Bull. Korean Math. Soc. **45** (2008), No. 3, pp. 485–494

A BIJECTIVE PROOF OF THE SECOND REDUCTION FORMULA FOR LITTLEWOOD-RICHARDSON COEFFICIENTS

SOOJIN CHO, EUN-KYOUNG JUNG, AND DONGHO MOON

ABSTRACT. There are two well known reduction formulae for structural constants of the cohomology ring of Grassmannians, i.e., Littlewood-Richardson coefficients. Two reduction formulae are a conjugate pair in the sense that indexing partitions of one formula are conjugate to those of the other formula. A nice bijective proof of the first reduction formula is given in the authors' previous paper while a (combinatorial) proof for the second reduction formula in the paper depends on the identity between Littlewood-Richardson coefficients of conjugate shape.

In this article, a *direct* bijective proof for the second reduction formula for Littlewood-Richardson coefficients is given. Our proof is independent of any previously known results (or bijections) on tableaux theory and supplements the arguments on bijective proofs of reduction formulae in the authors' previous paper.

1. Introduction

Littlewood-Richardson coefficients are structural constants of the cohomology ring of Grassmannians and the ring of Schur functions. They also describe how a tensor product of two irreducible polynomial representations of a general linear group is decomposed as the sum of other irreducible representations. A combinatorial model for Littlewood-Richardson coefficients was first formulated by D. E. Littlewood and A. R. Richardson in 1934 (see [14]): They are counted by the number of skew tableaux with certain properties.

Two reductive formulae for Littlewood-Richardson coefficients are described in [6] in geometric terms. In the language of tableaux, the first reduction formula is to eliminate one row from a tableau of certain type while the second formula is to eliminate one column.

Received October 8, 2007.

2000 *Mathematics Subject Classification.* Primary 05E10; Secondary 14M15.

Key words and phrases. reduction formulae, Littlewood-Richardson coefficient, Schubert calculus.

This research was done while authors were visiting Korea Institute for Advanced Study.

The first author was supported by the grant No.20073020 from Ajou University.

The third author was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD) (KRF-2007-313-C00019).

©2008 The Korean Mathematical Society

Bijections for the proof of reduction formulae for Littlewood-Richardson coefficients are given in [1]. For the first reduction formula, an explicit bijection is given. A well known symmetry of Littlewood-Richardson coefficients using conjugate partitions is used for the second reduction formula. This gives us a bijective proof for the second reduction formula since there is a bijection by P. Hanlon and S. Sundaram [7] between the Littlewood-Richardson tableaux of conjugate shapes. It, however does not give an explicit bijection since the bijection of Hanlon and Sundaram needs multitude of calculations relying on Schensted's row insertion. Due to the intricacy of Hanlon-Sundaram's algorithm, it has been expected that a direct bijection for the second reduction formula of eliminating one column from each Littlewood-Richardson tableau must be more complicated than that for the first reduction formula of eliminating one row.

Contrary to our previous expectations, we, in this article, show there exists a direct (and explicit) bijective proof for the second reduction formula, which is simpler than that for the first reduction formula. This proof supplements the results in [1] by the authors. It is still interesting to understand why the bijective proof for the column-wise reduction is more manageable than that for the row-wise reduction.

In Section 2, we provide backgrounds, definitions and notations, and give statement of reduction formulae. In Section 3, we provide an algorithm to obtain a combinatorial proof of the second reduction formula. We prove that our algorithm is well defined in Section 4. Some remarks on the related works are given in the final section.

2. Preliminaries

We briefly review some basic terminologies and results related to Littlewood-Richardson coefficients in this section. Then, we state reduction formulae. We refer readers to [15, 5] for more details on tableaux theory and [6, 5, 8] for Schubert calculus on Grassmannians.

A *partition* is a nonincreasing sequence of nonnegative integers with finite nonzero numbers. *Ferrer's diagram* (or *diagram*) of a partition λ is a left-justified array of boxes with λ_i boxes in the ith row. For a partition λ, the *size* $|\lambda|$ of λ is the sum of all parts of λ, and λ' is the *conjugate partition* of λ obtained by transposing the rows and columns in the diagram of λ. For two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$, we let $\lambda \subseteq \nu$ denote the condition $\lambda_i \leq \nu_i$ for all i. If $\lambda \subseteq \nu$, then the diagram of ν/λ is the collection of boxes in the diagram of ν but not in the diagram of λ.

