alpha^{i-i_n}(e_n)x^{i-i_n+j_n})(1-e) = 0$. But $(1-e) = x^{-i}(x^i - \alpha^{i-i_1}(e_1)x^{i-i_1+j_1} - \cdots - \alpha^{i-i_n}(e_n)x^{i-i_n+j_n})$. So

$$\begin{aligned} &[\alpha^{i-i_1}(e_1)x^{i-i_1+j_1} + \cdots + \alpha^{i-i_n}(e_n)x^{i-i_n+j_n}] \\ &\quad \cdot x^{-i}[x^i - \alpha^{i-i_1}(e_1)x^{i-i_1+j_1} - \cdots - \alpha^{i-i_n}(e_n)x^{i-i_n+j_n}] \\ &= [\alpha^{i-i_1}(e_1)x^{j_1-i_1} + \cdots + \alpha^{i-i_n}(e_n)x^{j_n-i_n}] \\ &\quad \cdot [x^i - \alpha^{i-i_1}(e_1)x^{i-i_1+j_1} - \cdots - \alpha^{i-i_n}(e_n)x^{i-i_n+j_n}]. \end{aligned}$$

Since $i_t = j_t$ and R is α-skew Armendariz, $\alpha^{i-i_t}(e_t)(1 - \alpha^{i-i_t}(e_t)) = 0$ and $\alpha^{i-i_t}(e_t)\alpha^{i-i_k}(e_k) = 0$ for each $k \ne t$.

(2) By (1) and (2) we have for each $k \ne t$, $\alpha^{i-i_k}(e_k) = \alpha^{i-i_t}(e_t)\alpha^{i-i_k}(e_k) = 0$, so $e_k = 0$, as α is injective. Thus $e = x^{-i_t}e_tx^{i_t}$. By Proposition 1, $\alpha^{i_t}(e_t) = e_t$, so $e = x^{-i_t}\alpha^{i_t}(e_t)x^{i_t} = x^{-i_t}x^{i_t}e_t = e_t$. Therefore the result follows. □

Corollary 8. *If R is an Armendariz ring and $e^2 = e \in R[x, x^{-1}]$, then $e \in R$.*

Corollary 9. *Let R be an α-skew Armendariz ring with α a monomorphism of R. Then $R[x, x^{-1}; \alpha]$ is an abelian ring.*

Corollary 10. *Let R be an Armendariz ring, then $R[x, x^{-1}]$ is an abelian ring.*

Recall that R is a Baer ring if the right annihilator of every non-empty subset of R is generated by an idempotent of R. These definitions are left-right symmetric. Kaplansky [13] defined an AW*-algebra as a C*-algebra with the stronger property that the right annihilator of the nonempty subset is generated by a projection. A ring R is called a *right* (*resp. left*) *p.p.*-ring if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of R is generated (as a right (resp. left) ideal) by an idempotent of R). R is called a p.p.-ring if it is both right and left p.p.

The next example shows that Baer property of a ring R doesn't extend, in general, to the polynomial ring $R[x]$ or Laurent polynomial ring $R[x, x^{-1}]$:

Example 11. From [14, p. 39], $M_2(\mathbb{Z}_3)$ is a Baer ring. But neither $M_2(\mathbb{Z}_3)[x]$ nor $M_2(\mathbb{Z}_3)[x, x^{-1}]$ is a Baer ring. In fact the right annihilator

$$r\left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \right)$$

cannot be generated (as a right ideal) by an idempotent.

Hong et al. in [11, Theorem 21] proved that, for an automorphism α of a ring R with $\alpha(e) = e$ for any $e^2 = e \in R$, if R is an α-skew Armendariz ring, then R is a Baer (resp. p.p.-)ring if and only if $R[x; \alpha]$ is a Baer (resp. p.p.-)ring.

Theorem 12. *Let R be an α-skew Armendariz ring and α a monomorphism of R. Then R is a Baer ring if and only if $R[x, x^{-1}; \alpha]$ is a Baer ring.*

