

RIGIDITY OF MINIMAL SUBMANIFOLDS WITH FLAT NORMAL BUNDLE

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ABSTRACT. Let M^n be a complete immersed super stable minimal submanifold in \mathbb{R}^{n+p} with flat normal bundle. We prove that if M has finite total L^2 norm of its second fundamental form, then M is an affine n -plane. We also prove that any complete immersed super stable minimal submanifold with flat normal bundle has only one end.

1. Introduction

Let M be an n -dimensional complete minimal submanifold in \mathbb{R}^{n+p} . When $n = 2$ and $p = 1$, do Carmo and Peng [3], Fischer-Colbrie and Schoen [5] independently showed that the only complete stable minimal surface is a plane. Recall that a minimal submanifold is stable if the second variation of its volume is always nonnegative for any normal variation with compact support. For higher dimensional minimal hypersurfaces, do Carmo and Peng [4] generalized the result mentioned as above. We will denote by A the second fundamental form of M .

Theorem ([4]). *Let M^n be a complete stable minimal hypersurface in \mathbb{R}^{n+1} satisfying $\int_M |A|^2 dv < \infty$. Then M must be a hyperplane.*

It was shown by Shen and Zhu [9] that if M is a stable minimal hypersurface with finite total scalar curvature (i.e., $\int_M |A|^n dv < \infty$), then M is a hyperplane. In higher codimensional cases, Spruck [11] proved that for a variation vector field $E = \varphi\nu$, the second variation of $\text{Vol}(M_t)$ satisfies

$$\frac{d^2 \text{Vol}(M_t)}{dt^2} \geq \int_M \left(|\nabla \varphi|^2 - |A|^2 \varphi^2 \right) dv,$$

where ν is the unit normal vector field and $\varphi \in W_0^{1,2}(M)$. Motivated by this, Wang [13] introduced the concept of super stability to prove that if M^n ($n \geq 3$) is a complete super stable minimal submanifold with finite total scalar curvature in \mathbb{R}^{n+p} , then M is an affine n -plane.

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Definition 1.1 ([13]). We call a minimal submanifold M in \mathbb{R}^{n+p} *super stable* if

$$\int_M (|\nabla\varphi|^2 - |A|^2\varphi^2)dv \geq 0$$

holds for all $\varphi \in W_0^{1,2}(M)$, the space of $W^{1,2}$ functions with compact support in M .

When $p = 1$, the definition of super stability is exactly the same as that of stability and the normal bundle is trivially flat. However, the normal bundle becomes complicated in higher codimension. We consider the simplest case when the normal bundle is flat. Submanifolds with flat normal bundle was studied by Terng [12]. Recently Smoczyk, Wang, and Xin [10] proved a Bernstein type theorem for minimal submanifolds in \mathbb{R}^{n+p} with flat normal bundle under a certain growth condition. We now state our first result as follows.

Theorem 1.2. *Let M^n be a complete immersed super stable minimal submanifold in \mathbb{R}^{n+p} with flat normal bundle satisfying that $\int_M |A|^2dv < \infty$. Then M is an affine n -plane.*

Theorem 1.2 can be regarded as generalization of the above theorem due to do Carmo and Peng. It was proved by Cao, Shen, and Zhu [1] that a complete immersed stable minimal hypersurface $M^n (n \geq 3)$ in \mathbb{R}^{n+1} has only one end. We also generalize their result.

Theorem 1.3. *Any complete immersed super stable minimal submanifold with flat normal bundle has only one end.*

2. Preliminaries

We follow the notations of Chern-do Carmo-Kobayashi [2].

Now we choose an orthonormal frame e_1, \dots, e_{n+p} in \mathbb{R}^{n+p} such that, restricted to M , the vectors e_1, \dots, e_n are tangent to M and e_{n+1}, \dots, e_{n+p} are normal to M . And we shall denote the second fundamental form by h_{ij}^α . Then we have $|A|^2 = \sum (h_{ij}^\alpha)^2$ and

$$2|A|\Delta|A| + 2|\nabla|A||^2 = \Delta|A|^2 = 2 \sum (h_{ijk}^\alpha)^2 + 2 \sum h_{ij}^\alpha \Delta h_{ij}^\alpha.$$

By Chern-do Carmo-Kobayashi ([2], (2.23)), we have

$$\sum h_{ij}^\alpha \Delta h_{ij}^\alpha = - \sum (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)(h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta) - \sum h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta.$$

Since M has flat normal bundle, we have $h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta = 0$. Therefore, we get

$$\sum h_{ij}^\alpha \Delta h_{ij}^\alpha = - \sum h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta.$$

