Commun. Korean Math. Soc. **23** (2008), No. 3, pp. 377–386

# ON THE STABILITY OF A CAUCHY-JENSEN FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we prove the stability of a Cauchy-Jensen functional equation

$$2f(x+y,\frac{z+w}{2}) = f(x,z)+f(x,w)+f(y,z)+f(y,w)$$

in the sense of Th. M. Rassias.

## 1. Introduction

In 1940, S. M. Ulam [6] raised a question concerning the stability of homomorphisms: Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D. H. Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Th. M. Rassias [5] gave a generalization. Recently, P. Găvruta [1] also obtained a further generalization of the Hyers-Ulam-Rassias theorem.

Throughout this paper, let $X$ be a normed space and $Y$ a Banach space. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if $g$ satisfies the functional equation $g(x+y) = g(x)+g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x)+g(y)$).

For a given mapping $f : X \times X \to Y$, we define

$$Cf(x,y,z,w) := 2f(x+y,\frac{z+w}{2}) - f(x,z) - f(x,w) - f(y,z) - f(y,w)$$

for all $x,y,z,w \in X$. A mapping $f : X \times X \to Y$ is called a Cauchy-Jensen mapping if $f$ satisfies the functional equation

$$Cf(x,y,z,w) = 0$$

Received October 1, 2007.

2000 *Mathematics Subject Classification.* Primary 39B52.

*Key words and phrases.* stability, Cauchy-Jensen mapping, functional equation.

for all $x, y, z, w \in X$ and the functional equation $Cf = 0$ is called a Cauchy-Jensen functional equation. In 2006, Park and Bae [4] obtained the generalized Hyers-Ulam-Rassias stability of the Cauchy-Jensen functional equation (see also [3]). From Theorem 7 in [4], we get the following theorem:

**Theorem 1.1.** *Let $0 < p < 1$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x,y,z,w)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

*for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - f(x,0) - F(x,y)\| \leq \sum_{j=1}^{\infty} \|\frac{2f(2^j x, 0)}{6^j}\| + \frac{15\varepsilon}{6-2^p}\|x\|^p + \frac{15\varepsilon}{6-3^p}\|y\|^p$$

*for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by*

$$F(x,y) := \lim_{j\to\infty} \frac{f(2^j x, 3^j y)}{6^j}$$

*for all $x, y \in X$.*

In this paper, we investigate the stability of a Cauchy-Jensen functional equation in the sense of Th. M. Rassias. We have better stability results than that of Theorem 1.1. We improve the stability results under weaker inequality condition by adapting different method in the proof.

## 2. Stability of a Cauchy-Jensen mapping

We need the following lemma to prove the main theorems.

**Lemma 2.1.** *Let $f : X \times X \to Y$ be a Cauchy-Jensen mapping. Then the following equalities hold;*

$$f(x,y) = 4^n f(\frac{x}{2^n}, \frac{y}{2^n}) - (4^n - 2^n) f(\frac{x}{2^n}, 0),$$
$$f(x,y) = \frac{f(2^n x, 2^n y)}{4^n} + (\frac{1}{2^n} - \frac{1}{4^n}) f(2^n x, 0),$$
$$f(x,y) = \frac{f(2^n x, 2^n y)}{4^n} + (2^n - 1) f(\frac{x}{2^n}, 0)$$

*for all $x, y \in X$ and $n \in \mathbb{N}$.*

*Proof.* Since

$$f(0,y) = -\frac{1}{2} Cf(0,0,y,y) = 0 \quad \text{and}$$
$$f(x,0) = \sum_{j=1}^{n} \frac{2^j}{4} Cf(\frac{x}{2^j}, \frac{x}{2^j}, 0, 0) + 2^n f(\frac{x}{2^n}, 0) = 2^n f(\frac{x}{2^n}, 0),$$

we get

$$\begin{aligned}
&4^j f(\frac{x}{2^j},\frac{y}{2^j}) - 4^{j+1} f(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}) + 2\cdot 4^j f(\frac{x}{2^{j+1}},0)\\
&= 4^j(\frac{1}{2}Cf(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{y}{2^j},\frac{y}{2^j}) - 2Cf(\frac{x}{2^{j+1}},0,\frac{y}{2^j},0)) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(2^j x,2^j y)}{4^j} - \frac{f(2^{j+1}x,2^{j+1}y)}{4^{j+1}} - \frac{f(2^{j+1}x,0)}{4^{j+1}}\\
&= \frac{1}{4^{j+1}}(-\frac{1}{2}Cf(2^j x,2^j x,2^{j+1}y,2^{j+1}y) + 2Cf(2^j x,0,2^{j+1}y,0)) = 0
\end{aligned}$$

