Commun. Korean Math. Soc. **23** (2008), No. 3, pp. 307–312

# REMARKS ON SOME COMBINATORIAL DETERMINANTS

Sangtae Jeong

Abstract. In this note we first give a simple, direct proof of a combinatorial determinant involving the usual higher derivatives and then obtain a corresponding result in positive characteristic.

## 1. Introduction

Let $k[[x]]$ be the ring of formal power series in a variable $x$ over the underlying field $k$. Over a field $k$ of characteristic 0, the following combinatorial determinant is known in [8]. Indeed, it concerns an evaluation of the Wronskian of powers of a single power series.

**Theorem A.** *For $f \in k[[x]]$, we have*

$$\det\left\{\left(\frac{d^j}{dx^j}f(x)^i\right)_{i,j=0}^{n}\right\} = 1!2!\cdots n!f'(x)^{n(n+1)/2} \quad (n = 0, 1, 2, \ldots).$$

This result follows from a property of the Wronskian:

$$\begin{aligned} & W_x(f_0(y), f_1(y), \ldots, f_n(y)) \\ = \;& (dy/dx)^{n(n+1)/2} W_y(f_0(y), f_1(y), \ldots, f_n(y)), \end{aligned}$$

where $W_x(f_0(y), f_1(y), \ldots, f_n(y))$ is defined as determinant of $(n+1) \times (n+1)$ matrix whose $i$-th row is the vector $\left(\frac{d^i}{dx^i}f_0(y), \frac{d^i}{dx^i}f_1(y), \ldots, \frac{d^i}{dx^i}f_n(y)\right)$ and $f_i (0 \le i \le n)$ are at least $n$ times differentiable functions. In what follows we shall deal with formal differentiability of a single function in $k[[x]]$ without mentioning convergence of it.

The purpose of this note is to first give a simple, direct proof of Theorem A and then to establish its analogue in positive characteristic. Besides, we derive an amusing identity among the coefficients of a twisted formal power series in the Frobenius endomorphism $\tau$. For Theorem A leads to a determinantal identity among the coefficients of a formal power series.

---

Received February 2, 2007.

2000 *Mathematics Subject Classification.* Primary 15A15.

*Key words and phrases.* determinants, Hasse-Teichmüller derivatives, Wronskian, formal power series.

This work was financially supported by KRF-2005-015-C00007.

## 2. Preliminaries

As it stands, the formula for $n \geq p$ in Theorem A is not interesting in the case where the underlying field $k$ is of prime characteristic $p$. In order to avoid this case we will consider the Hasse-Teichmüller operators $\mathsf{D}^{(j)}$, which are defined by

$$\mathsf{D}^{(j)}(\sum_{i=0}^{\infty} a_i x^i) = \sum_{i=0}^{\infty} \binom{i}{j} a_i x^{i-j}$$

on the ring $k[[x]]$ of formal power series with coefficients in a field $k$ of *arbitrary* characteristic. When the field $k$ has characteristic 0, $\mathsf{D}^{(j)}$ are closely related by the usual higher derivatives $\frac{d^j}{dx^j}$:

$$\mathsf{D}^{(j)} = \frac{1}{j!}\frac{d^j}{dx^j}. \tag{1}$$

Even when the field $k$ has positive characteristic $p$, it is easily seen that $\mathsf{D}^{(j)}$ is a non-trivial operator for each integer $j \geq p$. Like the ordinary higher derivatives, the Hasse-Teichmüller operators satisfy the product rule [10], quotient rule [2] and chain rule [3, 4]. We refer to [5] for a concise summary on all of these properties.

## 3. Equivalent statement

It is evident from the relation (1) that Theorem A is equivalent to the following statement.

**Theorem** B. *For $f \in k[[x]]$, we have*

$$\det\left\{\left(\mathsf{D}^{(j)} f(x)^i\right)_{i,j=0}^{n}\right\} = f'(x)^{n(n+1)/2} \quad (n = 0, 1, 2, \ldots).$$

To give a proof of Theorem B, we here state a special case of the product rule for $\mathsf{D}^{(j)}$ what we call, the power rule (see [10, 1]).

