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CENTRAL HILBERT ALGEBRAS

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ABSTRACT. The notion of central Hilbert algebras and central deductive systems is introduced, and related properties are investigated. We show that the central part of a Hilbert algebra is a deductive system. Conditions for a subset of a Hilbert algebra to be a deductive system are given. Conditions for a subalgebra of a Hilbert algebra to be a deductive system are provided.

1. INTRODUCTION

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication ($\rightarrow$) and the constant 1 which is considered as the logical value "true". Several authors discussed deductive systems of Hilbert algebras including their properties (see References). In this paper, we introduce the notion of central Hilbert algebras and central deductive systems of Hilbert algebras. We show that the central part of a Hilbert algebra is a deductive system. We give conditions for a subset of a Hilbert algebra to be a deductive system, and provide conditions for a subalgebra of a Hilbert algebra to be a deductive system.

2. PRELIMINARIES

A Hilbert algebra can be considered as a fragment of propositional logic containing only a logical connective implication "$\rightarrow$" and the constant 1 which is interpreted as the value "true". In the following, the logical connective implication "$\rightarrow$" will be denoted by "$\cdot$".

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Let $K(\tau)$ be the class of all algebras of type τ. A *Hilbert algebra* is a system $(H, \cdot, 1) \in K(\tau)$, where $\tau = (2, 0)$, such that

(H1) $(\forall a, b \in H)\ (a \cdot (b \cdot a) = 1)$.
(H2) $(\forall a, b, c \in H)\ ((a \cdot (b \cdot c)) \cdot ((a \cdot b) \cdot (a \cdot c)) = 1)$.
(H3) $(\forall a, b \in H)\ (a \cdot b = b \cdot a = 1 \Rightarrow a = b)$.

The element 1 is called the *unit* of H. Denote by $\mathbb{H}$ the collection of Hilbert algebras. For any $H \in \mathbb{H}$, we define a binary relation $\preceq$ in H by $a \preceq b$ if and only if $a \cdot b = 1$, then $\preceq$ is a partial order in $H := (H; \cdot, 1)$. A Hilbert algebra $H := (H; \cdot, 1)$ is said to be *commutative* if $(x \cdot y) \cdot y = (y \cdot x) \cdot x$ for all $x, y \in H$.

In a Hilbert algebra $H := (H; \cdot, 1)$, we have the following assertions:

(a1) $x \preceq y \cdot x$.
(a2) $x \cdot 1 = 1,\ 1 \cdot x = x$.
(a3) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$.
(a4) $x \preceq (x \cdot y) \cdot y$.
(a5) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.
(a6) $x \cdot y \preceq (y \cdot z) \cdot (x \cdot z)$.
(a7) $x \preceq y \Rightarrow z \cdot x \preceq z \cdot y,\ y \cdot z \preceq x \cdot z$.

The concept of a deductive system on a Hilbert algebra $H := (H; \cdot, 1)$ was also introduced by A. Diego [4] as a subset of H containing 1 and closed under a "deduction", i.e.:

Definition 2.1. A nonempty subset D of a Hilbert algebra $H := (H; \cdot, 1)$ is called a *deductive system* of H if it satisfies:

(b1) $1 \in D$,
(b2) $(\forall x \in D)\ (\forall y \in H)\ (x \cdot y \in D \Rightarrow y \in D)$.

3. Central Hilbert Algebras

Definition 3.1. Let $H \in \mathbb{H}$. An element $w \in H$ is said to be *central* if it satisfies:

$$(\forall x \in H)\,(w \neq x \Rightarrow w \cdot x = x\ \&\ x \cdot w = w). \tag{3.1}$$

Denote by $\mathcal{C}(H)$ the set of all central elements of $H \in \mathbb{H}$. We call $\mathcal{C}(H)$ the *central part* of H. Obviously, $1 \in \mathcal{C}(H)$.

Example 3.2. Let $H = \{1, a, b, c, d\}$ be a set with the following Cayley table.

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	d
c	1	a	b	1	d
d	1	1	1	c	1

Then $H := (H, \cdot, 1)$ is a Hilbert algebra and $\mathcal{C}(H) = \{1, c\}$.

Example 3.3. Let $H = \{1, a, b, c\}$ be a set with the following Cayley table.

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Then $H := (H, \cdot, 1)$ is a Hilbert algebra and $\mathcal{C}(H) = \{1, a, b, c\} = H$.

