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SOFT WS-ALGEBRAS

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ABSTRACT. Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. In this paper we apply the notion of soft sets by Molodtsov to the theory of subtraction algebras. The notion of soft WS-algebras, soft subalgebras and soft deductive systems are introduced, and their basic properties are derived.

1. Introduction

To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [8]. Maji et al. [7] and Molodtsov [8] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [7] described the application of soft set theory to a decision making problem. Maji et al. [6] also studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of

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set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [9]. In this paper we apply the notion of soft sets by Molodtsov to the theory of subtraction algebras. We introduce the notion of soft WS-algebras, soft WS*-algebras, soft subalgebras and soft deductive systems, and then derive their basic properties.

2. Basic results on subtraction algebras

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true.

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.
(a12) $(x - y) - z = (x - z) - (y - z)$.

As a weak form of a subtraction algebra, Jun et al. discussed the weak subtraction algebras as follows:

Definition 2.1 ([3]). By a *weak subtraction algebra* (*WS-algebra*), we mean a triplet $(W, -, 0)$, where W is a nonempty set, $-$ is a binary operation on W and $0 \in W$ is a nullary operation, called *zero element*, such that

(S3) $(\forall x, y, z \in W)$ $((x - y) - z = (x - z) - y)$,

(S4) $(\forall x \in W)\ (x - 0 = x,\ x - x = 0)$,
(a12) $(\forall x, y, z \in W)\ ((x - y) - z = (x - z) - (y - z))$.

Note that every subtraction algebra is a WS-algebra, but the converse is not true in general (see [3]).

A nonempty subset S of a WS-algebra X is called a *subalgebra* of X if $x - y \in S$ for all $x, y \in S$. A nonempty subset A of a WS-algebra X is called a *deductive system* of a WS-algebra X (it is also called an ideal in [5]) if it satisfies

(b1) $0 \in A$,
(b2) $(\forall x \in X)\ (\forall y \in A)\ (x - y \in A \Rightarrow x \in A)$.

A mapping $f : X \to Y$ of WS-algebras is called a *homomorphism* if it satisfies

$$(\forall x, y \in X)(f(x - y) = f(x) - f(y)). \tag{2.1}$$

For a homomorphism $f : X \to Y$ of WS-algebras, the *kernel* of f, denoted by $\ker(f)$, is defined to be the set

$$\ker(f) = \{x \in X \mid f(x) = 0\}.$$

Let X be a WS-algebra. A fuzzy set $\mu : X \to [0, 1]$ is called a *fuzzy subalgebra* of X if $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3. Basic results on soft sets

Molodtsov [8] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 3.1 ([8]). A pair (F, A) is called a *soft set* over U, where F is a mapping given by

$$F : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [8].

Definition 3.2 ([6]). Let (F, A) and (G, B) be two soft sets over a common universe U. The *intersection* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i) $C = A \cap B$,
(ii) $(\forall e \in C)$ ($H(e) = F(e)$ or $G(e)$, (as both are same set)).

In this case, we write $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

Definition 3.3 ([6]). Let (F, A) and (G, B) be two soft sets over a common universe U. The *union* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

Definition 3.4 ([6]). If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) AND (G, B)" denoted by $(F, A)\widetilde{\wedge}(G, B)$ is defined by $(F, A)\widetilde{\wedge}(G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 3.5 ([6]). If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) OR (G, B)" denoted by $(F, A)\widetilde{\vee}(G, B)$ is defined by $(F, A)\widetilde{\vee}(G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cup G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 3.6 ([6]). For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a *soft subset* of (G, B), denoted by $(F, A)\widetilde{\subset}(G, B)$, if it satisfies:

(i) $A \subset B$,
(ii) For every $\varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

4. Soft WS-algebras

In what follows let X and A be a WS-algebra and a nonempty set, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of X, that is, R is a subset of $A \times X$ unless otherwise specified. A set-valued function $F : A \to \mathscr{P}(X)$ can be defined as $F(x) = \{y \in X \mid xRy\}$ for all $x \in A$. The pair (F, A) is then a soft set over X.