For two positive integers $k < n$, we let $G(k,n)$ be the *Grassmannian* of k-dimensional subspaces of $\mathbb{C}^n$, where $\mathbb{C}$ is the field of complex numbers. Let $(n-k)^k = (n-k, \dots, n-k)$ be the partition having k nonzero parts of equal size $n-k$. Then the set of *Schubert classes* $\{\sigma_\lambda \mid \lambda \subseteq (n-k)^k\}$ forms a basis of the cohomology ring $H^*(G(k,n), \mathbb{Z})$. For partitions $\lambda, \mu \subseteq (n-k)^k$, let $\sharp(\sigma_\lambda \cdot \sigma_\mu)_{G(k,n)}$ be the *intersection number* of two Schubert classes in the Grassmannian $G(k,n)$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $1 \leq \alpha \leq k$, we use conventions that $\lambda - \lambda_\alpha = (\lambda_1, \ldots, \widehat{\lambda_\alpha}, \ldots, \lambda_k)$, the partition obtained by deleting the αth row of λ, and $\lambda \ominus \alpha = (\lambda_1 - 1, \ldots, \lambda_\alpha - 1, \lambda_{\alpha+1}, \ldots, \lambda_k)$, the partition obtained by deleting the αth column of λ. For given k and n, if $\nu \subseteq (n-k)^k$, then we let $\nu^c = (n-k-\nu_k, n-k-\nu_{k-1}, \ldots, n-k-\nu_1)$ denote the *complement partition* of ν (with respect to k and n). Two basic reduction formulae explain relations among intersection numbers of three Schubert classes (see [6, page 202]), and can be proved by calculating the dimensions of appropriate subspaces, following the definition of Schubert varieties in Grassmannians. We fix two positive integers $k < n$ in the rest of the article.

Proposition 2.1 (First reduction formula). *For any three indices $0 \leq \alpha, \beta, \gamma \leq k$ with $\alpha + \beta + \gamma = 2k + 1$,*

$$\sharp(\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^c})_{G(k,n)}$$
$$= \begin{cases} 0 & \text{if } \lambda_\alpha + \mu_\beta + (\nu^c)_\gamma > n-k, \\ \sharp(\sigma_{\lambda-\lambda_\alpha} \cdot \sigma_{\mu-\mu_\beta} \cdot \sigma_{\nu^c-(\nu^c)_\gamma})_{G(k-1,n-1)} & \text{if } \lambda_\alpha + \mu_\beta + (\nu^c)_\gamma = n-k. \end{cases}$$

The following can be obtained by applying the first reduction formula to $\sharp(\sigma_{\lambda'} \cdot \sigma_{\mu'} \cdot \sigma_{\nu'^c})_{G(n-k,n)}$, where λ' is the conjugate partition of λ.

Proposition 2.2 (Second reduction formula). *For any three nonnegative integers λ_α, μ_β and $(\nu^c)_\gamma$ with $\lambda_\alpha + \mu_\beta + (\nu^c)_\gamma \geq 2(n-k)+1$,*

$$\sharp(\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^c})_{G(k,n)}$$
$$= \begin{cases} 0 & \text{if } \alpha + \beta + \gamma > k, \\ \sharp(\sigma_{\lambda\ominus\alpha} \cdot \sigma_{\mu\ominus\beta} \cdot \sigma_{\nu^c\ominus\gamma})_{G(k,n-1)} & \text{if } \alpha + \beta + \gamma = k \text{ and } \lambda_\alpha > \lambda_{\alpha+1}, \\ & \quad \mu_\beta > \mu_{\beta+1},\ (\nu^c)_\gamma > (\nu^c)_{\gamma+1}. \end{cases}$$

To give a combinatorial formulation of the intersection numbers of three Schubert classes, we need to set up some terms on tableaux. For partitions $\lambda \subseteq \nu$, a *skew tableau* of shape ν/λ with content μ is a filling of the diagram of ν/λ with μ_i i's. For a skew tableau T, we let $T(\zeta, \eta)$ denote the entry in the box at (ζ, η)-position of the tableau T, i.e., the (ζ, η)-box. A skew tableau T is called *column strict* if each column of T is strictly increasing and each row of T is weakly increasing. The *reverse row word* $w(T)$ of a skew tableau T is the word obtained by reading the entries of rows of T from right to left starting from top to bottom. A word $\mathbf{a} = a_1 a_2 \cdots a_r$ of positive integers is a *lattice word* if any initial subword $a_1 \cdots a_j$ of $\mathbf{a}$ contains as many i's as $(i+1)$'s for all i.

