Efficient Piecewise-Cubic Polynomial Curve Approximation Using Uniform Metric

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Abstract—We present efficient algorithms for solving the piecewise-cubic approximation problems in the plane. Given a set D of n points in the plane, we find a piecewise-cubic polynomial curve passing through only the points of a subset S of D and approximating the other points using the uniform metric. The goal is to minimize the size of S for a given error tolerance ε, called the min-# problem, or to minimize the error tolerance ε for a given size of S, called the min-ε problem. We give algorithms with running times $O(n^2 \log n)$ and $O(n^3)$ for both problems, respectively.

Index Terms — piecewise-cubic, approximation, uniform metric.

I. INTRODUCTION

The polynomial curve fitting problem is to construct a piecewise-polynomial curve $p(x)$ passing through a sequence of data points $D = \{(x_i, y_i) : i = 0, 1, ..., n\}$ and satisfying the tangents at points, that is, $t_i = p'(x_i)$. This problem arises in many applications, such as CAD/CAM, CAGD (Computer Aided Geometric Design), Computer Graphics, etc. But it may be redundant to construct the curve passing through all p_i. We are interested in finding a piecewise-polynomial curve $p(x)$ to pass through a subset S of original data set D and to approximate the points in $D - S$. The measure of the approximating error is desired to be the well-known uniform metric, that is,

$$\|p - D\|_\infty = \max_{i \in \{1,2,...,n\}} |y_i - p(x_i)|,$$

which is also known as the L_∞ or *Chebychev metric.* In particular, we consider piecewise-cubic polynomial curves in this paper. Also we deal with two optimization problems, which are formally stated as follows:

The Min-# problem: Let a data set D in the plane and $\varepsilon \geq 0$ be given. Find a piecewise-cubic polynomial curve that passes through the smallest number of points in D among all approximating curves whose errors are at most ε.

The Min-ε problem: Let a data set D in the plane and an integer k be given. Find a piecewise-cubic polynomial curve that minimizes the error among all approximating curves passing through at most k points of D.

II. Related Work

The previous works are mainly focused on the piecewise-linear polynomial curves. That is, the polynomial segments of the curve are line segments.

Imai and Iri [5], and Melkman and O'Rourke [6] study the min-# and min-ε polygonal-lines approximation problems in the plane, that is, the 2-D space. For the L_2 metric, they achieve algorithms whose running times are $O(n^2 \log n)$ and $O(n^2 \log^2 n)$, respectively, and using $O(n^2)$ space, for the two problems. Chan and Chin [2] reduce the time complexities of both results to $O(n^2)$ and $O(n^2 \log n)$, respectively. Furthermore, Chen and Daescu [3] show that the algorithms of [2] can use only $O(n)$ space without increasing their running times.

Hakimi and Schmeichel [4] deal with L_∞ metric. They give algorithms whose running times are $O(n^2)$ and $O(n^2 \log n)$ for min-# and min-ε problems, respectively. For L_∞ metric, Varadarajan [8] studies the both problems for 2-D polygonal-lines that are monotone, i.e., any line parallel to the y axis intersects the line in a point. He proposes $O(n^{4/3+\delta})$ time and space algorithms for both problems, where $\delta \geq 0$ is an arbitrarily small constant.

For the 3-dimensional space, Barequet et.al. [1] give algorithms with running times $O(n^2 \log n)$ and $O(n^2 \log^3 n)$ for the min-# and min-ε problems in L_2 metric, respectively. For L_∞ metric, the times are reduced to $O(n^2)$ and $O(n^2 \log n)$, respectively.

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III. Min-# problem

The data points $p_i = (x_i, y_i)$ are given sorted, that is, $x_i < x_{i+1}$ and the tangents t_i at p_i are also given. For an error bound $\varepsilon \geq 0$, we construct a graph $G = (V, E)$, where each vertex v_i in V represents a point p_i, and each edge (v_i, v_j) is in G if and only if there is a cubic polynomial f such that $y_i = f(x_i), t_i = f'(x_i)$, $y_j = f(x_j), t_j = f'(x_j)$, and

$$|y_k - f(x_k)| \leq \varepsilon \text{ for all } k, i < k < j.$$

Then we will find a shortest path in G from v_1 to v_n, which corresponds to the approximating curve satisfying the error tolerance $\leq \varepsilon$ and having the minimum number of (cubic) polynomial segments. The shortest path can be easily found by a breadth-first search and the overall time is depended on the size of the graph G. So we concentrate on finding an efficient method for constructing G and we call the graph G a *shortcut graph* ([Fig. 1]).

