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A CHARACTERIZATION OF SOBOLEV SPACES BY SOLUTIONS OF HEAT EQUATION AND A STABILITY PROBLEM FOR A FUNCTIONAL EQUATION

Yun-Sung Chung, Young-Su Lee, Deok-Yong Kwon, and Soon-Yeong Chung

Abstract. In this paper, we characterize Sobolev spaces $\mathcal{H}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ by the initial value of solutions of heat equation with a growth condition. By using an idea in its proof, we also discuss a stability problem for Cauchy functional equation in the Sobolev spaces.

1. Introduction

The heat kernel method which was first introduced by T. Matsuzawa [11] is to use the heat kernel $E(x,t)$ (see Section 2) to represent functions or generalized functions as initial values of solutions of the heat equation. This approach has been turned out to be a very effective tool when we do harmonic analysis or solve functional equations via generalized function theories (see [3], [6], [5], [7] and so on).

Among the various spaces of generalized functions, Sobolev spaces $\mathcal{W}^{p,s}(\mathbb{R}^n)$, $s \geq 0$, $p \geq 1$ and $\mathcal{H}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ have provided a convenient framework for applying the theory of generalized functions to boundary value problems because of their Hilbert space structure. In the paper [4], the spaces $\mathcal{W}^{p,s}$ were characterized by initial values of solutions of the heat equation in the case of $p > 1$ by following the approach of [11], but there has not been any result for the spaces $\mathcal{H}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$.

The purpose of this paper is to characterize Sobolev spaces $\mathcal{H}^s(\mathbb{R}^n)$ by using the initial value of solutions of heat equation. The main theorem (Theorem 3.1) states that if $U(x,t)$ is a solution of heat equation satisfying that there exists $M > 0$ such that for each $\sigma \in \mathbb{R}$, there is $C_\sigma > 0$ satisfying

$$(1) \qquad \int_{\mathbb{R}^n} |\hat{U}(\xi,t)|^2 \, (1+|\xi|^2)^{s+\sigma} \, d\xi \leq M \, \max\{C_\sigma t^{-\sigma}, 1\}, \quad t \in (0,T)$$

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then its initial value must belong to the Sobolev space $\mathcal{H}^s(\mathbb{R}^n)$ and conversely, for every $u \in \mathcal{H}^s(\mathbb{R}^n)$, the function $(u * E)(x,t)$ is a heat solution satisfying (1). Here, and in what follows, $*$ means the convolution with respect to the space variable x.

The next topic of this paper is a stability problem for Cauchy functional equation in the space $\mathcal{H}^s(\mathbb{R})$. A functional equation of the form

$$f(x+y) = f(x) + f(y)$$

is said to be *Cauchy functional equation* and is naturally extended to the space of generalized functions as

$$u \circ A = u \circ P_1 + u \circ P_2,$$

where A, P_1 and P_2 are the functions such that

$$A(x,y) = x+y, \quad P_1(x,y) = x, \quad \text{and} \quad P_2(x,y) = y$$

for $x, y \in \mathbb{R}$ and $\circ$ denotes the distributional pullback (see Section 4). By using the fact shown in Theorem 3.1 that every element of the space $\mathcal{H}^s(\mathbb{R})$ corresponds to a solution of heat equation by convolving the heat kernel $E(x,t)$, we show that for every solution $u \in \mathcal{S}'(\mathbb{R})$ of the inequality

$$\|u \circ A - u \circ P_1 - u \circ P_2\|_{\mathcal{H}^s(\mathbb{R})} \le \epsilon,$$

the function $(u * E)(x,t)$ can be approximated to an additive function if $\epsilon \ge 0$ is sufficiently small. Here, and in what follows, $*$ means the convolution with respect to the space variable x.