Proposition 3.4. *Let $H \in \mathbb{H}$. If H forms a chain, then $\mathcal{C}(H) = \{1\}$.*

Proof. Let w be an element of H which is not the unit of H. If w satisfies $x \preceq w$ for some $x \in H$ (or, $w \preceq x$ for some $x(\neq 1) \in H$), then w can not be a central element of H. □

Definition 3.5. A Hilbert algebra H is said to be *central* if $\mathcal{C}(H) = H$.

Example 3.6. The Hilbert algebra H in Example 3.3 is central.

Example 3.7. Let $H = \{1, a, b, c, d\}$ be a set with the following Cayley table.

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	d
c	1	a	b	1	d
d	1	a	b	c	1

Then $H := (H, \cdot, 1)$ is a central Hilbert algebra.

Proposition 3.8. *Let $H \in \mathbb{H}$. If $|H| = 2$, then H is central.*

Proof. Straightforward. □

Theorem 3.9. *Let $H \in \mathbb{H}$. The central part of H is a deductive system of H.*

Proof. Note that $1 \in \mathcal{C}(H)$. Let $x, y \in H$ be such that $x \cdot y \in \mathcal{C}(H)$ and $x \in \mathcal{C}(H)$.

We will show $y \in \mathcal{C}(H)$. If $x = y$, then it is clear. Suppose $x \neq y$. Then $x \in \mathcal{C}(H)$ yields $y = x \cdot y \in \mathcal{C}(H)$. Hence $\mathcal{C}(H)$ is a deductive system of H. □

We give a condition for a subset D of $H \in \mathbb{H}$ to be a deductive system of H.

Theorem 3.10. *Let $H \in \mathbb{H}$. If D is a subset of H satisfying $1 \in D \subseteq \mathcal{C}(H)$, then D is a deductive system of H.*

Proof. Let $x, y \in H$ be such that $x \in D$ and $x \cdot y \in D$. Since x is central, we have $y = x \cdot y \in D$. Hence D is a deductive system of H. □

Corollary 3.11. *Let $H \in \mathbb{H}$. If H is central, then every subset of H containing the unit 1 is a deductive system of H.*

Proof. Straightforward. □

Corollary 3.12. *Let $H \in \mathbb{H}$ be central. Then*

$$(\forall n \in \mathbb{N})\ (|H \setminus \{1\}| = n \ \Rightarrow\ |Ds(H)| = 2^n)$$

where $Ds(H)$ is the set of all deductive systems of H.

Proof. Straightforward. □

Note that a subalgebra of $H \in \mathbb{H}$ may not be a deductive system of H. In fact, let $H = \{1, a, b, c, d\}$ be a set with the following Cayley table.

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	1	1	1	d
d	1	1	1	c	1

Then $H := (H, \cdot, 1)$ is a Hilbert algebra and $S := \{1, a, c\}$ is a subalgebra of H. But S is not a deductive system of H since $c \cdot b = 1 \in S$ and $b \notin S$.

We provide a condition for a subalgebra to be a deductive system.

Theorem 3.13. *Let $H \in \mathbb{H}$. If H is central, then every subalgebra of H is a deductive system of H.*

Proof. Assume that H is central. Let S be a subalgebra of H. Then $1 = x \cdot x \in S$ for all $x \in S$. Let $x, y \in H$ be such that $x \in S$ and $x \cdot y \in S$. Since x is central, it follows that $y = x \cdot y \in S$. Hence S is a deductive system of H. □

Definition 3.14. Let $H \in \mathbb{H}$. A deductive system D of H is said to be *central* if $\mathcal{C}(H) \subset D$.

Example 3.15. In Example 3.2, the sets $D_1 := \{1, a, c\}$, $D_2 := \{1, b, c\}$, $D_3 := \{1, a, b, c\}$ and X are central deductive systems of H. But $E_1 := \{1, a\}$, $E_2 := \{1, b\}$, $E_3 := \{1, a, b\}$ and $E_4 := \{1, a, b, d\}$ are deductive systems of H which are not central.

Note that a central Hilbert algebra does not contain proper central deductive systems.

Theorem 3.16. *Let $H \in \mathbb{H}$. If D_1 and D_2 are central deductive systems of H, then so is $D_1 \cap D_2$.*

Proof. Straightforward. □

In general, the union of two deductive systems of a Hilbert algebra H may not be a deductive system of H. But, if the union of two deductive systems of a Hilbert algebra H is a deductive system of H, then the union of two central deductive systems of a Hilbert algebra H is a central deductive system of H.

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