**Power rule.** *For all $f \in k[[x]]$ and $j > 0$,*

$$\mathsf{D}^{(j)}(f^i) = \sum_{e=1}^{j} \sum_{\lambda_1+\lambda_2+\cdots+\lambda_e=j, \lambda_l>0} \binom{i}{e} f^{i-e}\mathsf{D}^{(\lambda_1)}f \cdot \mathsf{D}^{(\lambda_2)}f \cdots \mathsf{D}^{(\lambda_e)}(f).$$

*Proof of Theorem* B. Put a matrix $b = (b_{i,j})$ by

$$b_{i,j} = (-1)^{i+j}\binom{i}{j} f^{i-j} \quad (i, j \geq 0)$$

and a matrix $c = (c_{i,j})$ by $c_{i,j} = \mathsf{D}^{(j)} f^i$. Then the theorem will follow from the fact that $bc$ is an upper triangular matrix with diagonal entries $(bc)_{i,i} = (\mathsf{D}^{(1)} f)^i = f'^i$. Now we use the power rule to compute $(bc)_{i,j}$ for $0 \leq j \leq i \leq n$ as follows:

$$
\begin{aligned}
&(bc)_{i,j}\\
&= \sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k} f^{i-k}\mathsf{D}^{(j)} f^k\\
&= \sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k} f^{i-k}\sum_{e=1}^{j}\sum_{\lambda_1+\lambda_2+\cdots+\lambda_e=j,\lambda_l>0}\binom{k}{e} f^{k-e}\mathsf{D}^{(\lambda_1)}f\cdot\mathsf{D}^{(\lambda_2)}f\cdots\mathsf{D}^{(\lambda_e)}(f)\\
&= \sum_{k=0}^{i}\sum_{e=1}^{j}\sum_{\lambda_1+\lambda_2+\cdots+\lambda_e=j,\lambda_l>0}(-1)^{i+k}\binom{i}{k}\binom{k}{e} f^{i-e}\mathsf{D}^{(\lambda_1)}f\cdot\mathsf{D}^{(\lambda_2)}f\cdots\mathsf{D}^{(\lambda_e)}(f)\\
&= \sum_{e=1}^{j}\sum_{k=0}^{i}\sum_{\lambda_1+\lambda_2+\cdots+\lambda_e=j,\lambda_l>0}(-1)^{i+k}\binom{i}{e}\binom{i-e}{i-k} f^{i-e}\mathsf{D}^{(\lambda_1)}f\cdot\mathsf{D}^{(\lambda_2)}f\cdots\mathsf{D}^{(\lambda_e)}(f)\\
&= \sum_{e=1}^{j}\sum_{\lambda_1+\lambda_2+\cdots+\lambda_e=j,\lambda_l>0}\binom{i}{e}\left(\sum_{k=0}^{i}(-1)^{i+k}\binom{i-e}{i-k}\right) f^{i-e}\mathsf{D}^{(\lambda_1)}f\cdot\mathsf{D}^{(\lambda_2)}f\cdots\mathsf{D}^{(\lambda_e)}(f)\\
&= \sum_{e=1}^{j}\sum_{\lambda_1+\lambda_2+\cdots+\lambda_e=j,\lambda_l>0}\binom{i}{e}(1-1)^{i-e} f^{i-e}\mathsf{D}^{(\lambda_1)}f\cdot\mathsf{D}^{(\lambda_2)}f\cdots\mathsf{D}^{(\lambda_e)}(f).
\end{aligned}
$$

The result follows now from the very last equality that $(bc)_{i,j} = 0$ for $i > j$ and $(bc)_{i,i} = f'^{i}$. □

As a special case, we immediately have the following corollary, which was first observed in [11]. We denote by $[x^j]f^i$ the coefficient of $x^j$ in $f^i$. Then it is easy to check that $\mathsf{D}^{(j)} f^i|_{x=0} = [x^j]f^i$.

**Corollary 1.** *Let $f(x) = 1 + a_1x + a_2x^2 + \cdots$ be a formal power series and define a matrix $c$ by*

$$c_{i,j} = [x^j]f^i \quad (i, j \geq 0).$$

*Then*

$$\det((c_{i,j})_{i,j=0}^{n}) = a_1^{n(n+1)/2} \quad (n = 0, 1, 2, \ldots).$$

As an application, we recall the Bernoulli numbers $B_j^{(i)}$ of higher order $i$, defined by

$$\left(\frac{x}{e^x - 1}\right)^i = \sum_{j=0}^{\infty}\frac{B_j^{(i)}}{j!}x^j.$$

The following corollary is then immediate from $B_1^{(1)} = B_1 = -\frac{1}{2}$.

