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SUPERSTABILITY OF A GENERALIZED EXPONENTIAL FUNCTIONAL EQUATION OF PEXIDER TYPE

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Abstract. We obtain the superstability of a generalized exponential functional equation

$$f(x+y) = E(x,y)g(x)f(y)$$

and investigate the stability in the sense of R. Ger [4] of this equation in the following setting:

$$\left|\frac{f(x+y)}{E(x,y)g(x)f(y)} - 1\right| \leq \varphi(x,y),$$

where $E(x,y)$ is a pseudo exponential function. From these results, we have superstabilities of exponential functional equation and Cauchy's gamma-beta functional equation.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [17]). Among those there was the question concerning the stability of homomorphisms : let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the next year, D. H. Hyers [5] answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized by Th. M. Rassias [15]. Since then, the stability problems of various functional equations have been investigated by many authors (see [3, 4, 6-13, 16]).

In this paper we define a generalized exponential functional equation

$$f(x+y) = E(x,y)g(x)f(y) \tag{1}$$

where $E(x,y)$ is a pseudo exponential function. And then we prove the superstability of (1) and the stability of (1) in the sense of R. Ger [4].

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2. Solution of a generalized exponential functional equation

If f and g are functions on R with $f = g$ and $E(x, y) = 1$, then the equation (1) is an exponential functional equation and so $f(x) = a^x$ is a solution of (1). Also if $E(x, y) = k$, then $f(x) = g(x) = \frac{1}{k}a^x$ is a solution of (1). In particular, if $E(x, y) = a^{xy-c_1}$ with $(a > 0)$, then $g(x) = a^{\frac{x^2}{2}+c_1}$ and $f(x) = a^{\frac{x^2}{2}+c_2}$ are solutions of the equation (1), where $c_1, c_2 \in R$.

Now consider Cauchy's gamma-beta functional equation studied in [14]. Note that the beta function $B(x, y)$ is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt\ .$$

If f is a function on $(0, \infty)$ and $E(x, y) = B(x, y)^{-1}$, then $f(x) = a^x\Gamma(x)$ is a solution of the equation (1). In particular, if f is a continuous solution with $f(1) = a > 0$, then f is a unique solution of the equation

$$f(x + y) = E(x, y)f(x)f(y).$$

3. Superstability of a generalized exponential functional equation

J. Baker et all [2] proved the Hyers-Ulam stability of Cauchy's exponential equation

$$f(x + y) = f(x)f(y).$$

That is, if the Cauchy difference $f(x+y) - f(x)f(y)$ of a real-valued function f defined on a real vector space is bounded for all x, y, then f is either bounded or exponential. Their result was generalized by J. Baker [1] : let S be a semigroup and let $f : S \to E$ be a mapping, where E is a normed algebra in which the norm is multiplicative. If f satisfies the functional inequality

$$\| f(xy) - f(x)f(y) \| \leq \delta$$

for all x, $y \in S$, then f is either bounded or multiplicative. In particular, such a phenomenon for some functional equation is called the superstability.

Definition. Let D be an additive subset of R; that is, $x + y \in D$ for any $x, y \in D$. A function $E : D \times D \to R$ is said to be *pseudo exponential* if $E(x, y)$ satisfies the following conditions;

(a) $E(x, y) = E(y, x) \quad (x, y \in D)$,
(b) $|E(x, y)| \geq 1 \quad (x, y \in D)$,
(c) $E(x, y)E(z, x + y) = E(x, y + z)E(y, z) \quad (x, y \in D)$.

Theorem 3.1. *Let D be an additive subset of R and $\varphi : D \to (0, \infty)$ a given function. Suppose that f and g are nonzero functions with $|g(m)| \geq \max\{2, 4\varphi(m)/|f(m)\}$ for some $m \in D$ and $E(x, y)$ a pseudo exponential function on $D \times D$ such that*

$$| f(x + y) - E(x, y)g(x)f(y) | \leq \varphi(x) \tag{2}$$

for all $x, y \in D$. *Then*

$$g(x+y) = E(x,y)g(x)g(y)$$

for all $x, y \in D$.

