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ON w-CHEBYSHEV SUBSPACES IN BANACH SPACES

Maram Shams, Hamid Mazaheri, and Sayed Mansour Vaezpour

Abstract. The purpose of this paper is to introduce and discuss the concept of w-Chebyshev subspaces in Banach spaces. The concept of quasi Chebyshev in Banach space is defined. We show that w-Chebyshevity of subspaces are a new class in approximation theory. In this paper, also we consider orthogonality in normed spaces.

1. Introduction

Let X be a real Banach space and let $Y \subset X$ be a closed subspace. Consider the set-valued mapping $P_Y : X \to 2^Y$ defined by $P_Y(x) = \{y \in Y : \|x - y\| = d(x, Y)\}$. We say that Y is proximinal in X if $P_Y(x) \neq \emptyset$ for every $x \in X$. If Y is proximinal, $P_Y(x)$ is called the best approximation set of x in Y. We say that Y is Chebyshev if $P_Y(x)$ is singleton for every $x \in X$ and Y is said to be quasi Chebyshev if $P_Y(x)$ is nonempty and compact for every $x \in X$ (see [1], [5-6]).

For a Banach space X we denote its unit sphere by S_X. For $x \in X$ with $d(x, Y) = 1$, let $Q_Y(x) = x - P_Y(x)$. It is easy to see that $Q_Y(x) = \{z \in S_X : f(z) = f(x) \ \forall f \in Y^\perp\}$.

For $f \in X^*$ we define the pre-duality map of X by

$$J_X(f) = \{z \in S_X : \ f(z) = \|f\|\}.$$

Suppose $Y \subset X$ is a proximinal hyperplane and suppose $f \in Y^\perp$, $\|f\| = 1$ and $Y = \ker f$. Now let $x \in S_X$, $d(x, Y) = 1$. Then we have $|f(x)| = 1$. Hence

$$Q_Y(x) = \begin{cases} J_X(f) & \text{if} \quad f(x) = 1, \\ J_X(-f) & \text{if} \quad f(x) = -1. \end{cases}$$

If we put,

$$\hat{Y} = \{x \in X : \|x\| = d(x, Y)\} = \{x \in X : 0 \in P_Y(x)\},$$

then $\hat{Y}$ is a closed set in X. We easily can prove that if $d(x, Y) = 1$, then

$$P_Y(x) = (x + (\hat{Y} \bigcap S_X)) \bigcap Y.$$

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2. w-Chebyshev subspaces

In this section we define w-Chebyshev subspaces in the normed spaces and characterize them.

Definition. Let X be a normed space, Y be a subspace of X. We say that W is a w-Chebyshev subspace, if for every $x \in X$, $x + (\hat{Y} \bigcap S_X)$ is a nonempty and weakly compact set in X.

Note that if Y is w-Chebyshev, then $P_Y(x)$ is weakly compact for $x \in X \backslash Y$.

Theorem 2.1. *Let X be a normed space, Y be a subspace of X with* $\operatorname{codim} Y = 1$. *Then the following statement are equivalent:*

i) *Y is w-Chebyshev.*
ii) *for every $f \in Y^{\perp}$, $J_X(f)$ is weakly compact.*
iii) *for every $x \in X$, $P_Y(x)$ is weakly compact.*

Proof. i) $\Rightarrow$ ii) suppose Y is w-Chebyshev, $f \in Y^{\perp}$ and $\{z_n\} \subset J_X(f)$. Put $h := f/\|f\|$ then we have $\|h\| = 1$, $h(z_n) = \|z_n\|$, hence $z_n \in \hat{Y} \bigcap S_X$. Thus the proof is complete.

ii) $\Rightarrow$ iii) Suppose $x \in X$ and $d(x, Y) = 1$. Let $\{z_n\}$ be a sequence in $P_Y(x)$ therefore $x - z_n \in Q_Y(x)$. Since for every $f \in Y^{\perp}$, $J_X(f)$ is weakly compact and in either case $J_Y(f)$ is weakly compact if $Q_Y(x)$ is, hence $P_Y(x)$ is weakly compact. Now suppose $d(x, Y) = \alpha$, then $d(\frac{1}{\alpha}x, Y) = 1$. By the above prove we have $P_Y(\frac{1}{\alpha}x)$ is weakly compact. Since $P_Y(\frac{1}{\alpha}x) = \frac{1}{\alpha}P_Y(x)$, the proof is complete.