Definition 4.1. Let (F, A) be a soft set over X. Then (F, A) is called a *soft WS-algebra* (resp. soft WS*-algebra) over X if $F(x)$ is a subalgebra (resp. deductive system) of X for all $x \in A$.

Let us illustrate this definition using the following example.

Example 4.2. Let $X = \{0, 1, 2, 3, 4\}$ be a WS-algebra with the following Cayley table:

−	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Let (F, A) be a soft set over X, where $A = X$ and $F : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$F(x) = \{y \in X \mid xRy \Leftrightarrow y - (y - x) \in \{0, 1\}\}$$

for all $x \in A$. Then $F(0) = F(1) = X$, $F(2) = \{0, 1, 3, 4\}$, $F(3) = \{0, 1, 2, 4\}$, and $F(4) = \{0, 1, 2, 3\}$ are deductive systems of X. Therefore (F, A) is a soft WS*-algebra over X.

Example 4.3. Let $X = \{0, a, b, c, d\}$ be a WS-algebra with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

(1) Let (F, A) be a soft set over X, where $A = X$ and $F : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$F(x) = \{y \in X \mid xRy \Leftrightarrow x - (x - y) \in \{0, b\}\}$$

for all $x \in A$. Then $F(0) = F(b) = X$, $F(a) = \{0, b\}$, and $F(c) = F(d) = \{0, a, b\}$ are deductive systems of X. Therefore (F, A) is a soft WS*-algebra over X.

(2) Let (G, A) be a soft set over X, where $A = X$ and $G : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$G(x) = \{y \in X \mid xRy \Leftrightarrow y - (y - x) \in \{0, b\}\}$$

for all $x \in A$. Then $G(0) = G(b) = X$, $G(a) = \{0, b, c, d\}$, and $G(c) = G(d) = \{0, b\}$ are subalgebras of X. Therefore (G, A) is a soft WS-algebra over X. But (G, A) is not a soft WS*-algebra over X because $G(a) = \{0, b, c, d\}$ is not a deductive system of X since $a - c = 0 \in G(a)$ and $a \notin G(a)$.

Obviously every soft WS*-algebra over X is a soft WS-algebra over X, but the converse is not true as seen in Example 4.3(2).

Let A be a fuzzy subalgebra of X with membership function μ_A. Let us consider the family of α-level sets for the function μ_A given by

$$F(\alpha) = \{x \in X \mid \mu_A(x) \geq \alpha\},\ \alpha \in [0, 1].$$

Then $F(\alpha)$ is a subalgebra of X. If we know the family F, we can find the functions $\mu_A(x)$ by means of the following formula:

$$\mu_A(x) = \sup\{\alpha \in [0, 1] \mid x \in F(\alpha)\}.$$

Thus, every fuzzy subalgebra A may be considered as the soft WS-algebra $(F, [0, 1])$.

Theorem 4.4. *Let (F, A) be a soft WS-algebra (resp. soft WS*-algebra) over X. If B is a subset of A, then $(F|_B, B)$ is a soft WS-algebra (resp. soft WS*-algebra) over X.*

Proof. Straightforward. □

The following example shows that there exists a soft set (F, A) over X such that

(i) (F, A) is not a soft WS*-algebra over X.
(ii) there exists a subset B of A such that $(F|_B, B)$ is a soft WS*-algebra over X.

Example 4.5. Let (G, A) be a soft set over X given in Example 4.3(2). Note that (G, A) is not a soft WS*-algebra over X. But if we take $B = \{b, c, d\} \subset A$, then $(G|_B, B)$ is a soft WS*-algebra over X.