Proof. Let $e(x,y) = E(x,y)^{-1}$ for all $x, y \in D$. Since $|e(x,y)| \leq 1$,

$$| e(x,y)f(x+y) - g(x)f(y) | \leq \varphi(x) \tag{3}$$

for all $x, y \in D$. If we replace x by m and also y by m in (3), respectively, we get

$$| e(m,m)f(2m) - g(m)f(m) | \leq \varphi(m).$$

An induction argument implies that for all $n \geq 2$

$$\begin{aligned} &\left| f(nm) \prod_{i=1}^{n-1} e(m,im) - g(m)^{n-1} f(m) \right| \\ &\leq \varphi(m) \left(\prod_{i=1}^{n-2} |e(m,im)| + |g(m)| \prod_{i=1}^{n-3} |e(m,im)| \right. \\ &\left. + |g(m)|^2 \prod_{i=1}^{n-4} |e(m,im)| + \cdots + |g(m)|^{n-2} \right). \end{aligned} \tag{4}$$

Indeed, if the inequality (4) holds, we have

$$\begin{aligned} &\left| f((n+1)m) \prod_{i=1}^{n} e(m,im) - g(m)^n f(m) \right| \\ &\leq \ | e(nm,m)f((n+1)m) - g(m)f(nm) | \prod_{i=1}^{n-1} |e(m,im)| \\ &\quad + |g(m)| \ | f(nm) \prod_{i=1}^{n-1} e(m,im) - g(m)^{n-1} f(m) | \\ &\leq \varphi(m) \left(\prod_{i=1}^{n-1} |e(m,im)| + |g(m)| \prod_{i=1}^{n-2} |e(m,im)| \right. \\ &\quad \left. + |g(m)|^2 \prod_{i=1}^{n-3} |e(m,im)| + \cdots + |g(m)|^{n-1} \right) \end{aligned}$$

for all $n \geq 2$. By(4), we get

$$\begin{aligned}
&\left| \frac{f(nm)\prod_{i=1}^{n-1} e(m,im)}{g(m)^{n-1}f(m)} - 1 \right| \\
&\leq \frac{\varphi(m)}{|f(m)|}\left(\frac{1}{|g(m)|^{n-1}} + \frac{1}{|g(m)|^{n-2}} + \cdots + \frac{1}{|g(m)|}\right) \\
&< \frac{\varphi(m)}{|f(m)||g(m)|}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) \\
&= \frac{2\varphi(m)}{|f(m)||g(m)|} \leq \frac{1}{2}
\end{aligned}$$

for all positive integer n. Thus we can easily show that

$$|f(nm)| \to \infty \quad \text{as} \quad n \to \infty.$$

By (3), we obtain

$$\left| \frac{e(x,nm)f(x+nm)}{f(nm)} - g(x) \right| \leq \frac{\varphi(x)}{|f(nm)|}$$

and thus we have

$$g(x) = \lim_{n\to\infty} \frac{f(x+nm)e(x,nm)}{f(nm)}$$

for all $x \in D$. Since $e(x+y,nm)e(x,y) = e(y,nm)e(x,y+nm)$,

$$\begin{aligned}
&|g(x+y)e(x,y) - g(x)g(y)| \\
&= \lim_{n\to\infty} \left| \frac{f(x+y+nm)e(x+y,nm)e(x,y)}{f(nm)} - g(x)\frac{f(y+nm)e(y,nm)}{f(nm)} \right| \\
&= \lim_{n\to\infty} \frac{|e(y,nm)|}{|f(nm)|}|f(x+y+nm)e(x,y+nm) - g(x)f(y+nm)| \\
&\leq \lim_{n\to\infty} \frac{\varphi(x)}{|f(nm)|} = 0
\end{aligned}$$

for all $x, y \in D$. Thus we have

$$g(x+y) = E(x,y)g(x)g(y)$$

for all $x, y \in D$. □

Theorem 3.2. *Let D be an additive subset of R and $\varphi : D \to (0,\infty)$ a given function. Suppose that f and g be nonzero functions with $|g(m)| \geq \max\{2, 4\varphi(m)/|f(m)|\}$ for some $m \in D$ and $E(x,y)$ a pseudo exponential function on $D \times D$ such that*

$$(5) \qquad | f(x+y) - E(x,y)g(x)f(y) | \leq \min\{\varphi(x), \varphi(y)\}$$

for all x, $y \in D$. Then

$$f(x+y) = E(x,y)g(x)f(y)$$

for all x, $y \in D$.

