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ON QUASI EINSTEIN MANIFOLDS

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ABSTRACT. The object of the present paper is to study some properties of a quasi Einstein manifold. A non-trivial concrete example of a quasi Einstein manifold is also given.

1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([1], p. 432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], pp. 432–433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation :

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1.2}$$

where $a, b \in \mathbf{R}$ and A is a non-zero 1-form such that

$$g(X, U) = A(X) \tag{1.3}$$

for all vector fields X.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [7] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (1.2). We shall call A the associated 1-form and U is called the generator of the manifold. In this paper we consider U as a unit vector field.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker

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spacetime are quasi Einstein manifolds [8]. Also quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [4]. So quasi Einstein manifolds have some importance in the general theory of relativity. Considering this aspect we are motivated to study such a manifold.

It is to be noted that M. C. Chaki and R. K. Maity [2] also introduced the notion of quasi Einstein manifolds which is different from that of R. Deszcz [7]. They took a and b as scalars and the generator U of the manifold as a unit vector field. The study of quasi Einstein manifolds was continued by M. C. Chaki [3], S. Guha [10], U. C. De and G. C. Ghosh [5], [6] and many others.

Section 2 of this paper contains a non-trivial concrete example of a quasi Einstein manifold. In Section 3 we enquire under what condition the associated 1-form A will be closed in a quasi Einstein manifold. In Section 4 we prove that if a quasi Einstein manifold satisfies cyclic Ricci tensor, then the integral curves of the vector field U are geodesic. In Section 5 it is shown that if U is a Killing vector field, then the quasi Einstein manifold satisfies cyclic Ricci tensor. Finally we study a quasi Einstein manifold admitting a conformal Killing vector field.

2. Example of a quasi Einstein manifold

In this section we construct a metric of quasi Einstein manifold.

Let E^5 be a Euclidean space with Cartesian coordinates (x^1, x^2, y^1, y^2, z) or, $(x^\alpha, y^\alpha, z)(\alpha = 1, 2)$. Let us consider

$$A = \sqrt{(\frac{3}{2b})}(dz - \sum_{\alpha=1}^{2} y^\alpha dx^\alpha), \text{ (where } b \text{ is a constant)}. \tag{2.1}$$

If we put

$$x^{\alpha^*} \equiv x^{2+\alpha} = y^\alpha, \quad x^\triangle = z, \quad \triangle = 5, \tag{2.2}$$

we have from (2.1) that

$$A_i = (-\sqrt{(\frac{3}{2b})}y^\alpha, 0, \sqrt{(\frac{3}{2b})}). \tag{2.3}$$

Now we consider a symmetric tensor field in E^5 defined by

$$g_{ij} = \begin{pmatrix} \frac{1}{4}(1+(y^1)^2) & \frac{y^1y^2}{4} & 0 & 0 & -\frac{1}{4}y^1 \\ \frac{1}{4}y^1y^2 & \frac{1}{4}(1+(y^2)^2) & 0 & 0 & -\frac{1}{4}y^2 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ -\frac{1}{4}y^1 & -\frac{1}{4}y^2 & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{2.4}$$

Then g_{ij} defines a positive definite Riemannian metric. The contravariant components of the tensor g^{ij} are given by

$$g^{ij} = \begin{pmatrix} 4 & 0 & 0 & 0 & 4y^1 \\ 0 & 4 & 0 & 0 & 4y^2 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 4y^1 & 4y^2 & 0 & 0 & 4(1+y^1+y^2) \end{pmatrix}. \tag{2.5}$$

We find the Christoffel symbols, by means of (2.4) and

$$[ij,r] = \frac{1}{2}[\frac{\partial g_{jr}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r}].$$