Theorem 4.6. *Let (F, A) and (G, B) be two soft WS-algebras (resp. soft WS*-algebra) over X. If $A \cap B \neq \emptyset$, then the intersection $(F, A)\tilde{\cap}(G, B)$ is a soft WS-algebra (resp. soft WS*-algebra) over X.*

Proof. Using Definition 3.2, we can write $(F, A)\tilde{\cap}(G, B) = (H, C)$, where $C = A \cap B$ and $H(x) = F(x)$ or $G(x)$ for all $x \in C$. Note that $H : C \to \mathscr{P}(X)$ is a mapping, and therefore (H, C) is a soft set over X. Since (F, A) and (G, B) are soft WS-algebras (resp. soft WS*-algebras) over X, it follows that $H(x) = F(x)$ is a subalgebra (resp. deductive system) of X or $H(x) = G(x)$ is a subalgebra (resp. deductive system) of X for all $x \in C$. Hence $(H, C) = (F, A)\tilde{\cap}(G, B)$ is a soft WS-algebra (resp. soft WS*-algebra) over X. □

Corollary 4.7. *Let (F, A) and (G, A) be two soft WS-algebras (resp. soft WS*-algebras) over X. Then their intersection $(F, A)\tilde{\cap}(G, A)$ is a soft WS-algebra (resp. soft WS*-algebra) over X.*

Proof. Straightforward. □

Theorem 4.8. *Let (F, A) and (G, B) be two soft WS-algebras (resp. soft WS*-algebras) over X. If A and B are disjoint, then the union $(F, A)\tilde{\cup}(G, A)$ is a soft WS-algebra (resp. soft WS*-algebra) over X.*

Proof. Using Definition 3.3, we can write $(F, A)\tilde{\cup}(G, B) = (H, C)$, where $C = A \cup B$ and for every $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \setminus B$ or $x \in B \setminus A$ for all $x \in C$. If $x \in A \setminus B$, then $H(x) = F(x)$ is a subalgebra (resp. deductive system) of X since (F, A) is a soft WS-algebra (resp. soft WS*-algebra) over X. If $x \in B \setminus A$, then $H(x) = G(x)$ is a subalgebra (resp. deductive system) of X since (G, B) is a soft WS-algebra (resp. soft WS*-algebra) over X. Hence $(H, C) = (F, A)\tilde{\cup}(G, A)$ is a soft WS-algebra (resp. soft WS*-algebra) over X. □

If A and B are not disjoint in Theorem 4.8, then Theorem 4.8 is not true in general as seen in the following example.

Example 4.9. Let $X = \{0, a, b, c, d\}$ be a WS-algebra which is given in Example 4.3

(1) Let $A = \{0, a, b, c\}$ be a subset of X and let $F : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$F(x) = \{y \in X \mid xRy \Leftrightarrow y - (y - x) \in \{0, b\}\}$$

for all $x \in A$. Then $F(0) = X$, $F(a) = \{0, b, c, d\}$, $F(b) = X$ and $F(c) = \{0, b\}$ are subalgebras of X. Hence (F, A) is a soft WS-algebra over X. Let $B = \{c\}$ be a subset of X and let $G : B \to \mathscr{P}(X)$ is a set-valued function defined by

$$G(x) = \{0, d\}$$

for all $x \in B$. Then $G(c) = \{0, d\}$ is a subalgebra of X. Hence (G, B) is a soft WS-algebra over X. But the union $(F, A)\tilde{\cup}(G, B)$ is not a soft WS-algebra over X because $F(c) \cup G(c) = \{0, b, d\}$ is not a subalgebra of X since $d - b = c \notin F(c) \cup G(c)$.

(2) Let $A = \{0, a, b, c\}$ be a subset of X and let $F : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$F(x) = \{y \in X \mid xRy \Leftrightarrow x - (x - y) \in \{0, b\}\}$$

for all $x \in A$. Then $F(0) = X$, $F(a) = \{0, b\}$ and $F(b) = X$ are deductive systems of X. Hence (F, A) is a soft WS*-algebra over X. Let $B = \{a, c\}$ be a subset of X and let $G : B \to \mathscr{P}(X)$ is a set-valued function defined by

$$G(x) = \{0, a, c\}$$

for all $x \in B$. But $G(a) = \{0, a, c\}$ and $G(c) = \{0, a, c\}$ are deductive systems of X. Hence (G, B) is a soft WS*-algebra over X. Then the union $(F, A)\tilde{\cup}(G, B)$ is not a soft WS*-algebra over X, since $F(a) \cup G(a) = \{0, a, b, c\}$ is not a deductive system of X because $d - c = b \in F(a) \cup G(a)$ and $d \notin F(a) \cup G(a)$.