Proof. By Theorem 3.1, $g(x+y) = E(x,y)g(x)g(y)$ for all $x, y \in D$. Then $g(nm) \geq g(m)^n$ for all n and so

$$|g(nm)| \to \infty \quad \text{as} \quad n \to \infty.$$

Thus we have

$$\left| \frac{f(nm+y)e(nm,y)}{g(nm)} - f(y) \right| \leq \frac{\varphi(y)}{|g(nm)|}$$

and so for all $y \in D$

$$f(y) = \lim_{n\to\infty} \frac{f(nm+y)e(nm,y)}{g(nm)}.$$

Since $e(x, nm+y)e(nm,y) = e(nm, x+y)e(x,y)$ with $e(x,y) = E(x,y)^{-1}$,

$$\begin{aligned}
&|f(x+y)e(x,y) - g(x)f(y)| \\
&= \lim_{n\to\infty} \left| \frac{f(nm+x+y)e(nm,x+y)e(x,y)}{g(nm)} - \frac{g(x)f(nm+y)e(nm,y)}{g(nm)} \right| \\
&= \lim_{n\to\infty} \frac{|e(nm,y)|}{|g(nm)|} |f(nm+x+y)e(x,nm+y) - g(x)f(nm+y)| \\
&\leq \lim_{n\to\infty} \frac{|e(nm,y)|\varphi(x)}{|g(nm)|} = 0
\end{aligned}$$

for all $x, y \in D$. Thus we have

$$f(x+y) = E(x,y)g(x)f(y)$$

for all $x, y \in D$. □

In Theorem 3.2, $f(x+y) = g(x+y)\frac{f(y)}{g(y)}$ for all $x, y \in D$ with $g(y) \neq 0$. Thus if $f(0) = g(0) = 1$, $f(x) = g(x)$ for all $x \in D$.

Definition. Let D be an additive subset of R containing 1. A function $\beta : D \times D \to R$ is said to be *beta-type* if $\beta(x,y)$ satisfies the following conditions;

(a) $\beta(x,y) = \beta(y,x) \quad (x, y \in D)$,
(b) $|\beta(n,m)| \leq 1 \quad (n, m \in Z_+)$,
(c) $\beta(x,y)\beta(z,x+y) = \beta(x,y+z)\beta(y,z) \quad (x, y \in D)$,
(d) $|\beta(x,n)| \leq \phi(x) \quad (\phi : D \to (0,\infty)$ is a function).

Note that the beta function $B(x,y)$ is a beta-type function with $B(x,m) < \phi(x) := \frac{1}{x}$. In [14], Y. W. Lee and B. M. Choi proved the superstability of Cauchy's gamma-beta functional equation

$$f(x+y) = B(x,y)^{-1}f(x)f(y).$$

The following theorem is a generalization of the result in [14].

Theorem 3.3. *Let D be an additive subset of R containing 1, and $\varphi : D \to (0,\infty)$ a given function. Suppose that f and g are nonzero functions with*

$|g(m)| \geq \max\{2, 4\varphi(m)/|f(m)|\}$ *for some* $m \in D$ *and* $\beta(x,y)$ *a beta-type function on* $D \times D$ *such that*

$$(6) \qquad |\, \beta(x,y)f(x+y) - g(x)f(y) \,| \leq \min\{\varphi(x), \varphi(y)\}$$

for all $x,\ y \in D$. *Then*

$$\beta(x,y)f(x+y) = g(x)f(y)$$

for all $x,\ y \in D$.

Proof. By the same techniques as in Theorem 3.1 and 3.2, we can prove the theorem. □

Corollary 3.4. *Let* $\delta > 0$, $a > 1$ *and* $\alpha \in (0,\infty)$ *be given. Suppose that* $f : [0,\infty) \to (0,\infty)$ *is a function with* $f(m) \geq \max(2, 2\sqrt{\delta})$ *for some positive integer* m *such that*

$$|\, f(x+y) - a^{xy+\alpha} f(x)f(y) \,| < \delta$$

for all $x,\ y \in (0,\infty)$. *Then*

$$f(x+y) = a^{xy+\alpha} f(x)f(y)$$

for all $x,\ y \in (0,\infty)$.