iii) $\Rightarrow$ i) Since $\operatorname{codim} Y = 1$, therefore there exists $f \in X^*$ and $Y = \ker f$. Let $x \in X$ and $\{z_n\}$ be a sequence in $x + (\hat{Y} \bigcap S_X)$. Therefore $d(z_n - x, Y) = \|z_n - x\| = 1$. On the other hand, $d(z_n - x, Y) = |f(z_n - x)|/\|f\|$. Hence $z_n - x \in J_Y(f) \bigcup J_Y(-f)$ for each n. Thus by the above discussion, $J_X(f)$ is weakly compact. Hence $\{z_n - x\}$ has a weakly convergent subsequence. □

We know that every quasi Chebyshev subspace with codimension one of X is a w-Chebyshev subspace of X. The following example shows that there exists a w-Chebyshev subspace of a Banach space X which is not quasi Chebyshev.

Example 2.2. Let $W = l^2$ and let $W_0 = \langle (-1)^n \rangle$ be the subspace of l^{∞}. Put $X = W \bigoplus W_0$ and define a norm on X by,

$$\|x + y\| = \max\{\|x\|_2, \|y\|_{\infty}\}$$

for all $x \in W$ and all $y \in W_0$.

It is clear that $\|\cdot\|$ is a norm on X, X is a Banach space with respect to this norm and $\operatorname{codim} W = 1$. If $y = (-1)^n \in W_0$, its not difficult to show that

$$P_W(y) = \{x \in W : \|x\|_2 \leq 1\} = B_W.$$

Hence $P_W(y)$ is not compact because W is reflexive. Therefore, W is not quasi Chebyshev in X, but by Theorem 2.1, W is w-Chebyshev.

Definition. A subspace Y of a normed space X is called w-boundedly compact if for every bounded sequence $\{y_n\}$ in Y, there exists $x_0 \in Y$ and a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \xrightarrow{w} x_0$.

Note that every bounded subset of a reflexive normed space is w-boundedly compact and therefore is weakly compact.

Theorem 2.3. *Let $\hat{Y}$ be w-boundedly compact. Then $P_Y(x)$ is weakly compact.*

Proof. Suppose $d(x,Y) = 1$ and $\{y_n\}$ is a sequence in $Q_Y(x)$. Therefore $\|y_n\| = 1$ and for every $f \in Y^\perp$, $f(y_n) = f(x)$. Since $Q_Y(x) \subset \hat{Y} \bigcap S_X$ and $\hat{Y}$ is w-boundedly compact, there exists the subsequence $\{y_{n_k}\}$ such that $y_{n_k} \xrightarrow{w} y_0$. Hence for all $f \in Y^\perp$, $f(y_0) = \lim f(y_{n_k}) = f(x)$. By Hahn Banach theorem, there exists $f_1 \in X^*$ such that $\|f_1\| = 1, f_1 \in Y^\perp$ and $f_1(x) = 1$. Since for every $f \in X^*$, $\|f\| = 1$, $|f(y_0)| \leq 1$ and $f_1(y_0) = 1$ therefore $\|y_0\| = 1$. Thus $Q_Y(x)$ is weakly compact and $P_Y(x)$ is weakly compact. Now for all $x \in X \backslash Y$ let $d(x,Y) = \alpha \neq 0$ then $d(\frac{1}{\alpha}x, Y) = 1$. Therefore $P_Y(\frac{1}{\alpha}x)$ is weakly compact. Since $P_Y(\frac{1}{\alpha}x) = \frac{1}{\alpha}P_Y(x)$, the proof is complete. □

Theorem 2.4. *$\hat{Y}$ is w-boundedly compact if and only if Y is w-Chebyshev.*

Proof. Suppose Y is w-Chebyshev subspace and $\{y_n\} \subset \hat{Y}$ is a bounded sequence. Then there exists α such that $\|y_n\| \leq \alpha$. Put $x_n := \frac{y_n}{\|y_n\|}$, therefore $x_n \in \hat{Y} \bigcap S_X$. Since Y is w-Chebyshev, there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \xrightarrow{w} x_0$. Also $\{\|y_{n_k}\|\}$ is a bounded sequence in C then $\|y_{n'}\| \to \beta$ thus $y_{n'} \xrightarrow{w} \beta x_0$. Also $\{\|y_{n_k}\|$ is a bounded sequence in the complex field C then $\|y_{n_K}\| \longrightarrow \beta$ thus $y_{n'} \xrightarrow{w} \beta x$. Then $\hat{Y}$ is w-boundedly compact.