For each α , let H_α denote the symmetric matrix (h_{ij}^α) , and set $S_{\alpha\beta} = \sum h_{ij}^\alpha h_{ij}^\beta$. Then the $(p \times p)$ matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal

for a suitable choice of e_{n+1}, \dots, e_{n+p} . Thus we have

$$\sum h_{ij}^\alpha \Delta h_{ij}^\alpha = - \sum S_{\alpha\alpha}^2 = - \sum_\alpha \left(\sum_{i,j} (h_{ij}^\alpha)^2 \right)^2.$$

Moreover

$$|A|^4 = (|A|^2)^2 = \left(\sum_\alpha \sum_{i,j} (h_{ij}^\alpha)^2 \right)^2 \geq \sum_\alpha \left(\sum_{i,j} (h_{ij}^\alpha)^2 \right)^2.$$

Hence we have

$$\begin{aligned} & 2|A|\Delta|A| + 2|\nabla|A||^2 \\ &= \Delta|A|^2 = 2 \sum (h_{ijk}^\alpha)^2 + 2 \sum h_{\alpha ij} \Delta h_{ij}^\alpha \geq 2 \sum (h_{ijk}^\alpha)^2 - 2|A|^4. \end{aligned}$$

Since $\sum (h_{ijk}^\alpha)^2 = |\nabla A|^2$, we get

$$(2.1) \quad |A|\Delta|A| + |A|^4 \geq |\nabla A|^2 - |\nabla|A||^2.$$

From (2.1) and curvature estimate by Y. Xin ([14], Lemma 3.1), we obtain

$$(2.2) \quad |A|\Delta|A| + |A|^4 \geq \frac{2}{n} |\nabla|A||^2.$$

We also need the following lemma.

Lemma 2.1 ([6]). *Let M^n be a complete immersed minimal submanifold in \mathbb{R}^{n+p} . Then the Ricci curvature of M satisfies*

$$\text{Ric}(M) \geq -\frac{n-1}{n} |A|^2.$$

3. Proofs of the theorems

Proof of Theorem 1.2. First we have

$$\frac{1}{2} \Delta(|A|^2) = \text{div}(|A|\nabla|A|) = |A|\Delta|A| + |\nabla|A||^2.$$

Therefore from (2.2) we get

$$\frac{1}{2} \Delta(|A|^2) + |A|^4 \geq \frac{n+2}{n} |\nabla|A||^2.$$

Fix a point $p \in M$ and for $R > 0$ choose a cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_p(R)$, $\varphi = 0$ on $M \setminus B_p(2R)$, and $|\nabla\varphi| \leq \frac{1}{R}$. Multiplying both sides by φ^2 and integrating over M , we have

$$\frac{1}{2} \int_M \varphi^2 \Delta(|A|^2) dv + \int_M \varphi^2 |A|^4 dv \geq \frac{n+2}{n} \int_M \varphi^2 |\nabla|A||^2 dv.$$

Using integration by parts, we get

$$(3.1) \quad -2 \int_M \varphi |A| \langle \nabla\varphi, \nabla|A| \rangle dv + \int_M \varphi^2 |A|^4 dv \geq \frac{n+2}{n} \int_M \varphi^2 |\nabla|A||^2 dv.$$

Since M is super stable, we have for any $\varphi \in W_0^{1,2}(M)$

$$\int_M |\nabla\varphi|^2 dv \geq \int_M |A|^2 \varphi^2 dv.$$

Replacing φ by $|A|\varphi$, we have

$$\begin{aligned} (3.2) \quad & \int_M \varphi^2 |\nabla|A||^2 dv + \int_M |A|^2 |\nabla\varphi|^2 dv + 2 \int_M \varphi |A| \langle \nabla\varphi, \nabla|A| \rangle dv \\ & \geq \int_M \varphi^2 |A|^4 dv. \end{aligned}$$

Combining (3.1) and (3.2), we obtain

$$\int_M \varphi^2 |\nabla|A||^2 dv + \int_M |A|^2 |\nabla\varphi|^2 dv \geq \frac{n+2}{n} \int_M \varphi^2 |\nabla|A||^2 dv.$$

Hence

$$\frac{2}{n} \int_M \varphi^2 |\nabla|A||^2 dv \leq \int_M |A|^2 |\nabla\varphi|^2 dv \leq \frac{1}{R^2} \int_M |A|^2 dv.$$

Letting $R \rightarrow \infty$, we obtain that $|A|$ is constant. Since the volume of a complete minimal submanifold is infinite, we conclude that M is an affine n -plane. \square

Proof of Theorem 1.3. By using the same arguments as in [1] and the following lemma which is an analogue of Schoen-Yau’s result ([8], Lemma 2), we can prove Theorem 1.3. \square

Lemma 3.1. *Let M^n be a complete super stable minimal submanifold in \mathbb{R}^{n+p} . Then any harmonic function with finite total energy has to be constant.*

Proof. We use the same arguments as in [8].