Proof. Let $E(x,y) = a^{xy+\alpha}$ for all $x,\ y \in (0,\infty)$. Then $E(x,y) \geq 1$ for all $x, y \in (0,\infty)$. Also

$$\frac{E(x,y)E(z,x+y)}{E(x,y+z)E(y,z)} = \frac{a^{xy}a^{z(x+y)}}{a^{x(y+z)}a^{yz}} = 1$$

for all $x, y, z \in (0,\infty)$. Thus $E(x,y)$ is a pseudo exponential function. By Theorem 3.2, we complete the proof. □

Corollary 3.5. *Let* $\delta > 0$ *and* $k \geq 1$ *be given. Suppose that* $f : R \to R$ *is a function with* $|f(m)| \geq \max(2, 2\sqrt{\delta})$ *for some positive integer* m *such that*

$$|\, f(x+y) - kf(x)f(y) \,| < \delta$$

for all $x,\ y \in R$. *Then*

$$f(x+y) = kf(x)f(y)$$

for all $x,\ y \in R$.

Proof. By Theorem 3.2 with $E(x,y) = k$, we complete the proof. □

4. Stability of the equation (1) in the sense of R. Ger [4]

R. Ger [4] suggested a new type of stability for the exponential equation

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leq \delta.$$

In this section, the stability problem in the sense of R. Ger [4] for the functional equation (1) shall be investigated. Throughout this section, we denote by D an additive subset of R and by $\varphi : D \times D \to [0, \infty)$ a functional such that

$$\varepsilon(x) := \sum_{i=0}^{\infty} \frac{ln(1+\varphi(2^i x, 2^i x))(1+\varphi(2^i x, s))(1+\varphi(s, 2^i x))}{2^{i+1}} < \infty$$

for all $x, s \in D$.

Theorem 4.1. *Let $E(x,y)$ be a pseudo exponential functional equation on $D \times D$. If $f, g : D \to (0, \infty)$ are functional such that*

$$\left| \frac{f(x+y)}{E(x,y)g(x)f(y)} - 1 \right| \leq \varphi(x,y) \tag{7}$$

for all $x, y \in D$ and $f(s), g(s) \geq 1$ for some $s \in D$, then there exists a unique function $H : D \to (0, \infty)$ such that

(i) $H(x+y) = E(x,y)H(x)H(y)$,

(ii) $\frac{1}{g(s)f(s)e^{\varepsilon(x)}} \leq \frac{H(x)}{f(x)} \leq g(s)f(s)e^{\varepsilon(x)}$,

(iii) $$\frac{1}{(1+\varphi(x,s))(1+\varphi(s,x))g(s)^2 f(s)^2 e^{\varepsilon(x)}} \leq \frac{H(x)}{g(x)} \leq (1+\varphi(x,s))(1+\varphi(s,x))g(s)^2 f(s)^2 e^{\varepsilon(x)}$$

for all $x, y \in D$.

Proof. Let $e(x,y) = E(x,y)^{-1}$ for all $x, y \in D$. Since $|e(x,y)| \leq 1$,

$$\left| \frac{e(x,y)f(x+y)}{g(x)f(y)} \right| \leq 1 + \varphi(x,y) \tag{8}$$

for all $x, y \in D$. If we define functions $G, F : D \to R$ by

$$G(x) = \ln g(x) \quad \text{and} \quad F(x) = \ln f(x)$$

for all $x \in D$, then the inequality (8) may be transformed into

$$|F(x+y) + \ln e(x,y) - G(x) - F(y)| \leq \ln(1 + \varphi(x,y)).$$

For $x = y$ the inequality (8) implies

$$|F(2x) + \ln e(x,x) - G(x) - F(x)| \leq \ln(1 + \varphi(x,x)) \tag{9}$$

for all $x \in D$. Letting $y = s$ in (8), we have

$$|F(x+s) + \ln e(x,s) - G(x)| \leq \ln(1+\varphi(x,s)) + |F(s)| \tag{10}$$

for all $x \in D$ and

$$|F(s+x) + \ln e(s,x) - F(x)| \leq \ln(1+\varphi(s,x)) + |G(s)| \tag{11}$$

for all $x \in D$. By (10) and (11),

$$|G(x) - F(x)| \leq \ln(1+\varphi(x,s))(1+\varphi(s,x)) + |G(s)| + |F(s)| \tag{12}$$

for all $x \in D$. Define a function u by

$$u(x) := \ln(1+\varphi(x,x))(1+\varphi(x,s))(1+\varphi(s,x)) + |G(s)| + |F(s)|$$