Conversely, suppose $\hat{Y}$ is w-boundedly compact, and $\{x_n\} \subset x + (\hat{W} \bigcap S_X)$. Then $d(x_n - x, W) = \|x_n - x\| = 1$ therefore $\{x_n - x\}$ is a bounded sequence in $\hat{Y}$ and has a w-convergence sequence $\{x_{n_k} - x\}$, $x_{n_k} - x \xrightarrow{w} x_0$. □

Theorem 2.5. *Let M be a proximinal subspace of X and W be a subspace of X containing M. If W is a w-Chebyshev, then W/M is w-Chebyshev.*

Proof. We know that for every $x \in X$, $d(x,W) = d(x + M, W/M)$. Suppose $\{y_n + M\} \subseteq \hat{y} + ((\widehat{W/M}) \bigcap S_{X/M})$. Then we have

$$d(y_n - y, W) = d((y_n - y) + M, W/M) = \|(y_n - y) + M\| = 1.$$

Since M is proximinal,

$$d(y_n - y, W) = \|(y_n - y) + m_n\|.$$

Then $m_n \in P_W(y_n - y)$, therefore $y_n - y - m_n \in \hat{W} \bigcap S_X$. Since W is w-Chebyshev there exists the subsequence $\{y_{n_k} - m_{n_k}\}$ such that $y_{n_k} - m_{n_k} \xrightarrow{w} x_0$. Since the canonical map $\pi : X \to X/M$ is continuous, therefore $\pi(y_{n_k} - m_{n_k}) \xrightarrow{w} \pi(x_0)$. Hence for all $f \in (X/M)^*$,

$$f(y_{n_k} + M) = f \circ \pi(y_{n_k}) = f \circ \pi(y_{n_k} - m_{n_k}) \to f \circ \pi(x_0) = f(x_0 + M).$$

Thus W/M is w-chebyshev. □

Theorem 2.6. *Let M be a proximinal w-boundedly compact subspace of a normed space X and W be a subspace of X containing M. If $(W\hat{/}M)\bigcap S_{X/M}$ is compact, then W is a w-Chebyshev subspace of X.*

Proof. Since for all $x \in X$ we have $d(x, W) = d(x + M, W/M)$. Therefore $\pi(\hat{W}\bigcap S_X) \subset (W\hat{/}M)\bigcap S_(X/M)$. Now suppose $\{x_n\}$ be a sequence in $x + (\hat{W}\bigcap S_X)$ then sequence $\{\pi(x_n - x)\}$ is in $(W\hat{/}M)\bigcap S_{X/M}$ where $\pi(x_n - x) = (x_n - x) + M$. Since $(W\hat{/}M)\bigcap S_{X/M}$ is compact, there exists $x_0 \in X$ and a subsequence such that $x_{n_k} + M \to x_0 + M$. Since M is proximinal, there exists a sequence $\{m_{n_k}\}$ in M such that $\|x_0 - x_{n_k} + x - m_{n_k}\| = d(x_0 - x_{n_k} + x, M)$. Therefore

$$\lim \|x_0 - x_{n_k} + x - m_{n_k}\| = \lim \|(x_{n_k} - x) + M - (x_0 + M)\| = 0.$$

Also M is w-boundedly compact, then there exists a m_0 such that $m_{n_k} \xrightarrow{w} m_0$. Put $y_0 = x_0 - m_0$, we have

$$\begin{aligned} |x^*(y_0) - x^*(x_{n_k} - x)| &= |x^*(x_0 - m_0 - x_{n_k} + m_{n_k} - m_{n_k} - x)| \\ &\leq |x^*(x_0 - x_{n_k} - x - m_{n_k})| + |x^*(m_{n_k} - m_0)| \\ &\leq \|x^*\|\|x_0 - x_{n_k} - x - m_{n_k}\|. \end{aligned}$$

Then $x_{n_k} - x \xrightarrow{w} y_0$. Since $\hat{W}\bigcap S_X$ is closed so $y_0 \in \hat{W}\bigcap S_X$ and the proof is complete. □

Definition. let X be a Banach space. A weak*-closed subspace $M \subset X^*$ is said to have property (W^*) if for every $x \in X\backslash M_\perp$,

$$D_x = \{y \in X : f(y) = f(x) \ \forall f \in M, \ \ \|y\| = \|x\|_M\}$$

is nonempty and weakly compact, where $\|x\|_M = \sup\{|f(x)| : \|f\| \leq 1, f \in M\}$.