Theorem 4.10. *If (F, A) and (G, B) are soft WS-algebras (resp. soft WS*-algebras) over X, then $(F, A)\tilde{\wedge}(G, B)$ is a soft WS-algebra (resp. soft WS*-algebra) over X.*

Proof. By Definition 3.4, we know that

$$(F, A)\tilde{\wedge}(G, B) = (H, A \times B),$$

where $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$. Since $F(x)$ and $G(y)$ are subalgebras (resp. deductive systems) of X, the intersection $F(x) \cap G(y)$ is also a subalgebra (resp. deductive system) of X. Hence $H(x, y)$ is a subalgebra (resp. deductive system) of X for all $(x, y) \in A \times B$, and therefore $(F, A)\tilde{\wedge}(G, B) = (H, A \times B)$ is a soft WS-algebra (resp. soft WS*-algebra) over X. □

Example 4.11. In Example 4.9, we note that $(F, A)\tilde{\wedge}(G, B)$ is a soft WS-algebra (resp. soft WS*-algebra) over X.

Definition 4.12. A soft WS-algebra (and/or soft WS*-algebra) (F, A) over X is said to be *trivial* (resp. *whole*) if $F(x) = \{0\}$ (resp. $F(x) = X$) for all $x \in A$.

Example 4.13. Let X be a WS-algebra. For any $a \in X$ and a deductive system I of X, let

$$(a;I) = \{x \in X \mid x - a \in I\}. \tag{4.1}$$

Note that $(a;I)$ is a deductive system of X. For $A = X$, let $F : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$F(x) = \{y \in X \mid xRy \Leftrightarrow y \in (0;\{0\})\}$$

for all $x \in A$. Then $F(x) = \{0\}$ for all $x \in A$, and so (F, A) is a trivial soft WS-algebra (and/or trivial soft WS*-algebra) over X. Now consider a WS-algebra $X = \{0, a, b, c, d\}$ in Example 4.3. For $A = X$, let $F : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$F(x) = \{y \in X \mid xRy \Leftrightarrow y \in (c;\{0, b\})\}$$

for all $x \in A$. Then $F(x) = X$ for all $x \in X$. Hence (F, A) is a whole soft WS-algebra (and/or whole soft WS*-algebra) over X.

Let $f : X \to Y$ be a mapping of WS-algebras. For a soft set (F, A) over X, $(f(F), A)$ is a soft set over Y, where $f(F) : A \to \mathscr{P}(Y)$ is defined by $f(F)(x) = f(F(x))$ for all $x \in A$.

Lemma 4.14. *Let $f : X \to Y$ be a homomorphism of WS-algebras. If (F, A) is a soft WS-algebra over X, then $(f(F), A)$ is a soft WS-algebra over Y. Moreover, if f is onto and (F, A) is a soft WS*-algebra over X, then $(f(F), A)$ is a soft WS*-algebra over Y.*

Proof. For every $x \in A$, we have $f(F)(x) = f(F(x))$ is a subalgebra of Y since $F(x)$ is a subalgebra of X and its homomorphic image is also a subalgebra of Y. Hence $(f(F), A)$ is a soft WS-algebra over Y. Assume that f is onto and (F, A) is a soft WS*-algebra over X. Then $F(x)$ is a deductive system of X. Since f is onto, it follows that $f(F)(x) = f(F(x))$ is a deductive system of Y for all $x \in A$ so that $(f(F), A)$ is a soft WS*-algebra over Y. □

Theorem 4.15. *Let $f : X \to Y$ be a homomorphism of WS-algebras and let (F, A) be a soft WS-algebra (resp. soft WS*-algebra) over X.*