2. Preliminaries

In this section, we briefly recall the definition and basic properties of the Sobolev spaces $\mathcal{H}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ and the *heat kernel* $E(x,t)$ which is the fundamental solution of the heat equation

$$(\partial_t - \Delta)U(x,t) = 0$$

in $\mathbb{R}^n \times (0,\infty)$. Here, we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$. We denote by $C^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$, by $C_o^\infty(\mathbb{R}^n)$ the set of all functions in $C^\infty(\mathbb{R}^n)$ which have compact supports, by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing functions in $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions in $\mathbb{R}^n$.

We recall the definition of Sobolev spaces $\mathcal{H}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ as below. For properties of the spaces, we refer to [9].

Definition. Let s be a real number. We denote by $\mathcal{H}^s(\mathbb{R}^n)$ the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\hat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}} \in L^2(\mathbb{R}^n),$$

equipped with the norm

$$\|u\|_s = \Big[\int |\hat{u}(\xi)|^2(1+|\xi|^2)^s\, d\xi\Big]^{\frac{1}{2}},$$

where $\hat{u}$ is the Fourier transform of u.

We denote by the n-dimensional heat kernel

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t) & t>0, \\ 0 & t \le 0. \end{cases}$$

Now, we give some basic properties of the heat kernel $E(x,t)$. Their proofs can be found in [11].

Theorem 2.1. (i) *$E(\cdot,t)$ is an entire function for every $t>0$.*
(ii) *We have*

$$\int_{\mathbb{R}^n} E(x,t)\; dx = 1, \quad t>0.$$

(iii) *There exist $C>0$ and $a>0$ such that*

$$|\partial_x^\alpha E(x,t)| \le C^{|\alpha|} t^{-\frac{n+|\alpha|}{2}} \alpha!^{1/2} \exp\Big[-\frac{a|x|^2}{4t}\Big], \quad t>0, \tag{2}$$

where a can be taken as close as desired to 1 and $0<a<1$.

The following result of Cauchy problem for the heat equation is well known and will be very useful to prove the main result of this paper. For its proof, we refer to [8].

Theorem 2.2. *Let $T>0$ and $U(x,t)$ be a continuous function on $\mathbb{R}^n \times [0,T)$ with the following properties*

(i) $(\partial_t - \Delta)U(x,t) = 0$, $(x,t) \in \mathbb{R}^n \times (0,T)$,
(ii) *There exist $C>0$ and $k>0$ such that*

$$|U(x,t)| \le C\; e^{k|x|^2}, \quad (x,t) \in \mathbb{R}^n \times [0,T).$$

Then, $U(x,t)$ is uniquely determined by

$$U(x,t) = U(x,0) * E(x,t).$$

3. A characterization of Sobolev spaces

In what follows, for a given function $U(x,t)$ in $\mathbb{R}^n \times (0,T)$, the notations $\hat{U}(\xi,t)$ or $(\mathcal{F}_x U)(\xi,t)$ denote the Fourier transform of a function $U(x,t)$ with respect to the space variable x.

We are now in a position to state and prove the main result of this paper.

Theorem 3.1. *Let $T>0$ and $s\in\mathbb{R}$. For every $u \in \mathcal{H}^s(\mathbb{R}^n)$, the function $U(x,t) = (u*E)(x,t)$ is a well defined C^∞ function in $\mathbb{R}^n \times (0,T)$ satisfying that*

(i) $(\partial_t - \Delta)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T)$,

(ii) *there exists* $M > 0$ *such that for each* $\sigma \in \mathbb{R}$, *there is* $C_\sigma > 0$ *satisfying*

$$\int_{\mathbb{R}^n} |\hat{U}(\xi,t)|^2 \ (1+|\xi|^2)^{s+\sigma} \ d\xi \le M \ \max\{C_\sigma t^{-\sigma}, 1\}, \quad t \in (0,T), \tag{3}$$

(iii) $U(\cdot,t) \to u$ *in* $\mathcal{H}^s(\mathbb{R}^n)$ *as* $t \to 0^+$.

Conversely, if a function $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,T))$ *satisfies the conditions* (i) *and* (ii), *then there exists* $u \in \mathcal{H}^s(\mathbb{R}^n)$ *such that*

$$U(x,t) = (u * E)(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T).$$

Furthermore, $U(\cdot,t) \to u$ *in* $\mathcal{H}^s(\mathbb{R}^n)$ *as* $t \to 0^+$.