for all $x,y\in X$ and $j\in\mathbb{N}$. Hence we have

$$\begin{aligned}
&f(x,y) - 4^n f(\frac{x}{2^n},\frac{y}{2^n}) + (4^n-2^n) f(\frac{x}{2^n},0)\\
&= \sum_{j=0}^{n-1}(4^j f(\frac{x}{2^j},\frac{y}{2^j}) - 4^{j+1} f(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}) + 2\cdot 4^j f(\frac{x}{2^{j+1}},0)) = 0,\\
&f(x,y) - \frac{f(2^n x,2^n y)}{4^n} - (\frac{1}{2^n}-\frac{1}{4^n}) f(2^n x,0)\\
&= \sum_{j=0}^{n-1}(\frac{f(2^j x,2^j y)}{4^j} - \frac{f(2^{j+1}x,2^{j+1}y)}{4^{j+1}} - \frac{2f(2^j x,0)}{4^{j+1}}) = 0 \quad\text{and}\\
&\frac{f(2^n x,2^n y)}{4^n} - f(x,y) + (2^n-1) f(\frac{x}{2^n},0)\\
&= \frac{f(2^n x,2^n y)}{4^n} - f(x,y) + (\frac{1}{2^n}-\frac{1}{4^n}) f(x,0) = 0
\end{aligned}$$

for all $x,y\in X$ and $n\in\mathbb{N}$. □

**Theorem 2.2.** *Let $p<1$ and $\varepsilon>0$. Let $f: X\times X\to Y$ be a mapping such that*

$$\|Cf(x,y,z,w)\| \le \varepsilon(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)$$

*for all $x,y,z,w\in X\backslash\{0\}$. Then there exists a unique Cauchy-Jensen mapping $F: X\times X\to Y$ such that*

$$\|f(x,y)-F(x,y)\| \le \frac{\varepsilon}{2-2^p}\|x\|^p + \varepsilon\|y\|^p \tag{2.1}$$

*for all $x,y\in X\backslash\{0\}$. Moreover, the mapping $F: X\times X\to Y$ is given by*

$$F(x,y) := \lim_{j\to\infty}\frac{1}{2^j} f(2^j x,y)$$

*for all $x,y\in X$.*

*Proof.* Since

$$\begin{aligned}\|\frac{1}{2^j}f(2^jx,y)-\frac{1}{2^{j+1}}f(2^{j+1}x,y)\| &= \frac{1}{2^{j+2}}\|Cf(2^jx,2^jx,y,y)\| \\ &\le \frac{1}{2^{j+1}}(2^{jp}\varepsilon\|x\|^p+\varepsilon\|y\|^p)\end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and all $j \in \mathbb{N}$, we get

$$\|\frac{1}{2^l}f(2^lx,y)-\frac{1}{2^m}f(2^mx,y)\| \le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}(2^{jp}\varepsilon\|x\|^p+\varepsilon\|y\|^p) \tag{2.2}$$

for given integers $l, m$ $(0 \le l < m)$ and all $x, y \in X\backslash\{0\}$. The sequence $\{\frac{1}{2^j}f(2^jx,y)\}$ is a Cauchy sequence for all $x, y \in X\backslash\{0\}$. Since $Y$ is complete, the sequence $\{\frac{1}{2^j}f(2^jx,y)\}$ converges for all $x, y \in X\backslash\{0\}$. Define $F_1 : X\times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty}\frac{1}{2^j}f(2^jx,y)$$

for all $x, y \in X\backslash\{0\}$. Putting $l = 0$ and taking $m \to \infty$ in (2.2), one can obtain the inequality (2.1). We easily obtain

$$\lim_{j\to\infty}\frac{1}{2^j}f(2^jx,0) = \frac{F_1(x,y)+F_1(x,-y)}{2} + \lim_{j\to\infty}\frac{1}{2^{j+2}}Cf(2^jx,2^jx,y,-y)$$

and

$$\lim_{j\to\infty}\frac{1}{2^j}f(0,y) = 0$$

for all $x, y \in X\backslash\{0\}$. Hence we can define $F : X \times X \to Y$ by