Definition 2.3. For three partitions λ, μ, ν such that $\lambda \subseteq \nu$, a column strict tableau T of shape ν/λ and content μ is a *Littlewood-Richardson tableau* (*LR-tableau*) if its reverse row word $w(T)$ is a lattice word. For λ, μ and ν, the corresponding *Littlewood-Richardson coefficient* (*LR-coefficient*) is the number of LR-tableau of shape ν/λ and content μ, and is denoted by $c^\nu_{\lambda,\mu}$.

The following gives a combinatorial description of the intersection numbers of Schubert varieties on Grassmannians [14].

Proposition 2.4. *For three partitions $\lambda, \mu, \nu \subseteq (n-k)^k$,*

$$c^{\nu}_{\lambda,\mu} = \sharp(\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^c})_{G(k,n)} .$$

Now, we restate reduction formulae in combinatorial way, i.e., in terms of number of LR-tableaux. The first part of Propositions 2.1 and 2.2 can be easily verified using properties of LR-tableaux (see [1]), and we only restate the second part of reduction formulae.

Theorem 2.5 (First reduction formula). *Let λ, μ and ν be partitions such that $\lambda, \mu, \nu \subseteq (n-k)^k$, $|\nu| = |\lambda| + |\mu|$ and $\lambda, \mu \subseteq \nu$. Suppose that there are indices α, β, γ such that $\alpha + \beta - \gamma = k$ and $\lambda_\alpha + \mu_\beta = \nu_\gamma$. Then the number of* LR*-tableaux of shape ν/λ and content μ is equal to the number of* LR*-tableaux of shape $(\nu - \nu_\gamma)/(\lambda - \lambda_\alpha)$ and content $(\mu - \mu_\beta)$.*

Theorem 2.6 (Second reduction formula). *Let λ, μ and ν be partitions such that $\lambda, \mu, \nu \subseteq (n-k)^k$, $|\nu| = |\lambda| + |\mu|$ and $\lambda, \mu \subseteq \nu$. Suppose that there are indices α, β, γ such that $\alpha + \beta + \gamma = k$ and $\lambda_\alpha + \mu_\beta + \nu^c_\gamma \geq 2(n-k) + 1$. We also assume that $\lambda_{\alpha+1} < \lambda_\alpha$, $\mu_{\beta+1} < \mu_\beta$ and $\nu^c_{\gamma+1} < \nu^c_\gamma$. Then the number of* LR*-tableaux of shape ν/λ and content μ is equal to the number of* LR*-tableaux of shape $\nu \ominus (k-\gamma)/\lambda \ominus \alpha$ and content $\mu \ominus \beta$.*

A nice bijection for Theorem 2.5 is given in [1] while the proof of Theorem 2.6 in [1] relies on the bijection by Hanlon and Sundaram between two sets of LR-tableaux of conjugate shapes and contents [7]. It, therefore, does not give an explicit and direct bijective proof of the reduction formula. Our main concern in this article is to give a direct bijective proof for Theorem 2.6 as we did in [1] for Theorem 2.5.

3. Algorithm

In this section, we define an algorithm for a bijection to prove Theorem 2.6. Throughout the section, we assume that partitions λ, μ, ν and indices α, β, γ satisfy the condition for Theorem 2.6 for given $k < n$. Note first that for α, β, γ such that $\alpha + \beta + \gamma = k$, $\lambda_\alpha + \mu_\beta + \nu^c_\gamma \geq 2(n-k) + 1$ if and only if

$$\lambda_\alpha - \nu_{\alpha+\beta+1} + \mu_\beta \geq n - k + 1. \tag{3.1}$$

The well-definedness of our bijective map depends on the following lemma whose proof is given in Section 4.