(i) *If $F(x) = \ker(f)$ for all $x \in A$, then $(f(F), A)$ is the trivial soft WS-algebra (resp. soft WS*-algebra) over Y.*

(ii) *If f is onto and (F, A) is whole, then $(f(F), A)$ is the whole soft WS-algebra (resp. soft WS*-algebra) over Y.*

Proof. (i) Assume that $F(x) = \ker(f)$ for all $x \in A$. Then $f(F)(x) = f(F(x)) = \{0_Y\}$ for all $x \in A$. Hence $(f(F), A)$ is the trivial soft WS-algebra (resp. soft WS*-algebra) over Y by Lemma 4.14 and Definition 4.12.

(ii) Suppose that f is onto and (F, A) is whole. Then $F(x) = X$ for all $x \in A$, and so $f(F)(x) = f(F(x)) = f(X) = Y$ for all $x \in A$. It follows from Lemma 4.14 and Definition 4.12 that $(f(F), A)$ is the whole soft WS-algebra (resp. soft WS*-algebra) over Y. □

Definition 4.16. Let (F, A) and (G, B) be two soft WS-algebras over X. Then (F, A) is called a *soft subalgebra* (resp. *soft deductive system*) of (G, B), denoted by $(F, A)\tilde{<}(G, B)$ (resp. $(F, A)\tilde{\prec}(G, B)$), if it satisfies:

(i) $A \subset B$,
(ii) $F(x)$ is a subalgebra (resp. deductive systems) of $G(x)$ for all $x \in A$.

Example 4.17. Let (F, A) be a soft WS-algebra over X which is given in Example 4.2. Let $B = \{1, 3, 4\}$ be a subset of A and let $G : B \to \mathscr{P}(X)$ be a set-valued function defined by

$$G(x) = \{y \in X \mid xRy \Leftrightarrow y - (y - x) \in \{0, 1\}\}$$

for all $x \in B$. Then $G(1) = X$, $G(3) = \{0, 1, 2, 4\}$ and $G(4) = \{0, 1, 2, 3\}$ are WS-subalgebras of $F(1)$, $F(3)$ and $F(4)$, respectively. Hence (G, B) is a soft subalgebra of (F, A). Moreover it is a soft deductive system of (F, A).

Theorem 4.18. *Let (F, A) and (G, A) be two soft WS-algebras over X.*

(i) *If $F(x) \subset G(x)$ for all $x \in A$, then $(F, A)\tilde{<}(G, A)$.*
(ii) *If $B = \{0\}$ and (H, B), (F, X) are soft WS-algebras over X, then $(H, B)\tilde{<}(F, X)$.*

Proof. Straightforward. □

Theorem 4.19. *Let (F, A) be a soft WS-algebra over X and let (G_1, B_1) and (G_2, B_2) be soft subalgebras* (resp. *soft deductive system*) *of (F, A). Then*

(i) $(G_1, B_1)\tilde{\cap}(G_2, B_2)\tilde{<}(F, A)$ (resp. $(G_1, B_1)\tilde{\cap}(G_2, B_2)\tilde{\prec}(F, A)$).
(ii) *If $B_1 \cap B_2 = \emptyset$, then $(G_1, B_1)\tilde{\cup}(G_2, B_2)\tilde{<}(F, A)$* (resp. $(G_1, B_1)\tilde{\cup}(G_2, B_2)\tilde{\prec}(F, A)$).

Proof. (i) Using Definition 3.2, we can write

$$(G_1, B_1)\tilde{\cap}(G_2, B_2) = (G, B),$$

where $B = B_1 \cap B_2$ and $G(x) = G_1(x)$ or $G_2(x)$ for all $x \in B$. Obviously, $B \subset A$. Let $x \in B$. Then $x \in B_1$ and $x \in B_2$. If $x \in B_1$, then $G(x) = G_1(x)$ is a subalgebra (resp. deductive system) of $F(x)$ since $(G_1, B_1)\tilde{<}(F, A)$ (resp. $(G_1, B_1)\tilde{\prec}(F, A)$). If $x \in B_2$, then $G(x) = G_2(x)$ is a subalgebra (resp. deductive system) of $F(x)$ since $(G_2, B_2)\tilde{<}(F, A)$ (resp. $(G_2, B_2)\tilde{<}(F, A)$). Hence $(G_1, B_1)\tilde{\cap}(G_2, B_2) = (G, B)\tilde{<}(F, A)$ (resp. $(G_1, B_1)\tilde{\cap}(G_2, B_2) = (G, B)\tilde{\prec}(F, A)$).