Proof. Assume that R is a Baer ring. Since R is α-skew Armendariz, it is abelian by Corollary 9. But abelian Baer rings are reduced by [4, Corollary 1.15]. By Theorem 2, reduced α-skew Armendariz rings are α-rigid. Thus $A(R; \alpha)$ is α-rigid, as in the proof of Theorem 3. Since by [13], $R[x, x^{-1}; \alpha] \simeq A(R, \alpha)[x, x^{-1}; \alpha]$, we will assume that α is an automorphism of R and R is an α-rigid Baer ring. Since α is an automorphism of R, we can take each element of $R[x, x^{-1}; \alpha]$ as $f = x^r a_r + x^{r+1} a_{r+1} + \cdots + a_0 + a_1 x + \cdots + a_n x^n$, where r and n are integers. Let I be a nonempty subset of $R[x, x^{-1}; \alpha]$ and I_0 be the set of all coefficients of elements of I. Then I_0 is a nonempty subset of R and so $r_R(I_0) = eR$ for some idempotent $e \in R$. Using Proposition 4, we see that $e \in r_{R[x,x^{-1};\alpha]}(I)$, hence we get $eR[x, x^{-1}; \alpha] \subseteq r_{R[x,x^{-1};\alpha]}(I)$. Now, we let $0 \neq g = b_k x^k + b_{k+1} x^{k+1} + \cdots + b_0 + \cdots + b_m x^m \in r_{R[x,x^{-1};\alpha]}(I)$. Then $Ig = 0$ and hence $fg = 0$ for any $f \in I$.

Let $f = x^r a_r + x^{r+1} a_{r+1} + \cdots + a_0 + a_1 x + \cdots + a_n x^n \in I$, where r and n are integers. Then we have $a_r b_k = 0$ and $a_r b_{k+1} + \alpha(a_{r+1})\alpha(b_k) = 0$. This implies that $a_r b_{k+1} \alpha(a_r) = 0$ and that $a_r b_{k+1} = 0$. Assume inductively that $a_r b_k = a_r b_{k+1} = \cdots = a_r b_{t-1} = 0$. Now we show that $a_r b_t = 0$. We have $a_r b_t + \alpha(a_{r+1} b_{t-1}) + \alpha^2(a_{r+2} b_{t-2}) + \cdots + \alpha^{t-k}\alpha(a_{r+t-k} b_k) = 0$. Thus we have $a_r b_t \alpha(a_r) = 0$ and so $a_r b_t = 0$. Therefore $a_r b_j = 0$ for all $k \leq j \leq m$. Now we have $(x^{r+1} a_{r+1} + \cdots + a_0 + a_1 x + \cdots + a_n x^n) g = 0$. The same argument as above shows that $a_{r+1} b_j = 0$ for all $k \leq j \leq m$. Repeating this process it implies that $a_i b_j = 0$ for all $r \leq i \leq n$ and $k \leq j \leq m$. Thus $b_j \in r_R(I_0) = eR$ for $k \leq j \leq m$, and so $g = eg \in eR[x, x^{-1}; \alpha]$. Consequently $eR[x, x^{-1}; \alpha] = r_{R[x,x^{-1};\alpha]}(I)$. Therefore $R[x, x^{-1}; \alpha]$ is a Baer ring. □

Conversely, assume that $R[x, x^{-1}; \alpha]$ is a Baer ring. Let $U \subseteq R$. Then Theorem 7 implies that $r_{R[x,x^{-1};\alpha]}(U) = eR[x, x^{-1}; \alpha]$ for some idempotent element $e \in R$. Thus

$$r_R(U) = r_{R[x,x^{-1};\alpha]}(U) \cap R = eR[x, x^{-1}; \alpha] \cap R = eR,$$

and the result follows.

Corollary 13. *If R is an Armendariz ring, then R is a Baer ring if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is a Baer ring.*

Notice that in [5, Lemma 1.7] Birkenmeier, Kim, and Park in order to characterize some idempotents of $R[x; x^{-1}]$ or $R[[x; x-1]]$ and hence study the Baerness of either $R[x; x^{-1}]$ or $R[[x; x-1]]$, involves a long and quite technical calculation.

Corollary 14. *If R is a reduced ring, then R is a Baer ring if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is a Baer ring.*

Theorem 15. *Let R be an α-skew Armendariz ring and α a monomorphism of R. Then R is a p.p.-ring if and only if $R[x, x^{-1}; \alpha]$ is a p.p.-ring.*