Lemma 3.2. *Assume that T is an* LR*-tableau of shape ν/λ and content μ. For each $i = 1, 2, \ldots, \beta$, there is a j with $\nu_{\alpha+\beta+1} + 1 \leq j \leq \lambda_\alpha$ such that*

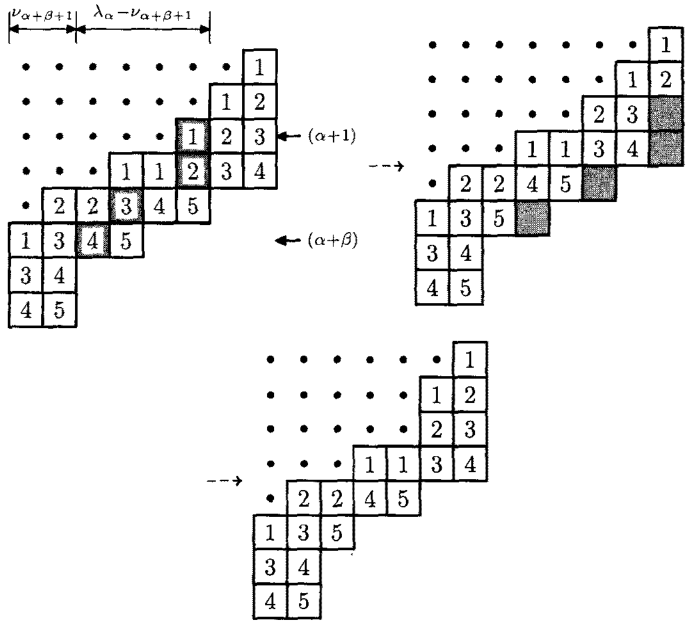
$$i = T(\alpha + i, j).$$

Now we provide a bijective map Φ:

Definition 3.3. For a given LR-tableau T of shape ν/λ and content μ, the reduced LR-tableau $\Phi(T)$ of shape $(\nu \ominus (k-\gamma)) \,/\, (\lambda \ominus \alpha)$ and content $\mu \ominus \beta$ is obtained by applying the following algorithm:

Step 1: **for** $i = 1$ **to** β **do**
  Empty a box in the $(\alpha+i)$th row and between the $(\nu_{\alpha+\beta+1}+1)$st column and the λ_αth column of T which is containing an i.
end for
Step 2: Slide the empty boxes right to reach the end of the row.
Step 3: **for** $i = 1$ **to** α **do**
  Move every box in the ith row to one box left.
end for
Step 4: Remove the empty boxes.

Example 3.4. Assume we are given partitions $\lambda = (7,6,5,3,1,0,0,0)$, $\mu = (6,5,5,5,3,0,0,0)$ and $\nu = (8,8,8,8,6,4,2,2)$. Therefore, we have $k = 8$ and $n-k = 8$. For $\alpha = 2$, $\beta = 4$ and $\gamma = 2$, we have $\lambda_2+\mu_4+\nu_2^c = 6+5+6 = 2\cdot 8+1$. The following shows how our reduction algorithm works to obtain $\Phi(T)$.



The inverse Ψ of Φ is easily obtained by inserting a box containing i, for $i = 1, \ldots, \beta$, appropriately. The precise definition of Ψ will appear in Section 4. Also a proof to show that Ψ is the inverse of Φ is given in Section 4.

4. Proofs

In this section, we give a proof that Definition 3.3 gives a well-defined bijection. We show that the tableaux we obtain by applying Algorithm 3.3 and its inverse (respectively) are LR-tableaux of expected shapes and contents. The proof that Φ and Ψ are inverse functions is immediate from the definitions.

Throughout this section, we fix n, k, partitions λ, μ, ν, and indices α, β, γ satisfying the conditions in Theorem 2.6. We also assume that T is an LR-tableau of shape ν/λ with content μ, and U is an LR-tableau of shape $\nu \ominus (k-\gamma)/\lambda \ominus \alpha$ with content $\mu \ominus \beta$. Whenever we refer to the order of a content in an LR-tableau, we mean the order in the corresponding reverse row word of the tableau. In a Young diagram, a box B' at (i', j')-position is on the *South-west* of the box B at (i, j)-position, if $i < i'$ and $j \geq j'$.

Lemma 4.1. *For any i and m, the mth $(i+1)$ is on the South-west of the mth i in T.*

Proof. We use an induction on m for a fixed i.

For $m = 1$, it is trivial because T is an LR-tableau.