Proof. ($\Rightarrow$) Since $E(\cdot,t)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ for each $t > 0$, the function $U(x,t) = (u * E)(x,t)$ is well defined in $\mathbb{R}^n \times (0,T)$. It is easy to see that $U(x,t)$ is a C^∞ function in $\mathbb{R}^n \times (0,T)$ satisfying the heat equation (i). The condition (ii) can be shown by considering the following estimate

$$\begin{aligned}\int |\hat{U}(\xi,t)|^2(1+|\xi|^2)^{s+\sigma} d\xi &= \int |\hat{u}(\xi)\hat{E}(\xi,t)|^2(1+|\xi|^2)^{s+\sigma} d\xi \\ &= \int |\hat{u}(\xi)|^2(1+|\xi|^2)^s \left[e^{-2t|\xi|^2}(1+|\xi|^2)^\sigma\right] d\xi \\ &\le \|u\|_s^2 \cdot \sup_\xi \left[e^{-2t|\xi|^2}(1+|\xi|^2)^\sigma\right]\end{aligned}$$

for each $t \in (0,T)$, with the following inequalities

$$\sup_\xi \left[e^{-2t|\xi|^2}(1+|\xi|^2)^\sigma\right] \le \max\{\ (\sigma/2)^\sigma t^{-\sigma}, 1\ \}, \quad t \in (0,T)$$

for each $\sigma > 0$ and

$$\sup_\xi \left[e^{-2t|\xi|^2}(1+|\xi|^2)^\sigma\right] = 1, \quad t \in (0,T)$$

for each $\sigma \le 0$, which can be obtained easily. Finally, by using (2) in Theorem 2.1 and the fact that $u \in \mathcal{S}'(\mathbb{R}^n)$, it is easy to see that

$$U(\cdot,t) \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \quad t \in (0,T)$$

and hence, by condition (ii), we have

$$U(\cdot,t) \in \mathcal{H}^s(\mathbb{R}^n), \quad t \in (0,T).$$

Moreover, we have

$$\begin{aligned}\|U(\cdot,t) - u\|_s^2 &= \int |\mathcal{F}_x(u * E)(\xi,t) - \hat{u}(\xi)|^2 \ (1+|\xi|^2)^s d\xi \\ &\le \int |\hat{u}(\xi)|^2(1+|\xi|^2)^s \ (e^{-t|\xi|^2} - 1)^2 d\xi,\end{aligned} \tag{4}$$

which converges to 0 as $t \longrightarrow 0^+$.
($\Leftarrow$) Take any $\sigma_0 > \frac{n}{2} - s$. Then by virtue of Hölder's inequality, we have

$$\int |\hat{U}(\xi,t)|\,d\xi = \int |\hat{U}(\xi,t)| \cdot (1+|\xi|^2)^{(s+\sigma_0)/2} \cdot \frac{1}{(1+|\xi|^2)^{(s+\sigma_0)/2}}\,d\xi$$
$$\le M'_{\sigma_0}\ \max\{\sqrt{C_{\sigma_0}}\,t^{-\frac{\sigma_0}{2}}, 1\}, \quad t > 0$$

for some $M'_{\sigma_0} > 0$. Then since $\hat{U}(\cdot,t) \in L^1(\mathbb{R}^n)$ for each $t > 0$, we have by Fourier inversion formula,

$$(5) \qquad |U(x,t)| = |\frac{1}{(2\pi)^n}\int \hat{U}(\xi,t)e^{i\xi\cdot x}d\xi|$$
$$\le M'_{\sigma_0}\ \max\{\sqrt{C_{\sigma_0}}\,t^{-\frac{\sigma_0}{2}}, 1\}, \quad (x,t) \in \mathbb{R}^n \times (0,T).$$