We can verify that

$$[\alpha\beta^*, \gamma] = \frac{1}{8}(\delta_{\gamma\beta}y^\alpha + \delta_{\alpha\beta}y^\gamma), \tag{2.6}$$

$$[\alpha\beta, \gamma^*] = -\frac{1}{8}(\delta_{\gamma\alpha}y^\beta + \delta_{\beta\gamma}y^\alpha), \tag{2.7}$$

$$[\alpha\beta^*, \triangle] = -\frac{\delta_{\alpha\gamma}}{8}, \quad [\alpha^*\triangle, \gamma] = -\frac{\delta_{\alpha\gamma}}{8}, \tag{2.8}$$

$$[\alpha\triangle, \gamma^*] = \frac{\delta_{\alpha\gamma}}{8}, \tag{2.9}$$

the other components are zero. We first show that the symmetry of the indices α, β^* of the Christoffel symbols. That is, we show that $[\alpha\beta^*, \gamma] = [\beta^*\alpha, \gamma]$. There are total 8 components which are $[13, 1]$, $[13, 2]$, $[23, 1]$, $[23, 2]$, $[14, 1]$, $[14, 2]$, $[24, 1]$, and $[24, 2]$.

We take any one of the above components, say, $[23, 1]$. Then

$$\begin{aligned} [23,1] &= \frac{1}{2}[\frac{\partial g_{31}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^3} - \frac{\partial g_{23}}{\partial x^1}] \\ &= \frac{1}{2}(0 + \frac{1}{4}y^2 - 0), \quad \text{using (2.2) and (2.4)} \\ &= \frac{1}{8}y^2 \end{aligned}$$

and

$$\begin{aligned} [32,1] &= \frac{1}{2}[\frac{\partial g_{21}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^2} - \frac{\partial g_{32}}{\partial x^1}] \\ &= \frac{1}{2}(\frac{1}{4}y^2 + 0 - 0), \quad \text{using (2.2) and (2.4)} \\ &= \frac{1}{8}y^2, \end{aligned}$$

that is,

$$[23,1] = [32,1] = \frac{1}{8}y^2.$$

We also show that the symmetry of the Christoffel symbols $[\alpha\delta, \gamma^*] = [\delta\alpha, \gamma^*]$. There are total 4 components which are $[15, 3]$, $[15, 4]$, $[25, 3]$, $[25, 4]$.

We take any one of the above components, say, $[15, 3]$. Then

$$\begin{aligned}[15,3] &= \frac{1}{2}[\frac{\partial g_{53}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^5} - \frac{\partial g_{15}}{\partial x^3}] \\ &= \frac{1}{2}(0 + 0 - (-\frac{1}{4})), \text{ using (2.2) and (2.4)} \\ &= \frac{1}{8}\end{aligned}$$

and

$$\begin{aligned}[51,3] &= \frac{1}{2}[\frac{\partial g_{13}}{\partial x^5} + \frac{\partial g_{53}}{\partial x^1} - \frac{\partial g_{51}}{\partial x^3}] \\ &= \frac{1}{2}(0 + 0 - (-\frac{1}{4})), \text{ using (2.2) and (2.4)} \\ &= \frac{1}{8},\end{aligned}$$

that is,

$$[15, 3] = [51, 3] = \frac{1}{8}.$$

The equations from (2.5) to (2.9) with the help of $\left\{ {h \atop i\ \ j} \right\} = g^{hr}[ij, r]$ implies that

$$\text{(2.10)}\quad \left\{ {\mu \atop \alpha\ \ \beta^*} \right\} = \frac{1}{2}\delta_{\mu\beta}y^\alpha, \quad \left\{ {\mu^* \atop \alpha\ \ \beta} \right\} = -\frac{1}{2}(\delta_{\alpha\mu}y^\beta + \delta_{\mu\beta}y^\alpha),$$

$$\text{(2.11)}\quad \left\{ {\mu^* \atop \alpha\ \ \triangle} \right\} = \frac{1}{2}\delta_{\alpha\mu}, \quad \left\{ {\triangle \atop \alpha\ \ \beta^*} \right\} = \frac{1}{2}(y^\alpha y^\beta - \delta_{\alpha\beta}),$$

$$\text{(2.12)}\quad \left\{ {\triangle \atop \triangle\ \ \beta^*} \right\} = -\frac{1}{2}y^\beta, \quad \left\{ {\mu \atop \alpha^*\ \ \triangle} \right\} = -\frac{1}{2}\delta_{\alpha\mu},$$

the other components are zero. After straightforward calculations, we obtain the independent components of the curvature tensor R_{ijkl} as follows