Suppose that the lemma is true for m: The mth i is in the (a, b)-box of T and the mth $(i+1)$ is in the (a', b')-box of T, where $a < a'$ and $b \geq b'$. Let the $(m+1)$st i be in the (c, d)-box and the $(m+1)$st $(i+1)$ be in the (c', d')-box of T. Note that $a \leq c$, $b > d$, $a' \leq c'$ and $b' > d'$ since T is an LR-tableau.

Suppose that $a = c$. Then we have $d = b - 1$. Therefore, we have $d = b - 1 \geq b' - 1 > d' - 1$ and $d \geq d'$. We also have $c = a < a' \leq c'$, and the $(m+1)$st $i + 1$ is on the South-west of the $(m+1)$st i if $a = c$.

Suppose that $a < c$. Then we have $c < c'$ because T is an LR-tableau. Hence we only need to show that $d \geq d'$. Assume that $d < d'$ on the contrary. Then $T(c, d') = i$ since $i = T(c, d) \leq T(c, d') < T(c', d') = i + 1$. This gives a contradiction since the i in the (c, d')-box is between two i's in the (a, b)-box and the (c, d)-box, which are the mth and the $(m+1)$st i respectively. Therefore, we have $d \geq d'$. □

The *reduction window of T* is a set of boxes which are in β consecutive rows from the $(\alpha+1)$st row to the $(\alpha+\beta)$th row and $(\lambda_\alpha - \nu_{\alpha+\beta+1})$ consecutive columns from the $(\nu_{\alpha+\beta+1} + 1)$st column to the λ_αth column. Observe that the reduction window contains all boxes of T whose column index is between $(\nu_{\alpha+\beta+1} + 1)$ and λ_α. Therefore, there must be at least one 1 in the reduction window of T; otherwise, the number of 1's in T can be at most $(n-k) - (\lambda_\alpha - \nu_{\alpha+\beta+1}) \leq \mu_\beta - 1 < \mu_1$ because of Equation (3.1). We assume that the first 1 in the reduction window of T is the ℓth 1 in $w(T)$. Then we have the following useful inequality.

Lemma 4.2. *We have the following inequality;*

$$\ell - 1 \leq n - k - \lambda_\alpha . \tag{4.3}$$

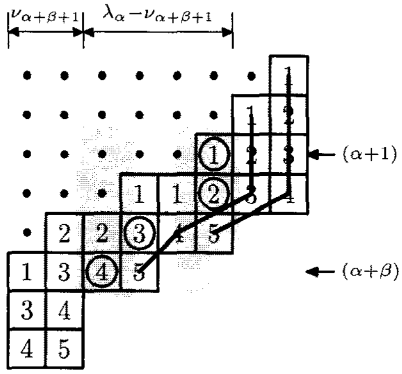
Proof. Note first that any number can appear at most once in each column of T since each column of T is strictly increasing. The right hand side of the inequality is the number of columns in which the first $(\ell - 1)$ 1's can appear and this finishes the proof. □

Lemma 4.4. *For $i = 1, \ldots, \beta$, the ℓth i is in the reduction window of T.*

Proof. The ℓth 1 is in the reduction window of T by the definition of ℓ.

Now suppose there is an integer $2 \leq i \leq \beta$ such that the ℓth i is not in the reduction window of T. Then the ℓth i should be placed in the jth column with $j \leq \nu_{\alpha+\beta+1}$ by Lemma 4.1. Therefore, we have $\mu_i \leq \nu_{\alpha+\beta+1} + \ell - 1$ since there can be at most one i in each column of T. Now, we have $\mu_i \leq \nu_{\alpha+\beta+1} + n - k - \lambda_\alpha \leq \mu_\beta - 1$ because of Equations (3.1) and (4.3), which is a contradiction. □

In the following picture, one may check the validity of Lemmas 4.1 and 4.4.



Note that Lemmas 4.1 and 4.4 imply Lemma 3.2 and Step 1 of Definition 3.3 is justified. We now show that $\Phi(T)$ is an LR-tableau. Clearly, the rows of $\Phi(T)$ are weakly increasing. Hence we prove that the columns of $\Phi(T)$ are strictly increasing and $w(\Phi(T))$ is a lattice word.