$$F(x,y) := \lim_{j\to\infty}\frac{1}{2^j}f(2^jx,y)$$

for all $x, y \in X$. Using $F(0,x) = 0$ and the definition of $F$ for all $x, y \in X$, we get

$$CF(x,y,z,w) = \lim_{j\to\infty}\frac{1}{2^j}Cf(2^jx,2^jy,z,w) = 0 \tag{2.3}$$

for all $x, y, z, w \in X\backslash\{0\}$. Now we prove that $CF(x,y,z,w) = 0$ for all $x, y, z, w \in X$. In fact, using (2.3) with the equality $F(2x,y) = 2F(x,y)$ for all

$x, y \in X$, we get

$$\begin{aligned} CF(x,y,z,0) &= CF(x,y,\frac{z}{2},\frac{z}{2}) - \frac{1}{2}CF(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{3z}{2}) + \frac{1}{2}CF(\frac{x}{2},\frac{x}{2},-\frac{z}{2},\frac{3z}{2}) \\ &\quad - \frac{1}{2}CF(\frac{x}{2},\frac{x}{2},-\frac{z}{2},\frac{z}{2}) + \frac{1}{2}CF(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}) - \frac{1}{2}CF(\frac{y}{2},\frac{y}{2},\frac{z}{2},\frac{3z}{2}) \\ &\quad + \frac{1}{2}CF(\frac{y}{2},\frac{y}{2},-\frac{z}{2},\frac{3z}{2}) - \frac{1}{2}CF(\frac{y}{2},\frac{y}{2},-\frac{z}{2},\frac{z}{2}) + \frac{1}{2}CF(\frac{y}{2},\frac{y}{2},\frac{z}{2},\frac{z}{2}) \\ &= 0, \\ CF(x,0,z,w) &= CF(\frac{x}{2},\frac{x}{2},z,w) = 0, \\ CF(x,0,z,0) &= -\frac{1}{2}CF(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{3z}{2}) + \frac{1}{2}CF(\frac{x}{2},\frac{x}{2},-\frac{z}{2},\frac{3z}{2}) \\ &\quad - \frac{1}{2}CF(\frac{x}{2},\frac{x}{2},-\frac{z}{2},\frac{z}{2}) + \frac{1}{2}CF(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}) = 0, \\ CF(x,y,0,0) &= -CF(\frac{x}{2},\frac{x}{2},-z,z) - CF(\frac{y}{2},\frac{y}{2},-z,z) + CF(x,y,-z,z) = 0 \end{aligned}$$

for all $x, y, z, w \in X\backslash\{0\}$. Similarly we can prove the other cases. Hence $F$ is a Cauchy-Jensen mapping satisfying (2.1). Finally, we prove the uniqueness. Let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.1). Then we have

$$\begin{aligned} &\|F(x,y) - F'(x,y)\| \\ &\le \frac{1}{2^n}\|f(2^n x,y) - F(2^n x,y)\| + \frac{1}{2^n}\|f(2^n x,y) - F'(2^n x,y)\| \\ &\le \frac{2^{np}}{2^n}\frac{2\varepsilon}{2-2^p}\|x\|^p + \frac{2\varepsilon}{2^n}\|y\|^p \end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in X\backslash\{0\}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X\backslash\{0\}$. Since $F, F'$ are Cauchy-Jensen mappings,

$$\begin{aligned} &F(0,y) = 0 = F'(0,y), \\ &F(x,0) = \frac{1}{2}[F(x,y) + F(x,-y)] = \frac{1}{2}[F'(x,y) + F'(x,-y)] = F'(x,0) \end{aligned}$$

for all $x, y \in X\backslash\{0\}$. This completes the proof of uniqueness. □

**Corollary 2.3.** *Let $\delta > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x,y,z,w)\| \le \delta$$

*for all $x, y, z, w \in X\backslash\{0\}$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le \frac{\delta}{2}$$

*for all $x, y \in X\backslash\{0\}$.*

**Theorem 2.4.** *Let $p > 2$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x,y,z,w)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \tag{2.4}$$

*for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le \left(\frac{\varepsilon}{2^p - 2} + \frac{4\varepsilon}{2^p - 4}\right)\|x\|^p + \frac{5 \cdot 2^p \varepsilon}{2^p - 4}\|y\|^p \tag{2.5}$$