**Corollary 2.**

$$\det((B_j^{(i)})_{i,j=0}^{n}) = (-1/2)^{n(n+1)/2}1!2!\cdots n! \quad (n = 0, 1, 2, \ldots).$$

We now introduce another combinatorial determinant involving the operator $(x\frac{d}{dx})^j$, which is less well known but has a number of applications to series in calculus. The operator has a nice expansion formula in terms of the usual higher derivatives (see [7]).

$$(2) \qquad (x\frac{d}{dx})^j f(x) = \sum_{e=1}^{j} S(j,e)x^e \frac{d^e f}{dx^e},$$

where $S(j,e)$ are Stirling numbers of the second kind. Together with the power rule and (1) we are able to use (2) to derive Theorem C analogous to Theorem A, in the same way we did in the proof of Theorem B.

**Theorem C.**

$$\det\left\{ \left( (x\frac{d}{dx})^j f(x)^i \right)_{i,j=0}^{n} \right\} = 1!2!\cdots n!(xf'(x))^{n(n+1)/2} \quad (n = 0,1,2,\ldots).$$

Along this work, there is another combinatorial determinant in the case where multiplication is replaced by composition.

**Theorem D.** *Let $f(x) = x + a_1x^2 + a_2x^3 + \cdots$ be a formal power series and define $f^{(0)} = x$ and $f^{(i)} = f(f^{(i-1)})$ for $i > 1$. Define a matrix $c$ by*

$$c_{i,j} = [x^{j+1}]f^{(i)} \quad (i,j \geq 0).$$

*Then*

$$\det((c_{i,j})_{i,j=0}^{n}) = 1!2!\cdots n! a_1^{n(n+1)/2} \quad (n = 0,1,2,\ldots).$$

*Proof.* See [6] □

## 4. An analogue in positive characteristic

We are now in a position of obtaining the analogue of Theorem D in characteristic $p > 0$. To this end, let $K$ denote a field of prime characteristic $p$ in which a subfield $\mathbb{F}_q$ of order $q = p^m$ is imbedded, and let $\tau$ be the Frobenius endomorphism on $G_a/K$, that is, $\tau(x) = x^q$. Let $K\{\{\tau\}\}$ be the ring of formal power series in $\tau$. Then multiplication in $K\{\{\tau\}\}$ is twisted by the commutation relation

$$\tau\alpha = \alpha^q\tau, \quad \alpha \in K.$$

More precisely, multiplication of two formal power series $f(\tau) = \sum_{i=0}^{\infty} a_i\tau^i$, $g(\tau) = \sum_{k=0}^{\infty} b_i\tau^i$ in $K\{\{\tau\}\}$ is given by

$$f(\tau)g(\tau) = \sum_{i=0}^{\infty} c_k\tau^k \quad \text{with} \quad c_k = \sum_{i+j=k} a_i b_j^{q^i}.$$

Let $A_{\mathbb{F}_q}(K)$ denote the set of $\mathbb{F}_q$-additive formal power series with coefficients in $K$. Then it is well known in [9] that $A_{\mathbb{F}_q}(K)$ is a ring with standard addition and multiplication being given by composition and that the map $K\{\{\tau\}\} \to A_{\mathbb{F}_q}(K), f(\tau) \mapsto f(\tau)(x)$ is isomorphic. We state the analogue of Theorem D for $K\{\{\tau\}\}$.

**Theorem E.** *Let $f(\tau) = \tau^0 + a_1\tau + a_2\tau^2 + \cdots$ be a twisted formal power series in $K\{\{\tau\}\}$ and define a matrix $c$ by*

$$c_{i,j} = [\tau^j]f^i \quad (i, j \geq 0).$$

*Then*

$$\det((c_{i,j})_{i,j=0}^n) = a_1^{e(n)} \quad (n = 0, 1, 2, \ldots),$$

*where*

$$e(n) = \frac{1}{q-1}(\frac{q^{n+1}-1}{q-1} - (n+1)).$$

By the isomorphism above, Theorem E is equivalent to the following statement.