(ii) Assume that $B_1 \cap B_2 = \emptyset$. We can write $(G_1, B_1)\tilde{\cup}(G_2, B_2) = (G, B)$ where $B = B_1 \cup B_2$ and

$$G(x) = \begin{cases} G_1(x) & \text{if } x \in B_1 \setminus B_2, \\ G_2(x) & \text{if } x \in B_2 \setminus B_1, \\ G_1(x) \cup G_2(x) & \text{if } x \in B_1 \cap B_2 \end{cases}$$

for all $x \in B$. Since $(G_i, B_i))\tilde{<}(F, A)$ (resp. $(G_i, B_i))\tilde{\prec}(F, A)$) for $i = 1, 2$, $B = B_1 \cup B_2 \subset A$ and $G_i(x)$ is a subalgebra (resp. deductive system) of $F(x)$ for all $x \in B_i$, $i = 1, 2$. Since $B_1 \cap B_2 = \emptyset$, $G(x)$ is a subalgebra (resp. deductive system) of $F(x)$ for all $x \in B$. Therefore $(G_1, B_1)\tilde{\cup}(G_2, B_2) = (G, B)\tilde{<}(F, A)$ (resp. $(G_1, B_1)\tilde{\cup}(G_2, B_2) = (G, B)\tilde{\prec}(F, A)$). □

Theorem 4.20. *Let $f : X \to Y$ be a homomorphism of WS-algebras and let (F, A) and (G, B) be soft WS-algebras over X. Then*

$$(F, A)\tilde{<}(G, B) \Rightarrow (f(F), A)\tilde{<}(f(G), B).$$

Proof. Assume that $(F, A)\tilde{<}(G, B)$. Let $x \in A$. Then $A \subset B$ and $F(x)$ is a subalgebra of $G(x)$. Since f is a homomorphism, $f(F)(x) = f(F(x))$ is a subalgebra of $f(G(x)) = f(G)(x)$, and thus $(f(F), A)\tilde{<}(f(G), B)$. □

Example 4.21. Let $X = \{0, a, b, c, d\}$ be a WS-algebra which is given in Example 4.3. Let $Y = \{0, 1, 2, 3\}$ be a WS-algebra with the following Cayley table [3]:

$-$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

If we define a mapping f from the WS-algebra X into the WS-algebra Y by $f(0) = 0$, $f(a) = 0$ $f(b) = 2$ $f(c) = 0$ and $f(d) = 2$, then it is a homomorphism of WS-algebras. In Example 4.3, it is shown that (G, A) is a soft WS-algebra. Let $B = \{0, b\}$ be a subset of A and let $H : B \to \mathscr{P}(X)$ be a set-valued function defined by

$$H(x) = \{y \in X \mid xRy \Leftrightarrow y * x \in \{0, a\}\}$$

for all $x \in B$. Then $H(0) = \{0, a\}$ and $H(b) = \{0, a, b\}$ are subalgebras of X. Hence (H, B) is a soft WS-algebra. Moreover, it can be shown that $H(0) = \{0, a\}$ and $H(b) = \{0, a, b\}$ are subalgebras of $G(0)$ and $G(b)$ respectively. Hence (H, B) is a soft subalgebra of (G, A). On the other hand, $f(H)(0) = f(H(0)) = \{0\}$ and $f(H)(b) = f(H(b)) = \{0, 2\}$ are subalgebras of $f(G)(0) = f(G(0)) = \{0, 2\}$ and $f(G)(b) = f(G(b)) = \{0, 2\}$ respectively. Therefore $(f(H), B)\tilde{<}(f(G), A)$.