*for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by*

$$F(x,y) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, 0\right) + \lim_{n\to\infty} 4^n\left(f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - f\left(\frac{x}{2^n}, 0\right)\right)$$

*for all $x, y \in X$.*

*Proof.* Apply (2.4) with $x = y = z = w = 0$ to get $f(0,0) = 0$. Since

$$\begin{aligned}\|2^n f\left(\frac{x}{2^n}, 0\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}, 0\right)\| &= 2^{n-1}\|Cf\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0, 0\right)\| \\ &\le \left(\frac{2}{2^p}\right)^{n+1}\frac{\varepsilon}{2}\|x\|^p\end{aligned}$$

and

$$\begin{aligned}&\|4^n\left(f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - f\left(\frac{x}{2^n}, 0\right)\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) - f\left(\frac{x}{2^{n+1}}, 0\right)\| \\ &= 2 \cdot 4^{n-1}\|Cf\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, \frac{y}{2^n}, \frac{y}{2^n}\right) - Cf\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0, 0\right) \\ &\quad + 2Cf\left(0, 0, \frac{y}{2^n}, \frac{y}{2^n}\right) - 4Cf\left(\frac{x}{2^{n+1}}, 0, \frac{y}{2^n}, 0\right)\| \\ &\le \frac{4^n\varepsilon}{2^{(n+1)p}}(4\|x\|^p + 5 \cdot 2^p\|y\|^p)\end{aligned}$$

for all $x, y \in X$, we get

$$\|2^l f\left(\frac{x}{2^l}, 0\right) - 2^m f\left(\frac{x}{2^m}, 0\right)\| \le \frac{1}{2} \cdot \sum_{j=l+1}^{m} \frac{2^j\varepsilon}{2^{jp}}\|x\|^p \tag{2.6}$$

and

$$\begin{aligned}&\|4^l\left(f\left(\frac{x}{2^l}, \frac{y}{2^l}\right) - f\left(\frac{x}{2^l}, 0\right)\right) - 4^m\left(f\left(\frac{x}{2^m}, \frac{y}{2^m}\right) - f\left(\frac{x}{2^m}, 0\right)\right)\| \\ &\le \sum_{j=l}^{m-1} \frac{4^j\varepsilon}{2^{(j+1)p}}(4\|x\|^p + 5 \cdot 2^p\|y\|^p)\end{aligned} \tag{2.7}$$

for all $x, y \in X$ and given integers $l, m$ $(0 \le l < m)$. Hence the sequences $\{2^n f(\frac{x}{2^n}, 0)\}$ and $\{4^n(f(\frac{x}{2^n}, \frac{y}{2^n}) - f(\frac{x}{2^n}, 0))\}$ are Cauchy sequences for all $x, y \in X$. Since $Y$ is complete, the sequences $\{2^n f(\frac{x}{2^n}, 0)\}$ and $\{4^n(f(\frac{x}{2^n}, \frac{y}{2^n}) - f(\frac{x}{2^n}, 0))\}$ converge for all $x, y \in X$. Define the maps $F_1, F_2 : X \times X \to Y$ by

$$\begin{aligned}F_1(x,y) &= \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, 0\right), \\ F_2(x,y) &= \lim_{n\to\infty} 4^n\left(f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - f\left(\frac{x}{2^n}, 0\right)\right)\end{aligned}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.6) and (2.7), one can obtain the inequalities

$$\|f(x,0) - F_1(x,0)\| \le \frac{\varepsilon}{2^p - 2}\|x\|^p,$$
$$\|f(x,y) - f(x,0) - F_2(x,y)\| \le \frac{4\varepsilon}{2^p - 4}\|x\|^p + \frac{5 \cdot 2^p \varepsilon}{2^p - 4}\|y\|^p$$

for all $x, y, z \in X$. Since

$$CF_2(x,y,z,w) = \lim_{n\to\infty} 4^n\left(Cf\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) - Cf\left(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0\right)\right) = 0,$$
$$CF_1(x,y,z,w) = \lim_{n\to\infty} 2^n Cf\left(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0\right) = 0$$

for all $x, y \in X$, $F$ is a Cauchy-Jensen mapping satisfying (2.5), where $F$ is defined by