Let f be a harmonic function on M with finite total energy. The super stability implies that

$$\int_M |A|^2 \varphi^2 dv \leq \int_M |\nabla\varphi|^2 dv.$$

Replacing φ by $|\nabla f|\varphi$ and integrating by parts, we have

$$\begin{aligned} (3.3) \quad & \int_M |A|^2 |\nabla f|^2 \varphi^2 dv \\ & \leq \int_M |\nabla\varphi|^2 dv \\ & = \int_M |\nabla f|^2 |\nabla\varphi|^2 + 2\varphi |\nabla f| \langle \nabla|\nabla f|, \nabla\varphi \rangle + |\nabla|\nabla f||^2 \varphi^2 dv \\ & = \int_M |\nabla f|^2 |\nabla\varphi|^2 dv - \frac{1}{2} \int_M \varphi^2 \Delta(|\nabla f|^2) dv + \int_M |\nabla|\nabla f||^2 \varphi^2 dv. \end{aligned}$$

Using Bochner formula for the harmonic function f , we know

$$\frac{1}{2} \Delta(|\nabla f|^2) = |\nabla df|^2 + \text{Ric}(\nabla f, \nabla f).$$

Then Lemma 2.1 gives

$$(3.4) \quad \frac{1}{2}\Delta(|\nabla f|^2) \geq |\nabla df|^2 - \frac{n-1}{n}|A|^2|\nabla f|^2.$$

By (3.3) and (3.4), we have

$$\begin{aligned} \int_M |A|^2|\nabla f|^2\varphi^2 dv &\leq \int_M |\nabla f|^2|\nabla\varphi|^2 dv + \int_M |\nabla|\nabla f||^2\varphi^2 dv \\ &\quad + \frac{n-1}{n} \int_M |A|^2|\nabla f|^2\varphi^2 dv - \int_M |\nabla df|^2\varphi^2 dv. \end{aligned}$$

Therefore we get

$$(3.5) \quad \begin{aligned} 0 &\leq \frac{1}{n} \int_M |A|^2|\nabla f|^2\varphi^2 dv \\ &\leq \int_M |\nabla f|^2|\nabla\varphi|^2 dv + \int_M (|\nabla|\nabla f||^2 - |\nabla df|^2)\varphi^2 dv. \end{aligned}$$

Let $\{e_i\}$ be an orthonormal frame in a neighborhood of $p \in M$. On the other hand, we have

$$\begin{aligned} |\nabla df|^2 - |\nabla|\nabla f||^2 &= \sum f_{ij}^2 - \frac{\sum_l (\sum_k f_k f_{kl})^2}{\sum_m f_m^2} \\ &= \frac{1}{2\sum_m f_m^2} \sum_{i,j,k} (f_i f_{kj} - f_k f_{ij})^2 \\ &\geq \frac{1}{2\sum_m f_m^2} \sum_{i,j} (f_i f_{jj} - f_j f_{ij})^2 \\ (\text{by Schwarz inequality}) &\geq \frac{1}{2n\sum_m f_m^2} \sum_i \left(\sum_j (f_i f_{jj} - f_j f_{ij}) \right)^2. \end{aligned}$$

Since f is harmonic, i.e., $\sum f_{jj} = 0$, we obtain

$$|\nabla df|^2 - |\nabla|\nabla f||^2 \geq \frac{1}{2n\sum_m f_m^2} \sum_i \left(\sum_j f_j f_{ij} \right)^2 = \frac{1}{2n} |\nabla|\nabla f||^2.$$

Hence the inequality (3.5) becomes

$$\frac{1}{2n} \int_M |\nabla|\nabla f||^2\varphi^2 dv \leq \int_M |\nabla f|^2|\nabla\varphi|^2 dv.$$

Choosing φ as in the proof of Theorem 1.2 we get

$$\frac{1}{2n} \int_{B_R(p)} |\nabla|\nabla f||^2 dv \leq \frac{1}{R^2} \int_M |\nabla f|^2 dv.$$

Letting $R \rightarrow \infty$, we obtain that $|\nabla f|$ is constant. Since f has finite total energy and the volume of M is infinite, it follows that f is constant. \square

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