Theorem 4.5. *The columns of $\Phi(T)$ are strictly increasing.*

Proof. For $1 \leq j \leq \nu_{\alpha+\beta+1}$, the jth column of $\Phi(T)$ is the jth column of T. Moreover, for $\lambda_\alpha \leq j \leq n-k-1$, the jth column of $\Phi(T)$ is the $(j+1)$st column of T. Therefore, we only have to care about the jth column for $\nu_{\alpha+\beta+1} + 1 \leq j \leq \lambda_\alpha - 1$. Note that every such column consists of boxes lying on between the $(\alpha+1)$st row and the $(\alpha+\beta)$th row.

For $i = 1, \ldots, \beta$ and $j = \nu_{\alpha+\beta+1} + 1, \ldots, \lambda_\alpha - 1$, either $(\Phi(T))(\alpha+i, j) = T(\alpha+i, j)$ or $(\Phi(T))(\alpha+i, j) = T(\alpha+i, j+1)$ is satisfied. Therefore, the only case that might cause a trouble is when $(\Phi(T))(\alpha+i, j) = T(\alpha+i, j+1)$ but $(\Phi(T))(\alpha+i+1, j) = T(\alpha+i+1, j)$. This, however, can not happen because of Lemma 4.1. □

Theorem 4.6. *The reverse row word $w(\Phi(T))$ is a lattice word.*

Proof. Note that $w(\Phi(T))$ is obtained by removing the ℓth $1, 2, \ldots, \beta$ in $w(T)$. Therefore, the subword of $w(\Phi(T))$ with entries $1, 2, \ldots, \beta$ is a lattice word, and the subword of $w(\Phi(T))$ with entries $\beta+1, \ldots, k$ is a lattice word also. Therefore, we only have to care about the relations between the number of β's and $(\beta+1)$'s on the left of each position of $w(\Phi(T))$. Here, we claim that the ℓth $(\beta+1)$ appears after the $(\ell+1)$st β in $w(T)$, which finishes the proof.

Assume the contrary. Lemma 4.4 and column increasingness of T imply that the ℓth β is on the $(\alpha+\beta)$th row of T. Thus, the ℓth $(\beta+1)$ can not be in the reduction window of T because of Lemma 4.1. Therefore, the ℓth $(\beta+1)$ must be in the first $\nu_{\alpha+\beta+1}$ columns of T. Because of the assumption, there is no β in the column where the ℓth $(\beta+1)$ is. Hence, the number of columns where β can appear is strictly less than $\ell+\nu_{\alpha+\beta+1}$. However, by Equations (3.1) and (4.3), we have $\ell-1 \leq n-k-\lambda_\alpha \leq \mu_\beta - \nu_{\alpha+\beta+1}-1$, and hence $\ell+\nu_{\alpha+\beta+1} \leq \mu_\beta$. This implies a contradiction. □

In the following, we show that $\Psi(U)$ is an LR-tableau of shape ν/λ and of content μ, that is, the reverse map of Φ is well-defined. It is easy to check that following two steps define $\Psi(U)$.

Step 1: **for** $i=1$ **to** α **do**
    Move every box in the ith row of U to one box right (so that we have one more dot in each row).
  end for

Step 2: **for** $i=1$ **to** β **do**
    Insert an extra box containing i in the $(\alpha+i)$th row so that each row is weakly increasing.
  end for

Obviously, rows of $\Psi(U)$ are weakly increasing and the reverse row word $w(\Psi(U))$ forms a lattice word. We, therefore, only need to check that columns of $\Psi(U)$ are strictly increasing.

Theorem 4.7. *The columns of $\Psi(U)$ are strictly increasing.*

Proof. It is clear that the columns of the first α rows and the last $(n-\alpha-\beta)$ rows of $\Psi(U)$ are strictly increasing. Without loss of generality, for $i=1,\ldots,\beta$, we may assume that the new box containing i is the rightmost i in the $(\alpha+i)$th row of $\Psi(U)$. Then, it is easy to see that the new box containing $(i+1)$ must be on the South-west of the new box containing i. This proves that the columns of β rows from the $(\alpha+1)$st to the $(\alpha+\beta)$th row of $\Psi(U)$ are strictly increasing.

Now, we care about the relation between the $(\alpha+\beta)$th row and the $(\alpha+\beta+1)$st row of $\Psi(U)$. Observe that $(\Psi(U))(\alpha+\beta,j) \leq U(\alpha+\beta,j)$ for all possible j since we insert an extra β on the $(\alpha+\beta)$th row of U so that the row is weakly increasing. Moreover, there is no box right below the last box of the $(\alpha+\beta)$th row of $\Psi(U)$. This shows that the column strictness is satisfied for the $(\alpha+\beta)$th row and the $(\alpha+\beta+1)$st row of $\Psi(U)$.