Proof. Assume that R is a p.p.-ring. Since R is α-skew Armendariz, it is abelian by Theorem 5. But abelian p.p.-rings are reduced by [4, Corollary 1.15]. By Theorem 2, reduced α-skew Armendariz rings are α-rigid. As the proof of Theorem 3, the Jordan extension $A(R, \alpha)$ is α-rigid, and we will assume that α is an automorphism and R is α-rigid. Since α is an automorphism of R, we can take each element of $R[x, x^{-1}; \alpha]$ as $f = x^r a_r + x^{r+1} a_{r+1} + \cdots + a_0 + a_1 x + \cdots + a_n x^n$, where r and n are integers. Let $f = x^r a_r + x^{r+1} a_{r+1} + \cdots + a_0 + a_1 x + \cdots + a_n x^n \in R[x, x^{-1}; \alpha]$. So there exists idempotents $e_i \in R$ such that $r_R(a_i) = e_i R$ for $i = r, \ldots, n$. Let $e = e_r e_{r+1} \cdots e_n$. Since R is abelian, $e^2 = e \in R$. We show that $r_{R[x,x^{-1};\alpha]}(f) = eR[x, x^{-1}; \alpha]$. Since $R[x, x^{-1}; \alpha]$ is abelian by Corollary 9, and by Proposition 4 we have $\alpha(e) = e$ for each idempotent $e \in R$, whence $feR[x, x^{-1}; \alpha] = 0$. Thus $eR[x, x^{-1}; \alpha] \subseteq r_{R[x,x^{-1};\alpha]}(f)$. Now suppose that $g = b_k x^k + b_{k+1} x^{k+1} + \cdots + b_0 + \cdots + b_m x^m \in r_{R[x,x^{-1};\alpha]}(f)$. Then we have $fg = 0$. Since R is α-rigid, by the same argument as in the proof of Theorem 12, we deduce that $a_i b_j = 0$ for all $r \leq i \leq n$ and $k \leq j \leq m$. Thus for each $k \leq j \leq m$, $b_j \in r_R(a_i)$ for all $r \leq i \leq n$. Hence $b_j = eb_j$ for all $k \leq j \leq m$. Thus $eR[x, x^{-1}; \alpha] \supseteq r_{R[x,x^{-1};\alpha]}(f)$. Therefore $eR[x, x^{-1}; \alpha] = r_{R[x,x^{-1};\alpha]}(f)$. The converse is similar to the proof of Theorem 12. □

Corollary 16. *If R is an Armendariz ring, then R is a p.p.-ring if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is a p.p.-ring.*

Corollary 17. *If R is a reduced ring, then R is a p.p.-ring if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is a p.p.-ring.*

4. Some extensions of α-skew-Armendariz rings

Let R be a ring and let

$$T(R,n) := \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\},$$

with $n \geq 2$. Then $T(R,n)$ is a subring of the triangular matrix ring $T_n(R)$. We can denote elements of $T(R,n)$ by $(a_1, a_2, \ldots, a_n)$. In the case $n = 2$ it is the trivial extension of R, and is denoted by $T(R,R)$. For an endomorphism α of R, the natural extension $\overline{\alpha} : T(R,n) \to T(R,n)$ defined by $\overline{\alpha}((a_i)) = (\alpha(a_i))$ is an endomorphism of $T(R,n)$.

Theorem 18. *Let R be a ring and α a monomorphism of R. Then the following are equivalent*:

(1) *R is α-rigid.*
(2) *For some $n \geq 3$, $T(R,n)$ is an α-skew Armendariz ring.*
(3) *For each n, $T(R,n)$ is an α-skew Armendariz ring.*

Proof. (1) $\Rightarrow$ (3). Suppose that R is α-rigid. Now observe that $T(R,n)[x;\alpha] \cong T(R[x;\alpha],n)$, given by $Ax^j \to (a_1x^j, a_2x^j, \ldots, a_nx^j)$, where $A = (a_1, a_2, \ldots, a_n)$. Assume that $fg = 0$ for $f, g \in T(R,n)[x;\alpha]$ with $f(x) = A_0 + A_1x + \cdots + A_tx^t$ and $g(x) = B_0 + \cdots + B_mx^m$ with $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and $B_j = (b_{j1}, b_{j2}, \ldots, b_{jn})$. Then using the above isomorphism we have for each $0 \leq i \leq n$, $0 \leq j \leq n-i+1$, $f_ig_j = 0$ with $f_i(x) = a_{0i} + a_{1i}x + \cdots + a_{ti}x^t$ and $g_j = b_{0j} + \cdots + b_{mj}x^m \in R[x;\alpha]$. Since R is α-rigid, by Theorem 2, $a_{0i}b_{sj} = 0$, for each $0 \leq i \leq t$, $0 \leq j \leq t-i+1$ and each s. Thus $A_0B_s = 0$ for each $0 \leq s \leq m$. (2) $\Rightarrow$ (1). Assume that for some $n \geq 3$, $T(R,n)$ is an α-skew Armendariz ring. To show that R is α-rigid, let $r \in R$ and $r\alpha(r) = 0$. Consider $h(x) = (0,0,1,0,\ldots,0) - (0,\alpha(r),0,\ldots,0)x$ and $k(x) = (0,0,\ldots,0,1,0) + (0,0,\ldots,\alpha(r),0,0)x$ in the ring $T(R,n)[x,x^{-1};\alpha]$. We have $h(x)k(x) = 0$ and $T(R,n)$ is an α-skew Armendariz ring, so $(0,0,1,0,\ldots,0)(0,0,\ldots,\alpha(r),0,0) = 0$. Hence $\alpha(r) = 0$ and $r = 0$, since α is a monomorphism. □