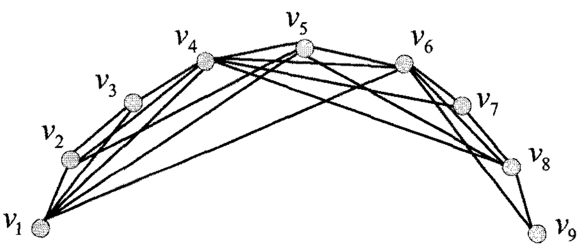

Fig. 1 Shortcut graph

Suppose a cubic polynomial segment is denoted by $f(x) = ax^3 + bx^2 + cx + d$. We shall test whether there is an edge (v_i, v_j) in G. First, we see that $y_i = f(x_i)$ and $t_i = f'(x_i)$. From these equations, there are the linear functions g and h of (a,b) such that $c = g(a,b)$ and $d = h(a,b)$. If the segment satisfies the error tolerance, then the following inequalities are satisfied. For all $k, i < k < j$,

$$y_k - \varepsilon \leq ax_k^3 + bx_k^2 + g(a,b)x_k + h(a,b) \leq y_k + \varepsilon$$

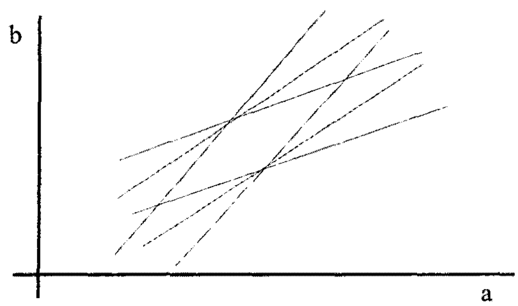

Fig. 2 Intersection of corridors

Each of the above inequalities represents the intersection of two half-planes, say a *corridor* in (a,b)-plane([Fig. 2]). A point (a,b) satisfying all the inequalities belongs to the intersection of the corridors. From the equations $y_j = f(x_j)$ and $t_j = f'(x_j)$, we can get a point (a_0, b_0) in (a,b)-plane. Thus there is an edge (v_i, v_j) in G if and only if the point (a_0, b_0) belongs to the intersection of the corridors. This is a special case of the point location problem for convex polygonal regions, which is solved in $O(\log n)$ time [7]. Also we can easily show that it is solved in $O(\log n)$ time to get the intersection of corridors when the corridors are given incrementally. Therefore, for a fixed node v_i, we can test whether there is an edge from v_i to v_j for all $j > i$, totally in $O(n \log n)$ time.

Theorem 1 For given n points and an error bound ε, we can construct the shortcut graph G in $O(n^2 \log n)$ time.

Proof. Let a node v_i be fixed. Then we incrementally consider the nodes v_j, $j > i$. Suppose it is time to test whether there is an edge from v_i to v_j. Then we already get the intersection I_{j-2} of corridors from the inequalities for the points p_k, $i < k < j-1$. Thus we can find the intersection I_{j-1} of I_{j-2} and the corridor for the point p_{j-1} in $O(\log n)$ time, because we can find the intersection points of I_{j-2} and the supporting lines of the corridor by the binary search on the boundary chain of I_{j-2}. Then the point location of (a_0, b_0) for v_j on I_{j-1} is solved in $O(\log n)$ time.

From the above argument, we can find the all edges going from a node in $O(n \log n)$ time. So all edges in G can be obtained in $O(n^2 \log n)$ time.

From Theorem 1, we get the shortcut graph G. Then we can find a shortest path in G from v_1 to v_n in $O(n^2)$ time. So the overall time is $O(n^2 \log n)$.

Theorem 2 For given n data points and an error bound ε, we can solve the min-# problem in $O(n^2 \log n)$ time.

IV. Min-ε problem

In this section, the min-ε problem is considered. Suppose we are given a set D of (sorted) points $p_i = (x_i, y_i)$ with tangents t_i in the plane and a fixed integer k. Then we wish to find the smallest $\varepsilon \geq 0$ such that there is a piecewise-cubic polynomial curve that has the error $\leq \varepsilon$ and that passes through at most k points of D, satisfying the tangents at the points to be passed.

For any polynomial f, let $d_{ij}(f)$ be defined by

$$d_{ij}(f) = \max_{i \leq k \leq j} |y_k - f(x_k)|.$$

Then for the (unique) polynomial f satisfying that $y_i = f(x_i), t_i = f'(x_i), y_j = f(x_j), t_j = f'(x_j)$, the error constant ε_{ij} is determined by $\varepsilon_{ij} = d_{ij}(f)$. This is easily obtained in $O(n)$ time. Thus the constants ε_{ij} $(i < j)$ are the candidates of the smallest error bound ε for the min-ε problem. Then for a candidate ε_{ij}, we solve the min-# problem. If the curve of the solution passes through the points $> k$, then $\varepsilon_{ij} < \varepsilon$. Otherwise, $\varepsilon_{ij} \geq \varepsilon$. So we can perform the binary search over the ε_{ij}'s, the number of which is $O(n^2)$.

Theorem 3 For given n points and an integer $k \geq 0$, we can solve the min-ε problem in $O(n^3)$ time.

Proof. From the above statements, we can get all the constants ε_{ij} for $i < j$ in $O(n^3)$. Since we can solve the min-# problem in $O(n^2 \log n)$, it takes totally $O(n^2 \log^2 n)$ time to perform the binary search over the ε_{ij}'s. So we can find the smallest error bound ε for the min-ε problem. It takes the overall time of $O(n^3)$.

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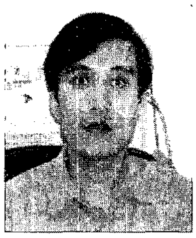

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