Finally, consider the αth and the $(\alpha+1)$st rows of $\Psi(U)$. Note that the αth row of $\Psi(U)$ is obtained by adding one dot on the left of the αth row of U, and the $(\alpha+1)$st row of $\Psi(U)$ is obtained by adding one 1 on the left of the $(\alpha+1)$st row of U. Therefore, column strictness is satisfied for these two rows of $\Psi(U)$. □

5. Remarks

(1) Theorem 2.5, the first reduction formula, is a special case of the conjecture by R. King, C. Tollu and F. Toumazet on factorization of Littlewood-Richardson polynomials [9, 10], which is now a theorem (see [11, 1, 2]). Since Theorem 2.6, the second reduction formula, is a conjugated version of the first reduction formula, one may presume that Theorem 2.6 is also a special case of the factorization theorem by King, Tollu and Toumazet of LR-polynomials. It, however, is not a special case of the factorization theorem of LR-polynomials by King, Tollu and Toumazet.

(2) An analogous reduction formula to the first reduction formula, removing two rows at a time, is also proved by the authors as a special case of the factorization theorem of King, Tollu and Toumazet in combinatorial way (see [2]).

(3) There are many combinatorial models for LR-coefficients other than LR-tableaux: hives [12], puzzles [13], checker boards [16], Mondrian tableaux [3, 4]. It would be interesting to find bijective proofs of reduction formulae in terms of different models of LR-coefficients.

References

[1] S. Cho, E.-K. Jung, and D. Moon, *A combinatorial proof of the reduction formula for Littlewood-Richardson coefficients*, J. Combin. Theory Ser. A **114** (2007), no. 7, 1199–1219.

[2] ———, *Some cases of King's conjecture on factorization of Littlewood-Richardson polynomials*, preprint, 2007.

[3] I. Coskun, *A Littlewood-Richardson rule for two-step flag varieties*, Preprint, `http://www-math.mit.edu/~coskun/reviki51.pdf`.

[4] I. Coskun and R. Vakil, *Geometric positivity in the cohomology of homogeneous spaces and generalized Schubert calculus*, Preprint, `http://www-math.mit.edu/~coskun/seattleoct17.pdf`.

[5] W. Fulton, *Young Tableaux*, With applications to representation theory and geometry. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[6] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978.

[7] P. Hanlon and S. Sundaram, *On a bijection between Littlewood-Richardson fillings of conjugate shape*, J. Combin. Theory Ser. A **60** (1992), no. 1, 1–18.

[8] J. Harris, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995.

[9] R. C. King, C. Tollu, and F. Toumazet, *Stretched Littlewood-Richardson and Kostka coefficients*, Symmetry in physics, CRM Proc. Lecture Notes, vol. 34, Amer. Math. Soc., Providence, RI, 2004, pp. 99–112.

[10] ———, *The hive model and the polynomial nature of stretched Littlewood-Richardson coefficients*, Sém. Lothar. Combin. **54A** (2006), 1–19.

[11] ———, *Factorization of Littlewood-Richardson coefficients*, preprint, 2007.

[12] A. Knutson and T. Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbf{C})$ *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

[13] A. Knutson, T. Tao, and C. Woodward, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48.

[14] D. E. Littlewood and A. R. Richardson, *Group characters and algebra*, Phi. Trans. A (1934), 99–141.

[15] R. P. Stanley, *Enumerative Combinatorics. Vol. 2*, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.

[16] R. Vakil, *A geometric Littlewood-Richardson rule*, Appendix A written with A. Knutson. Ann. of Math. (2) **164** (2006), no. 2, 371–421.

Soojin Cho
Department of Mathematics
Ajou University
Suwon 443-749, Korea
E-mail address: chosj@ajou.ac.kr

Eun-Kyoung Jung
Department of Mathematics
Ajou University
Suwon 443-749, Korea
E-mail address: ejung@ajou.ac.kr

Dongho Moon
Department of Applied Mathematics
Sejong University
Seoul 143-747, Korea
E-mail address: dhmoon@sejong.ac.kr