Theorem 2.7. *Let Y be w-Chebyshev subspace of X. Then $Y^\perp$ has the property W^*.*

Proof. let $\|x\| = 1$ and $\{y_n\}$ is a sequence in D_x. Therefore for all $f \in Y^\perp$, $f(y_n) = f(x)$ and $\|y_n\| = \|x\|_M$. Then $\{y_n\} \subset Q_Y(x) \subset \hat{Y}\bigcap S_X$, hence there exists the subsequence $\{y_{n_k}\}$ such that $y_{n_k} \xrightarrow{w} y_0$. □

3. Orthogonality in normed linear spaces

Suppose X is a normed linear space and $x, y \in X$, x is said to be orthogonal to y and is denoted by $x \perp y$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all scalar α. If M_1 and M_2 are subsets of X, it is defined $M_1 \perp M_2$ if and only if for all $g_1 \in M_1, g_2 \in M_2$, $g_1 \perp g_2$. (see [3-5]). It is defined,

$$\breve{M} = \{x \in X : \ M \perp x\},$$

the cometric set of M.

At first we state lemmas which is needed in the proof of the main results.

Lemma 3.1 (Papini and Singer [6]). *Let G be a linear subspace of a normed linear space X and $x \in X \backslash G$. The following statements are equivalent:*

(i) *For every $G \in g$, $\|g - g_0\| \leq \|x - g\|$.*
(ii) *For each $g \in G$ there exists a functional $f^g \in X^*$ such that $\|f^g\| = 1$, $f^g(x) = f^g(g_0)$ and $f^g(g) = \|g\|$.*

Theorem 3.2. *Let X be a Banach space, M be a linear subspace of X and $F \subseteq X$. If $\breve{M}$ is a convex subset of X, then the following statements are equivalent:*

(i) *$M \perp F$.*
(ii) *For all $g \in M$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f|_F = 0$.*

Proof. i) $\Rightarrow$ ii). Since $M \perp \breve{M}$, then for each $g \in M$ there exists $f \in X^*$ such that $\|f\| = 1$, $f|_{\breve{M}} = 0$ and $f(g) = \|g\|$. Now $F \subseteq \breve{M}$, it follows that $f|_F = 0$.

ii) $\Rightarrow$ i). Suppose for all $g \in M$ there exists $f \in X^*$ such that $\|f\| = 1$, $f|_F = 0$ and $f(g) = \|g\|$. For all $g \in M$ and all $x \in F$, we have

$$\begin{aligned} \|g\| &= f(g) \\ &= f(g + \alpha x) \\ &\leq \|f\|\|g + \alpha x\| \\ &= \|g + \alpha x\| \end{aligned}$$

for all scalar α. Therefore $M \perp F$. □

Corollary 3.3. *Let X be a Banach space, M be a linear subspace of X and $x \in X$. If $\breve{M}$ is a convex subset of X, then the following statements are equivalent:*

(a) *$M \perp x$.*
(b) *For all $g \in M$, $\|g\|_{\Gamma_x} = \|g\|$, where*

$$\Gamma_x = \{f \in X^* : \ f(x) = 0\}, \text{ and } \|g\|_{\Gamma_x} = \sup_{f \in \Gamma_x} |f(g)|.$$

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MARAM SHAMS
DEPARTMENT OF MATHEMATICS
YAZD UNIVERSITY
YAZD, IRAN
E-mail address: shams@spu.yazduni.ac.ir

HAMID MAZAHERI
DEPARTMENT OF MATHEMATICS
YAZD UNIVERSITY
YAZD, IRAN
E-mail address: hmazaheri@yazduni.ac.ir

SAYED MANSOUR VAEZPOUR
DEPARTMENT OF MATHEMATICS
YAZD UNIVERSITY
YAZD, IRAN
E-mail address: vaez@yazduni.ac.ir