$$F(x,y) = F_1(x,y) + F_2(x,y)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.5). By Lemma 2.1, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\le \|4^n F(\tfrac{x}{2^n}, \tfrac{y}{2^n}) - (4^n - 2^n)F(\tfrac{x}{2^n}, 0) - 4^n F'(\tfrac{x}{2^n}, \tfrac{y}{2^n}) + (4^n - 2^n)F'(\tfrac{x}{2^n}, 0)\| \\
&\le 4^n \|F(\tfrac{x}{2^n}, \tfrac{y}{2^n}) - f(\tfrac{x}{2^n}, \tfrac{y}{2^n})\| + 4^n\|f(\tfrac{x}{2^n}, \tfrac{y}{2^n}) - F'(\tfrac{x}{2^n}, \tfrac{y}{2^n})\| \\
&\quad + 4^n(\|F(\tfrac{x}{2^n}, 0) - f(\tfrac{x}{2^n}, 0)\| + \|f(\tfrac{x}{2^n}, 0) - F'(\tfrac{x}{2^n}, 0)\|) \\
&\le \frac{4^n}{2^{np}}\left(\left(\frac{4\varepsilon}{2 - 2^p} + \frac{16\varepsilon}{4 - 2^p}\right)\|x\|^p + \frac{10 \cdot 2^p\varepsilon}{4 - 2^p}\|y\|^p\right)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we have $F(x,y) = F'(x,y)$ as desired. □

**Theorem 2.5.** *Let $1 < p < 2$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x,y,z,w)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

*for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le \left(\frac{\varepsilon}{2^p - 2} + \frac{4\varepsilon}{4 - 2^p}\right)\|x\|^p + \frac{5 \cdot 2^p\varepsilon}{4 - 2^p}\|y\|^p \tag{2.8}$$

*for all $x, y \in X$. Moreover, the mapping $F : X \times X \to Y$ is given by*

$$F(x,y) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, 0\right) + \lim_{n\to\infty} \frac{f(2^n x, 2^n y) - f(2^n x, 0)}{4^n}$$

*for all $x, y \in X$.*

*Proof.* In the proof of Theorem 2.4, we obtain a Cauchy Jensen mapping $F_1 : X \times X \to Y$ satisfying

$$\|f(x,0) - F_1(x,0)\| \le \frac{\varepsilon}{2^p - 2}\|x\|^p$$

for all $x, y \in X$, where $F_1$ is given by

$$F_1(x,y) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n}, 0)$$

for all $x, y \in X$. Since $f(0,0) = 0$ and

$$\begin{aligned}
&\|\frac{f(2^n x, 2^n y) - f(2^n x, 0)}{4^n} - \frac{f(2^{n+1}x, 2^{n+1}y) - f(2^{n+1}x, 0)}{4^{n+1}}\| \\
&= \frac{1}{2\cdot 4^{n+1}}\|Cf(2^n x, 2^n x, 2^{n+1}y, 2^{n+1}y) + 2Cf(0,0,2^{n+1}y, 2^{n+1}y) \\
&\quad - 4Cf(2^n x, 0, 2^{n+1}y, 0) - Cf(2^n x, 2^n x, 0, 0)\| \\
&\le \frac{\varepsilon}{4^{n+1}}(4\cdot 2^{np}\|x\|^p + 5\cdot 2^{(n+1)p}\|y\|^p)
\end{aligned}$$

for all $x, y \in X$, we get

$$\begin{aligned}
&\|\frac{f(2^l x, 2^l y) - f(2^l x, 0)}{4^l} - \frac{f(2^m x, 2^m y) - f(2^m x, 0)}{4^m}\| \\
&\le \sum_{j=l}^{m-1} \frac{\varepsilon}{4^{j+1}}(4\cdot 2^{jp}\|x\|^p + 5\cdot 2^{(j+1)p}\|y\|^p)
\end{aligned} \tag{2.9}$$

for given integers $l, m$ $(0 \le l < m)$ and all $x, y \in X$. Hence the sequence $\{\frac{f(2^n x, 2^n y) - f(2^n x, 0)}{4^n}\}$ is a Cauchy sequence for all $x, y \in X$. Since $Y$ is complete, the sequence $\{\frac{f(2^n x, 2^n y) - f(2^n x, 0)}{4^n}\}$ converges for all $x, y \in X$. Define the map $F_2 : X \times X \to Y$ by

$$F_2(x,y) = \lim_{n\to\infty} \frac{f(2^n x, 2^n y) - f(2^n x, 0)}{4^n}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.9), we can obtain the inequality