Theorem 4.22. *Let $f : X \to Y$ be an onto homomorphism of WS-algebras and let (F, A) and (G, B) be soft WS-algebras over X. Then*

$$(F, A)\tilde{\prec}(G, B) \Rightarrow (f(F), A)\tilde{\prec}(f(G), B).$$

Proof. Assume that f is onto and $(F, A)\tilde{\prec}(G, B)$. Then $A \subset B$ and $F(x)$ is a deductive system of $G(x)$ for any $x \in A$. Since f is an onto homomorphism, $f(F)(x) = f(F(x))$ is a deductive system of $f(G(x)) = f(G)(x)$, and hence $(f(F), A)\tilde{\prec}(f(G), B)$. □

If $f : X \to Y$ be a not onto homomorphism of WS-algebras in Theorem 4.22, then Theorem 4.22 need not be true in general as seen in the following example.

Example 4.23. Let $X = \{0, 1, 2, 3, 4\}$ be a WS-algebra which is given in Example 4.2. Let $Y = \{0, a, b, c\}$ be a WS-algebra with the following Cayley table [3]:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

If we define a mapping f the from WS-algebra X into the WS-algebra Y by $f(0) = 0$, $f(1) = 0$, $f(2) = b$, $f(3) = c$ and $f(4) = 0$, then it is to shown that a homomorphism of WS-algebras.

In Example 4.2, (F, A) is a soft WS-algebra. Let $B = \{0, 4\}$ be a subset of A and let $G : B \to \mathscr{P}(X)$ be a set-valued function defined by

$$G(x) = \{0, 3\}$$

for all $x \in B$. $G(0) = \{0, 3\}$ and $G(4) = \{0, 3\}$ are subalgebras of X. Hence (H, B) is a soft WS-algebra. We know that (G, B) is a soft deductive system of (G, A). But $(f(G), B)$ is not a deductive system of $(f(F), A)$ because $f(G)(4) = f(G(4)) = \{0, c\}$ is not deductive system of $f(F)(4) = f(F(4)) = \{0, b, c\}$ since $b - c = 0 \in f(G(4))$ and $b \notin\in f(G(4))$

Corollary 4.24. *Let $f : X \to Y$ be a homomorphism of WS-algebras and let (F, A) and (G, B) be soft WS-algebras over X. Then*

$$(F, A)\tilde{\prec}(G, B) \Rightarrow (f(F), A)\tilde{<}(f(G), B).$$

Proof. Straightforward. □

References

[1] J. C. Abbott, *Sets, Lattices and Boolean Algebras,* Allyn and Bacon, Boston 1969.

[2] D. Chen, E. C. C. Tsang, D. S. Yeung, and X. Wang, *The parametrization reduction of soft sets and its applications,* Comput. Math. Appl. **49** (2005), 757–763.

[3] Y. B. Jun, Y. H. Kim, and K. J. Lee, *Weak forms of subtraction algebras,* Bull. Korean Math. Soc. **45** (2008), no. 3, 419–426.

[4] Y. B. Jun, H. S. Kim, and E. H. Roh, *Ideal theory of subtraction algebras,* Sci. Math. Jpn. **61** (2005), no. 3, 459–464.

[5] Y. B. Jun, C. H. Park, and E. H. Roh, *Order systems, ideals and right fixed maps of subtraction algebras,* Commun. Korean Math. Soc. **23** (2008), no. 1, 1–10.

[6] P. K. Maji, R. Biswas, and A. R. Roy, *Soft set theory,* Comput. Math. Appl. **45** (2003), 555–562.

[7] P. K. Maji, A. R. Roy, and R. Biswas, *An application of soft sets in a decision making problem,* Comput. Math. Appl. **44** (2002), 1077–1083.

[8] D. Molodtsov, *Soft set theory - First results,* Comput. Math. Appl. **37** (1999), 19–31.

[9] L. A. Zadeh, *Fuzzy sets,* Inform. Control **8** (1965), 338–353.

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