**Theorem F.** *Let $f(\tau)(x) = x + a_1x^q + a_2x^{q^2} + \cdots$ be an $\mathbb{F}_q$-additive formal power series in $A_{\mathbb{F}_q}(K)$ and define*

$$f^{(0)}(\tau)(x) = x \quad \textit{and} \quad f^{(i)}(\tau)(x) = f(f^{(i-1)}(\tau)(x))$$

*for $i > 1$. Define a matrix $c$ by*

$$c_{i,j} = [x^{q^j}]f^{(i)}(\tau)(x) \quad (i, j \geq 0).$$

*Then*

$$\det((c_{i,j})_{i,j=0}^n) = a_1^{e(n)}, \quad (n = 0, 1, 2, \ldots)$$

*where $e(n)$ is as in Theorem E.*

*Proof.* We show that the matrix

$$b_{i,j} = (-1)^{i+j}\binom{i}{j} \quad (i, j \geq 0)$$

has the property that $bc$ is an upper triangular matrix. Put

$$g_i(x) = \sum_k (-1)^{i+k}\binom{i}{k} f^{(k)}(x).$$

Then $(bc)_{i,j} = [x^{q^j}]g_i(x)$, and the theorem follows from the fact that

$$g_i(x) = \sum_{j=0} b_j^{(i)} x^{q^{i+j}} \quad \text{with} \quad b_0^{(i)} = a_1^{1+q+\cdots+q^{i-1}},$$

which we prove by induction on $i$. For the case $i = 0$ it is obviously true. Assume that $g_{i-1}(x) = \sum_{j=0}^{\infty} b_j^{(i-1)} x^{q^{i+j-1}}$. Then we compute

$$\begin{aligned} g_i(x) &= g_{i-1}(f(x)) - g_{i-1}(x) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_j^{(i-1)} a_k^{q^{i+j-1}} x^{q^{i+j+k-1}} - \sum_{j=0}^{\infty} b_j^{(i-1)} x^{q^{i+j-1}} \quad \text{where} \quad a_0 = 1 \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_j^{(i-1)} a_k^{q^{i+j-1}} x^{q^{i+j+k-1}} - \sum_{j=0}^{\infty} b_j^{(i-1)} x^{q^{i+j-1}} \\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} b_j^{(i-1)} a_k^{q^{i+j-1}} x^{q^{i+j+k-1}}. \end{aligned}$$

Hence $b_0^{(i)} = b_0^{(i-1)} a_1^{q^{i-1}} = a_1^{1+q+\cdots+q^{i-1}}$, as desired. □

## References


[1] K. Conrad, *The digit principle*, J. of Number Theory **84** (2000), 230–257.

[2] R. Göttfert and H. Niederreiter, *Hasse-Teichmüller derivatives and products of linear recurring sequences*, Finite fields: theory, applications, and algorithms (Las Vegas, NV, 1993), 117–125, Contemp. Math. **168**, Amer. Math. Soc., Providence, RI, 1994.

[3] H. Hasse, *Theorie der höheren Differentiale in einem algebraischen Funktionenkörper mit vollkommenem Konstantenkörper bei beliebiger Charakteristik*, J. Reine Angew. Math. **175** (1936), 50–54.

[4] H. Hasse and F. K. Schmidt, *Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Funktionenkörper einer Unbestimmten*, J. Reine Angew. Math. **177** (1937), 215–237.

[5] S. Jeong, M. Kim, and J. Son, *On explicit formulae for Bernoulli numbers and their counterparts in positive characteristic*, J. Number Theory **113** (2005), 53–68.

[6] K. Kedlaya, *Another combinatorial determinant*, J. Combin. Theory Ser. A **90** (2000), no. 1, 221–223.

[7] P. Knoph, *The operator $(x\frac{d}{dx})^n$ and its applications to series*, Mathematics Magazine **76** (2003), no. 5, 364–371.

[8] L. Mina, *Formole generali delle derivate successive d'una funzione espresse mediante quelle della sua inverse*, Giornale di Mat. **43** (1904), 196–212.

[9] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics **210**, Springer-Verlag, New York, 2002.

[10] O. Teichmüller, *Differentialrechung bei Charakteristik p*, J. Reine Angew. Math. **175** (1936), 89–99.

[11] H. Wilf, *A combinatorial determinant*, (unpublished).



Department of Mathematics
Inha University
Incheon 402-751, Korea
*E-mail address*: stj@inha.ac.kr