$$\text{(2.13)}\quad R_{\delta\gamma\beta\alpha} = \frac{1}{16}(\delta_{\alpha\beta}y^\gamma y^\delta + \delta_{\beta\gamma}y^\alpha y^\delta - \delta_{\alpha\gamma}y^\beta y^\delta - \delta_{\beta\delta}y^\alpha y^\gamma),$$

$$\text{(2.14)}\quad R_{\delta^*\gamma^*\beta\alpha} = \frac{1}{16}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}), \quad R_{\delta\gamma\triangle\alpha} = (\delta_{\alpha\gamma}y^\delta - \delta_{\alpha\delta}y^\gamma),$$

$$\text{(2.15)}\quad R_{\delta\gamma^*\beta^*\alpha} = \frac{1}{16}(\delta_{\beta\gamma}y^\alpha y^\delta - 2\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}),$$

$$\text{(2.16)}\quad R_{\delta\triangle\triangle\alpha} = \frac{1}{16}\delta_{\alpha\delta}, \quad R_{\triangle\gamma^*\triangle\alpha^*} = -\frac{1}{16}\delta_{\alpha\gamma}, \quad R_{\delta^*\triangle\beta^*\alpha} = \frac{1}{16}\delta_{\beta\delta}y^\alpha,$$

the other independent components are zero. From (2.5), (2.13), (2.14), (2.15) and (2.16), it follows that the Ricci tensor has the following non-zero components.

$$R_{\delta\alpha} = -\frac{1}{2}(\delta_{\alpha\delta} - 2y^\alpha y^\delta), \quad R_{\delta^*\alpha} = 0, \quad R_{\triangle\triangle} = 1, \tag{2.17}$$

$$R_{\delta^*\alpha^*} = -\frac{\delta_{\alpha\delta}}{2}, \quad R_{\gamma^*\triangle} = 0, \quad R_{\gamma\triangle} = -y^\gamma. \tag{2.18}$$

Now from the equations (2.3), (2.4), (2.17) and (2.18) we have

$$R_{ij} = -2g_{ij} + bA_iA_j\,. \tag{2.19}$$

Hence E^5 with the metric (2.4) is a quasi Einstein manifold. Summing up we can state the following:

Theorem 2.1. *The E^5 with the metric* (2.4) *is a quasi Einstein manifold where the associated 1-form A is given by* (2.1).

3. Nature of the associated 1-form of a quasi Einstein manifold

A Riemannian manifold is said to satisfy Codazzi type of Ricci tensor [9] if its Ricci tensor S satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{3.1}$$

Now from (1.2) we obtain

$$(\nabla_X S)(Y,Z) = b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)].$$

Hence from (3.1) we have

$$b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)] = b[(\nabla_Y A)(X)A(Z) + (\nabla_Y A)(Z)A(X)]$$

or,

$$(\nabla_X A)(Y)A(Z) - (\nabla_Y A)(X)A(Z) + (\nabla_X A)(Z)A(Y) - (\nabla_Y A)(Z)A(X). \tag{3.2}$$

Since equation (3.2) holds for all Z, putting $Z = U$ and using $(\nabla_X A)(U) = 0$, since U is a unit vector, we have

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0, \tag{3.3}$$

or,

$$dA(X,Y) = 0. \tag{3.4}$$

Thus we can state the following theorem:

Theorem 3.1. *If a quasi Einstein manifold satisfies the Codazzi type of Ricci tensor, then the associated 1-form A is closed.*

4. Quasi Einstein manifold satisfying cyclic Ricci tensor

Let us suppose that the manifold satisfies cyclic Ricci tensor. Then

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{4.1}$$

It is known [11] that Cartan hypersurfaces are manifolds, with non-parallel Ricci tensor, satisfying the form (4.1). From (1.2) we obtain

$$(\nabla_X S)(Y,Z) = b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)].$$

Hence from (4.1) we have

$$\begin{aligned}&(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y) + (\nabla_Y A)(X)A(Z)\\ &+(\nabla_Y A)(Z)A(X) + (\nabla_Z A)(Y)A(X) + (\nabla_Z A)(X)A(Y) = 0.\end{aligned} \tag{4.2}$$