Corollary 19. *Let R be a ring and α a monomorphism of R. Then the following are equivalent*:

(i) *R is α-rigid.*
(ii) *For some $n \geq 3$, $R[x]/\langle x^n \rangle$ is an α-skew Armendariz ring.*
(iii) *For each n, $R[x]/\langle x^n \rangle$ is an α-skew Armendariz ring, where $\langle x^n \rangle$ is the ideal of $R[x]$ generated by x^n.*

Proof. Observe that $T(R,n) \cong R[x]/\langle x^n \rangle$, for each positive integer n. □

As a corollary of Theorem 18, we see that the trivial extension $T(R,R)$ is an α-skew Armendariz ring for every α-rigid ring R.

If R is any of the following examples of α-rigid rings, then the trivial extension $T(R,R)$ is a non reduced α-skew Armendariz ring:

Examples 20. (i) Let D be a domain and $R = D[x_1, \ldots, x_n]$ the polynomial ring over D, with indeterminates $x_1, \ldots, x_n$. Let α be an endomorphism on R given by $\alpha(x_i) = x_{i+1}$ for each $1 \le i \le n-1$ and $\alpha(x_n) = x_1$. Then R is an α-rigid ring.

(ii) Let D be a domain and $R = D[x_1, x_2, \ldots]$ the polynomial ring over D, with indeterminates $x_1, x_2, \ldots$. Let α be an endomorphism on R given by $\alpha(x_i) = x_{i+1}$ for each $i \ge 1$. Then R is an α-rigid ring.

Examples 21. Let R be a domain and α an endomorphism on the polynomial ring $R[x]$ given by $\alpha(f(x)) = f(0)$. Then $R[x]$ is a non-rigid α-skew Armendariz ring.

Examples 22. Let S be a right Ore domain and K its ring of fractions. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in S, b \in K \right\}.$$

For each non-zero element $c \in S$ consider the endomorphism $\alpha_c : R \to R$ given by

$$\alpha_c\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & bc^{-1} \\ 0 & a \end{pmatrix}.$$

Then R is an α-skew Armendariz ring. To see this, let $p = A_0 + A_1x + \cdots + A_nx^n$ and $q = B_0 + \cdots + B_mx^m \in R[x;\alpha]$, with $pq = 0$,

$$A_i = \begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix} \text{ and } B_j = \begin{pmatrix} e_j & f_j \\ 0 & e_j \end{pmatrix}.$$

If $A_0 = 0$, so $A_0B_j = 0$ for each $0 \le j \le m$. If $a_0 \ne 0$, then since $A_0B_0 = 0$, we have $a_0e_0 = 0$, and so $e_0 = 0$. Also $a_0f_0 + b_0e_0 = 0$ implies $f_0 = 0$. Hence $B_0 = 0$. By a similar argument since $A_0B_1 = 0$, we have $B_1 = 0$. Therefore $A_0B_j = 0$ for each $0 \le j \le m$. Now if $a_0 = 0$, and for each j, $e_j = 0$, then $A_0B_j = 0$ for each j. Thus assume that for some t, $e_t \ne 0$ and that $e_0 = e_1 = \cdots = e_{t-1} = 0$. Then we have $(a_0 + a_1x + \cdots + a_nx^n)(e_tx^t + e_{t+1}x^{t+1} + \cdots + e_mx^m) = 0$, since $pq = 0$. Thus we have $a_0e_{t+1} + a_1e_t = 0$. So $a_1e_t = 0$ and hence $a_1 = 0$. By the same method we can see that $a_i = 0$ for each i. Now we have $A_0B_t + A_1\alpha(B_{t-1}) + \cdots + A_t\alpha^t(B_0) = 0$. So $a_0f_t + b_0e_t + a_1(f_{t-1}/c) + b_1e_{t-1} + \cdots + a_t(f_0/c^t) + b_te_0 = 0$. Thus $b_0 = 0$ and $A_0 = 0$. Hence $A_0Bj = 0$ for each j. Therefore R is an α-skew Armendariz ring.

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