Now, consider a function

$$f(t) = \begin{cases} t^{m-1}/(m-1)! & t \ge 0, \\ 0 & t < 0, \end{cases}$$

where $m = [\sigma_0/2] + 1$ if $\sigma_0 > 0$ and $m = 1$ if $\sigma_0 \le 0$. Multiplying f with a suitable C^∞ function with compact support, it is possible to get the following relation

$$(6) \qquad \left(\frac{d}{dt}\right)^m v(t) = \delta(t) + w(t)$$

for every $t \in \mathbb{R}$ for suitable functions $v(t)$ and $w(t)$ such that $v(t) = f(t)$ for $t \le T/4$, $v(t) = 0$ for $t \ge T/2$ and $w(t) \in C_0^\infty(\mathbb{R})$ with supp $w \subset [T/4, T/2]$, where $\delta(t)$ is the Dirac delta function. Define

$$V(x,t) = \int_0^\infty U(x,t+r)v(r)dr.$$

Then, it is easily seen that $V(x,t)$ is a bounded and continuous function on $\mathbb{R}^n \times [0, T/2)$ satisfying the heat equation

$$(7) \qquad (\partial_t - \Delta)V(x,t) = 0, \quad 0 < t < T/2.$$

Moreover, it follows from (6) and (7) that

$$(-\Delta)^m V(x,t) = (-\partial_t)^m V(x,t)$$
$$= U(x,t) + \int_0^\infty U(x,t+r)w(r)dr$$

for $(x,t) \in \mathbb{R}^n \times (0,T)$. Now, put

$$W(x,t) = -\int_0^\infty U(x,t+r)w(r)dr.$$

Then $W(x,t)$ is also a bounded solution of the heat equation in $\mathbb{R}^n \times (0, T/2)$ which is continuously extended to $\mathbb{R}^n \times [0, T/2)$, and hence by virtue of Theorem 2.2, we have

$$U(x,t) = (-\Delta)^m V(x,t) + W(x,t)$$
$$= \Big[(-\Delta)^m V_0 + W_0\Big] * E(x,t)$$

for $(x,t) \in \mathbb{R}^n \times (0, T/2)$, where

$$V_0 = V(\cdot, 0) \quad \text{and} \quad W_0 = W(\cdot, 0).$$

Thus if we put

$$u = (-\Delta)^m V_0 + W_0,$$

then we have the following unique expression

$$U(x,t) = (u * E)(x,t) \tag{8}$$

for $(x,t) \in \mathbb{R}^n \times (0, T/2)$. Furthermore, by the uniqueness property in Theorem 2.2, (8) can be extended to $\mathbb{R}^n \times (0, T)$.

Now it remains to show that $u \in \mathcal{H}^s(\mathbb{R}^n)$. By virtue of the fact that $V(\cdot, 0)$ and $W(\cdot, 0)$ are bounded, it is easy to see that $u \in \mathcal{S}'(\mathbb{R}^n)$. Moreover, since

$$\hat{U}(\xi, t) = \hat{u}(\xi) e^{-t|\xi|^2},$$

it follows from the monotone convergence theorem that we have

$$\int |\hat{u}(\xi)|^2 \, (1 + |\xi|^2)^s d\xi = \int \lim_{t \to 0^+} |\hat{u}(\xi)|^2 \, e^{-2t|\xi|^2} \, (1 + |\xi|^2)^s d\xi$$
$$= \lim_{t \to 0^+} \int |\hat{U}(\xi, t)|^2 \, (1 + |\xi|^2)^s d\xi$$
$$\leq M \, \max\{C_0, 1\},$$

which indicates $u \in \mathcal{H}^s(\mathbb{R}^n)$. The convergence $U(\cdot, t) \to u$ in $\mathcal{H}^s(\mathbb{R}^n)$ as $t \to 0^+$ is easy to see by using the same method given in the inequality (4). This completes the proof. □