for all $x \in D$. By (9) and (12), we have

$$|F(2x) + \ln e(x,x) - 2F(x)| \leq u(x)$$

for all $x \in D$. That is,

$$\left|\frac{F(2x)}{2} + \ln e(x,x)^{\frac{1}{2}} - F(x)\right| \leq \frac{1}{2}u(x) \tag{13}$$

for all $x \in D$. We use induction on n to prove

$$\left|\frac{F(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - F(x)\right| \leq \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} u(2^i x) \tag{14}$$

for all $x \in D$. Indeed, on account of (13) the inequality (14) holds true for $n = 1$. Suppose that inequality (14) holds true for some $n > 1$. Then (13) and (14) imply

$$\begin{aligned}
&\left|\frac{F(2^{n+1}x)}{2^{n+1}} + \ln \prod_{i=0}^{n} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - F(x)\right| \\
&\leq \left|\frac{F(2^{n+1}x)}{2^{n+1}} + \ln \prod_{i=1}^{n} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - \frac{F(2x)}{2}\right| \\
&\quad + \left|\frac{F(2x)}{2} + \ln e(x,x)^{\frac{1}{2}} - F(x)\right| \\
&\leq \sum_{i=0}^{n} \frac{u(2^i x)}{2^{i+1}},
\end{aligned}$$

which ends the proof of (14). For any $x \in D$ and for every positive integer n we define

$$P_n(x) := \frac{F(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}}.$$

Let $m, n > 0$ be integers with $n > m$. By (14), we have

$$\begin{aligned}
&|P_n(x) - P_m(x)| \\
&= \frac{1}{2^m}\left|\frac{F(2^{n-m}(2^m x))}{2^{n-m}} + \ln\prod_{i=m}^{n-1} e(2^i x, 2^i x)^{\frac{1}{2^{i-m+1}}} - F(2^m x)\right| \\
&= \frac{1}{2^m}\left|\frac{F(2^{n-m}(2^m x))}{2^{n-m}} + \ln\prod_{i=0}^{n-m-1} e(2^i(2^m x), 2^i(2^m x)^{\frac{1}{2^{i+1}}} - F(2^m x)\right| \\
&\le \frac{1}{2^m}\sum_{i=0}^{n-m-1}\frac{u(2^i(2^m x))}{2^{i+1}} = \sum_{i=m}^{n-1}\frac{u(2^i x)}{2^{i+1}} \\
&\le \frac{G(s)+F(s)}{2^m} + \sum_{i=m}^{\infty}\frac{\ln(1+\varphi(2^i x, 2^i x))(1+\varphi(2^i x, s))(1+\varphi(s, 2^i x))}{2^{i+1}}
\end{aligned}$$

for all $x \in D$. Taking the limit as $m \to \infty$, we get

$$\lim_{m\to\infty} |P_n(x) - P_m(x)| = 0$$

for all $x \in D$. Therefore, the sequence $\{P_n(x)\}$ is a Cauchy sequence, and we may define a function

$$L(x) := \lim_{n\to\infty} P_n(x).$$

Note that

$$\begin{aligned}
e(x+y, x+y) &= \frac{e(x, x+y)e(y, y+2x)}{e(x,y)} \\
&= \frac{e(x,x)e(y,y)e(2x,2x)}{e(x,y)^2}
\end{aligned}$$

for all $x, y \in D$. Thus we have

$$\begin{aligned}
&\prod_{i=0}^{n-1}\left[\frac{e(2^i x, 2^i x)e(2^i y, 2^i y)}{e(2^i x + 2^i y, 2^i x + 2^i y)}\right]^{\frac{1}{2^{i+1}}} \\
&= \prod_{i=0}^{n-1}\left[\frac{e(2^i x, 2^i y)^2}{e(2^{i+1} x, 2^{i+1} y)}\right]^{\frac{1}{2^{i+1}}} \\
&= \left[\frac{e(x,y)^2}{e(2x,2y)}\right]^{\frac{1}{2}} \cdot \left[\frac{e(2x,2y)^2}{e(2^2 x, 2^2 y)}\right]^{\frac{1}{2^2}} \cdots \left[\frac{e(2^{n-1}x, 2^{n-1}y)^2}{e(2^n x, 2^n y)}\right]^{\frac{1}{2^n}} \\
&= \frac{e(x,y)}{e(2^n x, 2^n y)^{\frac{1}{2^n}}}
\end{aligned}$$