$$\|f(x,y) - f(x,0) - F_2(x,y)\| \le \frac{4\varepsilon}{4 - 2^p}\|x\|^p + \frac{5\cdot 2^p\varepsilon}{4 - 2^p}\|y\|^p$$

for all $x, y, z \in X$. Since

$$CF_2(x,y,z,w) = \lim_{n\to\infty} \frac{Cf(2^n x, 2^n y, 2^n z, 2^n w) - Cf(2^n x, 2^n y, 0, 0)}{4^n} = 0$$

for all $x, y \in X$, $F$ is a Cauchy-Jensen mapping satisfying (2.8), where

$$F(x,y) = F_1(x,y) + F_2(x,y)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.8). By Lemma 2.1, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\leq \|\frac{F(2^n x, 2^n y)}{4^n} + (2^n - 1)F(\frac{x}{2^n}, 0) - \frac{F'(2^n x, 2^n y)}{4^n} - (2^n - 1)F'(\frac{x}{2^n}, 0)\| \\
&\leq \frac{1}{4^n}\|F(2^n x, 2^n y) - f(2^n x, 2^n y)\| + 2^n\|F(\frac{x}{2^n}, 0) - f(\frac{x}{2^n}, 0)\| \\
&\quad + \frac{1}{4^n}\|f(2^n x, 2^n y) - f(2^n x, 2^n y)\| + 2^n\|f(\frac{x}{2^n}, 0) - F'(\frac{x}{2^n}, 0)\| \\
&\leq (\frac{2^{np}}{4^n} + \frac{2^n}{2^{np}})(\frac{2\varepsilon}{2^p - 2} + \frac{8\varepsilon}{4 - 2^p})\|x\|^p + \frac{2^{np}}{4^n}\frac{10 \cdot 2^p \varepsilon}{4 - 2^p}\|y\|^p
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we have $F(x,y) = F'(x,y)$ as desired. □

## 3. Supertability of a Cauchy-Jensen mapping

**Theorem 3.1.** *Let $p < 0$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x,y,z,w)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

*for all $x, y, z, w \in X\backslash\{0\}$. Then $f$ is a Cauchy-Jensen mapping.*

*Proof.* By Theorem 2.2, there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \leq \frac{\varepsilon}{2 - 2^p}\|x\|^p + \varepsilon\|y\|^p \tag{3.1}$$

for all $x, y \in X\backslash\{0\}$. From (3.1), we get

$$\begin{aligned}
&\|f(x,y) - F(x,y)\| \\
&= \frac{1}{2}\|(Cf - CF)((k+1)x, -kx, (k+2)y, -ky) \\
&\quad + (f - F)(-kx, -ky) + (f - F)((k+1)x, -ky) \\
&\quad + (f - F)((k+1)x, (k+2)y) + (f - F)(-kx, (k+2)y)\| \\
&\leq ((k+1)^p + k^p)\frac{4 - 2^p}{2(2 - 2^p)}\varepsilon\|x\|^p + ((k+2)^p + k^p)\frac{3}{2}\varepsilon\|y\|^p
\end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and $k \in \mathbb{N}$. Similarly we get

$$\begin{aligned}
&\|f(x,0) - F(x,0)\|((k+1)^p + k^p)\frac{4 - 2^p}{2(2 - 2^p)}\varepsilon\|x\|^p + 3k^p\varepsilon\|y\|^p, \\
&\|f(0,y) - F(0,y)\| \leq \frac{4 - 2^p}{2 - 2^p}k^p\varepsilon\|x\|^p + ((k+2)^p + k^p)\frac{3}{2}\varepsilon\|y\|^p, \\
&\|f(0,0) - F(0,0)\| \leq \frac{4 - 2^p}{2 - 2^p}k^p\varepsilon\|x\|^p + 3k^p\varepsilon\|y\|^p
\end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and $k \in \mathbb{N}$. As $k \to \infty$ in the above inequalities, we have $F(x,y) = f(x,y)$ for all $x, y \in X$. □

We can easily prove the following theorem by the similar method used in Theorem 2.2 and Theorem 3.1.

**Theorem 3.2.** *Let $p, q < 0$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x, y, z, w)\| \leq \varepsilon(\|x\|^p + \|y\|^p)(\|z\|^q + \|w\|^q)$$

*for all $x, y, z, w \in X \backslash \{0\}$. Then $f$ is a Cauchy-Jensen mapping.*

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