Corollary 3.2. *Suppose that $T > 0$ and $s \in \mathbb{R}$. Then for every $u \in \mathcal{H}^s(\mathbb{R}^n)$, the function $U(x,t) = (u*E)(x,t)$ is a well defined C^∞ solution of heat equation in $\mathbb{R}^n \times (0, T)$ satisfying that for each $\sigma > n/2 - s$, there exist $M_\sigma > 0$ and $C_\sigma > 0$ such that*

$$|U(x,t)| \leq M_\sigma \, \max\{C_\sigma \, t^{-\frac{\sigma}{2}}, 1\}, \quad (x,t) \in \mathbb{R}^n \times (0, T).$$

Proof. Since we have for each $t > 0$,

$$|U(x,t)| = \left| \frac{1}{(2\pi)^{n/2}} \int \hat{U}(\xi, t) e^{i\xi \cdot x} d\xi \right|$$
$$\leq \frac{1}{(2\pi)^{n/2}} \int \left| \, \hat{U}(\xi, t)(1 + |\xi|^2)^{(s+\sigma)/2} \cdot \frac{1}{(1 + |\xi|^2)^{(s+\sigma)/2}} \, \right| d\xi, \; x \in \mathbb{R}^n,$$

the result follows from Theorem 3.1. □

In the proof of the sufficient part of Theorem 3.1, we actually proved the following:

Corollary 3.3. *Let $T > 0$ and $s \in \mathbb{R}$. For every $u \in \mathcal{H}^s(\mathbb{R}^n)$, the function $U(x,t) = (u * E)(x,t)$ is a well defined C^∞ function in $\mathbb{R}^n \times (0,T)$ satisfying that*

(i) $(\partial_t - \Delta)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T)$,

(ii) *For each $\sigma \in \mathbb{R}$, there is $C_\sigma > 0$ satisfying*

$$\int_{\mathbb{R}^n} |\hat{U}(\xi,t)|^2 \, (1+|\xi|^2)^{s+\sigma} \, d\xi \leq \|u\|_s^2 \, \max\{C_\sigma t^{-\sigma}, 1\}, \quad t \in (0,T),$$

(iii) $U(\cdot,t) \to u$ *in* $\mathcal{H}^s(\mathbb{R}^n)$ *as* $t \to 0^+$.

Proof. The same as the proof of the sufficient part of Theorem 3.1. □

If we only consider the necessity part of Theorem 3.1, the condition (3) may be weakened as follows :

Corollary 3.4. *Let $T > 0$ and $s \in \mathbb{R}$. If a function $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,T))$ satisfies*

(i) $(\partial_t - \Delta)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T)$,

(ii) *There exist $\sigma > \frac{n}{2} - s$, and $C > 0$ such that*

$$\int_{\mathbb{R}^n} |\hat{U}(\xi,t)|^2 \, (1+|\xi|^2)^{s+\sigma} \, d\xi \leq Ct^{-\sigma} \quad \text{for sufficiently small } t > 0,$$

(iii) *There exists $M > 0$ such that*

$$\int_{\mathbb{R}^n} |\hat{U}(\xi,t)|^2 \, (1+|\xi|^2)^{s} \, d\xi \leq M \quad \text{for sufficient small } t > 0,$$

then there exists $u \in \mathcal{H}^s(\mathbb{R}^n)$ such that

$$U(x,t) = (u * E)(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T).$$

Furthermore, $U(\cdot,t) \to u$ in $\mathcal{H}^s(\mathbb{R}^n)$ as $t \to 0^+$.

Proof. Replace σ_0 with σ in the proof of the necessity part of the Theorem 3.1 and follow the proof to get the result. □

4. Stability of Cauchy equation in $\mathcal{H}^s$

Many functional equations have been studied in the spaces of some generalized functions such as Schwartz distributions, Gevrey distributions and Fourier hyperfunctions (see [1], [2], [3], [5], [7], [10]).