for all $x, y \in D$. Since $\varepsilon(x) < \infty$,

$$\frac{\ln(1+\varphi(2^n x, s))(1+\varphi(s, 2^n x))}{2^n} \to 0 \quad \text{and} \quad \frac{\ln(1+\varphi(2^n x, 2^n y))}{2^n} \to 0$$

as $n \to \infty$. By (12), we have

$$\begin{aligned}&\left|\frac{G(2^n x)}{2^n} - \frac{F(2^n x)}{2^n}\right| \\ &\leq \frac{|G(s)| + |F(s)|}{2^n} + \frac{\ln(1+\varphi(2^n x, s))(1+\varphi(s, 2^n x))}{2^n}\end{aligned}$$

for all $x \in D$. Thus we arrive at for all $x \in D$,

$$\begin{aligned}0 = &\lim_{n\to\infty}\left|\frac{F(2^n x + 2^n y)}{2^n} + \ln e(2^n x, 2^n y)^{\frac{1}{2^n}} - \frac{G(2^n x)}{2^n} - \frac{F(2^n y)}{2^n}\right| \\ = &\lim_{n\to\infty}\left|\frac{F(2^n x + 2^n y)}{2^n} + \ln e(2^n x, 2^n y)^{\frac{1}{2^n}} - \frac{F(2^n x)}{2^n} - \frac{F(2^n y)}{2^n}\right| \\ = &\lim_{n\to\infty}\left|\frac{F(2^n x + 2^n y)}{2^n} + \ln \prod_{i=0}^{n-1} e(2^i x + 2^i y, 2^i x + 2^i y)^{\frac{1}{2^{i+1}}}\right. \\ &- \left(\frac{F(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}}\right) \\ &- \left(\frac{F(2^n y)}{2^n} + \ln \prod_{i=0}^{n-1} e(2^i y, 2^i y)^{\frac{1}{2^{i+1}}}\right) \\ &\left.+ \ln \prod_{i=0}^{n-1}\left[\frac{e(2^i x, 2^i x)e(2^i y, 2^i y)}{e(2^i x + 2^i y, 2^i x + 2^i y)}\right]^{\frac{1}{2^{i+1}}} \cdot e(2^n x, 2^n y)^{\frac{1}{2^n}}\right| \\ = &|L(x+y) + \ln e(x,y) - L(x) - L(y)|.\end{aligned} \tag{15}$$

Let $H(x) := e^{L(x)}$ for all $x \in D$. Then by (15), we have

$$H(x+y) = E(x,y)H(x)H(y)$$

for all $x, y \in D$. Taking the limit in (14) as $n \to \infty$, we have

$$\begin{aligned}&|L(x) - F(x)| \\ &\leq \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{u(2^i x)}{2^{i+1}} = |G(s)| + |F(s)| + \varepsilon(x)\end{aligned}$$

for all $x \in D$. That is,

$$\left|\ln \frac{H(x)}{f(x)}\right| \leq \ln g(s)f(s)e^{\varepsilon(x)}$$

for all $x \in D$. Note that $g(s)f(s)e^{\varepsilon(x)} \geq 1$ for all $x \in D$. If $\frac{H(x)}{f(x)} \geq 1$,

$$1 \leq \frac{H(x)}{f(x)} \leq g(s)f(s)e^{\varepsilon(x)}$$

and if $0 < \frac{H(x)}{f(x)} < 1$,

$$\frac{1}{g(s)f(s)e^{\varepsilon(x)}} \leq \frac{H(x)}{f(x)} < 1$$

for all $x \in D$. Thus we have

$$\frac{1}{g(s)f(s)e^{\varepsilon(x)}} \leq \frac{H(x)}{f(x)} \leq g(s)f(s)e^{\varepsilon(x)}$$

for all $x \in D$. By (12) we get

$$\frac{1}{(1+\varphi(x,s))(1+\varphi(s,x))g(s)f(s)} \leq \frac{f(x)}{g(x)} \leq (1+\varphi(x,s))(1+\varphi(s,x))g(s)f(s)$$

for all $x \in D$. Then

$$\begin{aligned}
&\frac{1}{(1+\varphi(x,s))(1+\varphi(s,x))g(s)^2f(s)^2e^{\varepsilon(x)}} \\
&\leq \frac{H(x)}{g(x)} = \frac{H(x)}{f(x)}\frac{f(x)}{g(x)} \\
&\leq (1+\varphi(x,s))(1+\varphi(s,x))g(s)^2f(s)^2e^{\varepsilon(x)}
\end{aligned}$$