Contracting Y and Z in (4.2) we have

$$\begin{aligned} 2(\nabla_U A)(X) &= 0, \quad \text{since } (\nabla_X A)(U) = 0,\\ &\text{or,} \quad (\nabla_U A)(X) = 0\\ &\text{or,} \quad g(X, \nabla_U U) = 0 \quad \text{for all } X. \end{aligned} \tag{4.3}$$

Equation (4.3) implies $\nabla_U U = 0$. This leads to the following theorem:

Theorem 4.1. *If a quasi Einstein manifold satisfies cyclic Ricci tensor, then the integral curves of the vector field U are geodesic.*

5. The generator U as a Killing vector field

In this section let us consider the generator U of the manifold is a Killing vector field. Then we have

$$(\pounds_U g)(X,Y) = 0, \tag{5.1}$$

where $\pounds$ denotes the Lie derivative. From which we get

$$g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0. \tag{5.2}$$

Again since $g(\nabla_X U, Y) = (\nabla_X A)(Y)$, we get from (5.2)

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0 \quad \text{for all } X, Y. \tag{5.3}$$

Similarly, we have

$$(\nabla_X A)(Z) + (\nabla_Z A)(X) = 0 \quad \text{for all } X, Z \tag{5.4}$$

and

$$(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0 \quad \text{for all } Z, Y. \tag{5.5}$$

Now from (1.2) we get

$$\begin{aligned}&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)\\ &= \{(\nabla_X A)(Y) + (\nabla_Y A)(X)\}A(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}A(Y)\\ &\quad + \{(\nabla_Z A)(Y) + (\nabla_Y A)(Z)\}A(X).\end{aligned} \tag{5.6}$$

Putting (5.3), (5.4) and (5.5) in (5.6) we have

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

Thus we can state the following:

Theorem 5.1. *If the generator of the quasi Einstein manifold is a Killing vector field then the manifold satisfies cyclic Ricci tensor.*

6. Quasi Einstein manifold admitting a conformal Killing vector field

We need the following theorem which is due to Obata [12], [13], [14].

Theorem A. *If a complete Riemannian manifold M of dimension $n \geq 2$ admits a non-constant function ρ such that*

$$\nabla\nabla\rho = -c^2\rho g,$$

where c is a positive constant, then M is isometric to a sphere of radius $\frac{1}{c}$ in $(n+1)$-dimensional Euclidean space.

In this section we suppose that a quasi Einstein manifold admits a conformal Killing vector field [15]. Then we get

$$(\pounds_X g)(Y,Z) = 2\rho g(Y,Z), \tag{6.1}$$

where ρ is a non-zero scalar. If $\rho =$ constant, then the vector field is called a homothetic vector field. If X is a conformal Killing vector, then we have [15]

$$(\pounds_X S)(Y,Z) = -(n-2)(\nabla d\rho)(Y,Z) + \nabla\rho g(Y,Z). \tag{6.2}$$

Also from (1.2) we have

$$(\pounds_X S)(Y,Z) = a(\pounds_X g)(Y,Z) + b\pounds_X\{A(Y)A(Z)\}. \tag{6.3}$$

Now we suppose that

$$\pounds_X\{A(Y)A(Z)\} = \frac{2\rho}{n-1}g(Y,Z) \quad \text{and} \quad a > r. \tag{6.4}$$

Using (6.1), (6.2) and (6.4) in (6.3) we obtain

$$-(n-2)(\nabla d\rho)(Y,Z) + \nabla\rho g(Y,Z) = 2a\rho g(Y,Z) + \frac{2b\rho}{n-1}g(Y,Z). \tag{6.5}$$

Contracting (6.5) we have

$$\nabla\rho = \frac{(r-a)n}{n-1}\rho. \tag{6.6}$$

Putting the value of $\nabla\rho$ in (6.5) we get

$$(\nabla d\rho)(Y,Z) = -\frac{(a-r)}{n-1}\rho g(Y,Z). \tag{6.7}$$

Now using the theorem of Obata from (6.7) we see that a complete quasi Einstein manifold of dimension $n \geq 2$ is isometric to a sphere.

But a sphere is an Einstein manifold. Hence we can state the following:

Theorem 6.1. *A complete quasi Einstein manifold of dimension $n \geq 2$ can not admit a nonhomothetic conformal vector field.*

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