In this section, we consider a stability problem for Cauchy functional equation in Sobolev spaces $\mathcal{H}^s(\mathbb{R})$ as an application of Theorem 3.1. Making use of the pullback of generalized functions, Cauchy functional equation

$$f(x+y) = f(x) + f(y)$$

is extended to

$$u \circ A = u \circ P_1 + u \circ P_2 \tag{9}$$

when u is a generalized function (see [1], [3], [5]). Here A, P_1 and P_2 are functions defined by $A(x,y) = x + y, P_1(x,y) = x$ and $P_2(x,y) = y$. According to the result as in [5], if $u \in \mathcal{S}'(\mathbb{R})$ satisfies the equation (9), then $(u * E)(x,t)$ is a well defined C^∞ function in $\mathbb{R} \times (0,\infty)$ of the form

$$(u * E)(x,t) = ax + bt$$

for some $a, b \in \mathbb{R}$. Note that we actually get $b = 0$ in the above equality because $(u * E)(x,t)$ is a solution of the heat equation.

Now we consider that the Cauchy difference is in the Sobolev space. Using similar calculations as in Theorem 3.1, we have the following stability theorem of Cauchy functional equation in the sense of Sobolev space.

Theorem 4.1. *Suppose $u \in \mathcal{S}'(\mathbb{R})$ satisfies the inequality*

$$\|u \circ A - u \circ P_1 - u \circ P_2\|_{H^s(\mathbb{R}^2)} \le \epsilon. \tag{10}$$

Then there exist constants σ, C and unique a, b such that

$$|(u * E)(x,t) - (ax + bt)| \le C\epsilon t^{-2\sigma} \tag{11}$$

for all $(x,t) \in \mathbb{R} \times (0, \frac{\sigma}{2}]$.

Proof. For convenience, we denote $E(x,t)$ by $E_t(x)$. Let $v := u \circ A - u \circ P_1 - u \circ P_2$ and $U(\xi,t) = (u * E_t)(\xi)$. Convolving in v the tensor product $E_t(x)E_r(y)$ of the heat kernel we have

$$\begin{aligned}
[(u \circ A) * E_t(x)E_r(y)](\xi,\eta) &= \langle u \circ A,\, E_t(\xi - x)E_r(\eta - y)\rangle \\
&= \langle u_x, \int E_t(\xi - x + y)E_r(\eta - y)dy\rangle \\
&= \langle u_x, \int E_t(\xi + \eta - x - y)E_r(y)dy\rangle \\
&= \langle u_x, (E_t * E_r)(\xi + \eta - x)\rangle \\
&= \langle u_x, E_{t+r}(\xi + \eta - x)\rangle \\
&= U(\xi + \eta, t + r)
\end{aligned}$$

and similarly we get

$$\begin{aligned}
[(u \circ P_1) * E_t(x)E_r(y)](\xi,\eta) = U(\xi,t), \\
[(u \circ P_2) * E_t(x)E_r(y)](\xi,\eta) = U(\eta,r).
\end{aligned}$$

Also we note that

$$\begin{aligned}
&\iint |\mathcal{F}(v * E_t(x)E_r(y))|^2(1 + |\xi|^2 + |\eta|^2)^s d\xi d\eta \\
= &\iint |\widehat{v}(\xi,\eta)|^2 e^{-2t|\xi|^2} e^{-2r|\eta|^2}(1 + |\xi|^2 + |\eta|^2)^s d\xi d\eta \le \|v\|_{H^s(\mathbb{R}^2)}.
\end{aligned}$$

Thus inequality (10) is converted into

$$\|U(\xi+\eta, t+r) - U(\xi, t) - U(\eta, r)\|_{H^s(\mathbb{R}^2)} \le \epsilon.$$