for all $x \in D$. It remains to show that H is unique. Suppose that $W : D \to (0,\infty)$ is another such function with

$$W(x+y) = E(x,y)W(x)W(y)$$

and

$$\frac{1}{g(s)f(s)e^{\varepsilon(x)}} \leq \frac{W(x)}{f(x)} \leq g(s)f(s)e^{\varepsilon(x)}$$

for all $x, y \in D$. Note that for all $x \in D$

$$\frac{H(2x)}{W(2x)} = \frac{H(x)^2}{W(x)^2}, \ldots, \frac{H(2^nx)}{W(2^nx)} = \frac{H(x)^{2^n}}{W(x)^{2^n}}$$

and

$$\begin{aligned}
&\frac{\varepsilon(2^nx)}{2^{n-1}} \\
&= \frac{1}{2^{n-1}}\sum_{i=0}^{\infty}\frac{\ln(1+\varphi(2^{n+i}x,2^{n+i}x))(1+\varphi(2^{n+i}x,s))(1+\varphi(s,2^{n+i}x))}{2^{i+1}} \\
&= \sum_{i=n}^{\infty}\frac{\ln(1+\varphi(2^ix,2^ix))(1+\varphi(2^ix,s))(1+\varphi(s,2^ix))}{2^{i+1}} \to 0
\end{aligned}$$

as $n \to \infty$. Thus we have

$$\frac{1}{g(s)^{\frac{1}{2^{n-1}}}f(s)^{\frac{1}{2^{n-1}}}e^{\frac{\varepsilon(2^nx)}{2^{n-1}}}} \leq \frac{H(x)}{W(x)} = \frac{H(x)}{f(x)}\frac{f(x)}{W(x)} \leq g(s)^{\frac{1}{2^{n-1}}}f(s)^{\frac{1}{2^{n-1}}}e^{\frac{\varepsilon(2^nx)}{2^{n-1}}}$$

for all $n \geq 1$, and so $H(x) = W(x)$ for all $x \in D$. □

Corollary 4.2. *Let $\delta > 0$, $\alpha \in (0,\infty)$ and $a > 1$ be given. Suppose that $f, g : [0,\infty) \to (0,\infty)$ be functions such that*

$$\left| \frac{f(x+y)}{a^{xy+\alpha} g(x) f(y)} - 1 \right| < \delta$$

for all $x, y \in [0,\infty)$ and $f(s)$, $g(s) \geq 1$. Then there exists a unique function $H : (0,\infty) \to (0,\infty)$ such that

$$H(x+y) = a^{xy+\alpha} H(x) H(y),$$

$$(1+\delta)^{-3} \leq \frac{H(x)}{f(x)} \leq (1+\delta)^3,$$

$$(1+\delta)^{-5} f(s)^{-1} g(s)^{-1} \leq \frac{H(x)}{g(x)} \leq (1+\delta)^5 f(s) g(s)$$

for all $x, y \in (0,\infty)$.

Proof. Let $\varphi(x,y) = \delta$. Then

$$\varepsilon(x) = \sum_{i=0}^{\infty} \frac{\ln(1+\delta)^3}{2^{i+1}} = \ln(1+\delta)^3.$$

By Theorem 4.1 with $E(x,y) = a^{xy+\alpha}$, we complete the proof. □

Remark 4.3. Let $B(x,y)$ be a beta function. By the same techniques as in the proof of Theorem 4.1, we have following result : let $\delta > 0$ be given. If a function $f : (0,\infty) \to (0,\infty)$ satisfies the inequality

$$\left| \frac{B(x,y) f(x+y)}{g(x) f(y)} - 1 \right| \leq \delta$$

for all $x, y \in (0,\infty)$ then there exists a unique function $H : (0,\infty) \to (0,\infty)$ such that

$$B(x,y) H(x+y) = H(x) H(y),$$

$$(1+\delta)^{-3} \leq \frac{H(x)}{f(x)} \leq (1+\delta)^3,$$

$$(1+\delta)^{-5} g(s)^{-1} f(s)^{-1} \leq \frac{H(x)}{g(x)} \leq (1+\delta)^5 f(s) g(s).$$

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