By virtue of Fourier inversion formula and Hölder's inequality we have

$$\begin{aligned}
&|U(\xi+\eta, t+r) - U(\xi, t) - U(\eta, r)| \\
&\le \frac{1}{2\pi} \iint |\mathcal{F}[v * E_t(x)E_r(y)](\xi,\eta)| d\xi d\eta \\
&= \frac{1}{2\pi} \left[\iint |\mathcal{F}[v * E_t(x)E_r(y)](\xi,\eta)|^2 (1+|\xi|^2+|\eta|^2)^{s+\sigma} d\xi d\eta \right]^{1/2} \\
&\times \left[\iint \frac{1}{(1+|\xi|^2+|\eta|^2)^{s+\sigma}} d\xi d\eta \right]^{1/2}
\end{aligned}$$

for some sufficiently large $\sigma \in \mathbb{R}$. Following the similar calculations as in Theorem 3.1 we obtain

$$\begin{aligned}
&\iint |\mathcal{F}[v * E_t(x)E_r(y)](\xi,\eta)|^2 (1+|\xi|^2+|\eta|^2)^{s+\sigma} d\xi d\eta \\
&= \iint |\widehat{v}(\xi,\eta)|^2 e^{-2t|\xi|^2} e^{-2r|\eta|^2} (1+|\xi|^2+|\eta|^2)^{s+\sigma} d\xi d\eta \\
&\le \|v\|_{H^s} \sup_{\xi,\eta}\{e^{-2t|\xi|^2} e^{-2r|\eta|^2} (1+|\xi|^2+|\eta|^2)^{\sigma}\} \\
&\le \epsilon \sup_{\xi}\{e^{-2t|\xi|^2}(1+|\xi|^2)^{\sigma}\} \sup_{\eta}\{e^{-2r|\eta|^2}(1+|\eta|^2)^{\sigma}\} \\
&\le \epsilon(\sigma/2)^{2\sigma}(tr)^{-\sigma}
\end{aligned}$$

for all $t, r \in (0, \frac{\sigma}{2}]$. Thus we have

$$|U(x+y, t+r) - U(x,t) - U(y,r)| \le C\epsilon(tr)^{-\sigma}$$

for some constant C. Putting $y = x, r = t$ and dividing by 2 we get

$$|2^{-1}U(2x, 2t) - U(x,t)| \le 2^{-1}C\epsilon t^{-2\sigma}.$$

Making use of the induction argument and triangle inequality we obtain

$$|2^{-n}U(2^n x, 2^n t) - U(x,t)| \le C'\epsilon t^{-2\sigma} \tag{12}$$

for some constant C'. Now we set $g_n(x,t) := 2^{-n}U(2^n x, 2^n t)$. In view of the inequality (12) it is easy to see that $g_n(x,t)$ is a uniform Cauchy sequence and hence $g(x,t) := \lim_{n\to\infty} g_n(x,t)$ exists. Replacing x, y, t, r by $2^n x, 2^n y, 2^n t, 2^n r$, respectively and then dividing by 2^n we get

$$g(x+y, t+s) = g(x,t) + g(y,s) \tag{13}$$

as taking $n \to \infty$. It is well known that the continuous solution of the equation (13) is of the form

$$g(x,t) = ax + bt$$

for some $a, b \in \mathbb{R}$. Letting $n \to \infty$ in (12) we get (11).
Finally we prove the uniqueness. Suppose that $h(x,t)$ satisfies (12) and (13). Note that

$$g(kx, kt) = kg(x,t) \tag{14}$$

for any rational number k. It follows from (14) and triangle inequality we get

$$\begin{aligned} k|g(x,t) - h(x,t)| &= |g(kx,kt) - h(kx,kt)| \\ &\leq |g(kx,kt) - U(kx,kt)| + |h(kx,kt) - U(kx,kt)| \\ &\leq 2C'\epsilon t^{-2\sigma}. \end{aligned}$$

Now letting $k \to \infty$, we must have $g = h$. □

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Yun-Sung Chung
BK21 Math Modeling HRD Div.
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
E-mail address: ysjung93@skku.edu

Young-Su Lee
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Daejeon 305-701, Korea
E-mail address: masuri@kaist.ac.kr

Deok-Yong Kwon
Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail address: dykwon@sogang.ac.kr

Soon-Yeong Chung
Department of Mathematics and Program of Integrated Biotechnology
Sogang University
Seoul 121-742, Korea
E-mail address: sychung@sogang.ac.kr