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A KUROSH-AMITSUR LEFT JACOBSON RADICAL FOR RIGHT NEAR-RINGS

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ABSTRACT. Let R be a right near-ring. An R-group of type-5/2 which is a natural generalization of an irreducible (ring) module is introduced in near-rings. An R-group of type-5/2 is an R-group of type-2 and an R-group of type-3 is an R-group of type-5/2. Using it $J_{5/2}$, the Jacobson radical of type-5/2, is introduced in near-rings and it is observed that $J_2(R) \subseteq J_{5/2}(R) \subseteq J_3(R)$. It is shown that $J_{5/2}$ is an ideal-hereditary Kurosh-Amitsur radical (KA-radical) in the class of all zero-symmetric near-rings. But $J_{5/2}$ is not a KA-radical in the class of all near-rings. By introducing an R-group of type-(5/2)(0) it is shown that $J_{(5/2)(0)}$, the corresponding Jacobson radical of type-(5/2)(0), is a KA-radical in the class of all near-rings which extends the radical $J_{5/2}$ of zero-symmetric near-rings to the class of all near-rings.

1. Introduction

Near-rings considered are right near-rings and R stands for a right near-ring. Many generalizations of the Jacobson radical of rings to near-rings were introduced and studied. Let $\nu \in \{0, 1, 2\}$. J_ν, the Jacobson radical of type-ν, was introduced and studied by Betsch [1] and J_3, the Jacobson radical of type-3, was introduced and studied by Holcombe [2]. In this paper an R-group of type-5/2 is introduced as a natural generalization of an irreducible (ring) module. The corresponding Jacobson radical $J_{5/2}$ is also introduced in near-rings. Moreover, $J_2(R) \subseteq J_{5/2}(R) \subseteq J_3(R)$. $J_{5/2}$ is an ideal-hereditary Kurosh-Amitsur radical (KA-radical) in the class of all zero-symmetric near-rings. But $J_{5/2}$ is not a KA-radical in the class of all near-rings. By introducing an R-group of type-(5/2)(0) it is proved that $J_{(5/2)(0)}$, the corresponding Jacobson radical of type-(5/2)(0), is a KA-radical in the class of all near-rings which extends the radical $J_{5/2}$ of zero-symmetric near-rings to the class of all near-rings.

We recall some of the definitions related to R-groups and Jacobson radicals of near-rings.

Let G be an R-group and R_0 be the zero-symmetric part of R. Then G is

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(i) *monogenic* if there is a $g \in G$ such that $Rg = G$.
(ii) *strongly monogenic* if G is monogenic and for each $g \in G$ either $Rg = 0$ or G.
(iii) an R-group of *type-0* if $G \neq 0$ and is a monogenic simple R-group.
(iv) an R-group of *type-1* if G is of type-0 and strongly monogenic.
(v) an R-group of *type-2* if $G \neq 0$, monogenic and R_0-simple.
(vi) an R-group of *type-3* if G is an R-group of type-2 and $x, y \in G$ and $rx = ry$ for all $r \in R$ implies $x = y$.

If I is an ideal of R, then it is denoted by $I \lhd R$.

Let Q be a mapping which assigns to each near-ring R an ideal $Q(R)$ of R. Such mappings are called ideal-mappings. We consider the following properties which Q may satisfy:

(H1) $h(Q(R)) \subseteq Q(h(R))$ for all homomorphisms h of R;
(H2) $Q(R/Q(R)) = \{0\}$ for all R;
Q is *r-hereditary* if $I \cap Q(R) \subseteq Q(I)$ for all ideal I of R;
Q is *s-hereditary* if $Q(I) \subseteq I \cap Q(R)$ for all ideals I of R;
Q is *ideal-hereditary* if it is both r-hereditary and s-hereditary, that is, if $Q(I) = I \cap Q(R)$ for all ideals I of R;
Q is *idempotent* if $Q(Q(R)) = Q(R)$ for all R;
Q is *complete* if $Q(I) = I$ and I is an ideal of R implies $I \subseteq Q(R)$.

With Q we associate two classes of near-rings $\mathbb{R}_Q$ and $\mathbb{S}_Q$ defined by $\mathbb{R}_Q := \{R \mid Q(R) = R\}$, $\mathbb{S}_Q := \{R \mid Q(R) = 0\}$ and are called Q-radical class and Q-semisimple class respectively.

- An ideal-mapping Q is a *Hoehnke* radical (H-radical) if it satisfies conditions (H1) and (H2).
- An ideal-mapping Q is a *Kurosh-Amitsur* radical (KA-radical) if it is a complete idempotent H-radical.

Let $\mathbb{M}$ be a class of near-ring. Classes of near-rings always assumed to be abstract, that is, they contains the one element near-ring and are closed under isomorphic copies. With every near-ring R, we associate two ideals of R, depending on $\mathbb{M}$. These ideals are defined by:

$$\mathbb{M}(R) := \Sigma\{I \mid I \text{ is an ideal of } R \text{ and } I \in \mathbb{M}\} \text{ and}$$

$$(R)\mathbb{M} := \cap\{I \mid I \text{ is an ideal of } R \text{ and } R/I \in \mathbb{M}\}.$$

$\mathbb{M}$ is called *regular* if $0 \neq I \lhd R \in \mathbb{M}$ implies that $0 \neq I/K \in \mathbb{M}$ for some $K \lhd I$; *hereditary* if $I \lhd R \in \mathbb{M}$ implies $I \in \mathbb{M}$ and; *c-hereditary* if I is a left invariant ideal of $R \in \mathbb{M}$, then $I \in \mathbb{M}$. (An ideal I of R is *left invariant* if $RI \subseteq I$.)

A class of near-rings $\mathbb{M}$ is a *Kurosh-Amitsur radical class* (KA-radical class) if it satisfies the following:

(R1) $\mathbb{M}$ is closed under homomorphic images;
(R2) $\mathbb{M}(R) \in \mathbb{M}$ for all near-rings R;
(R3) $\mathbb{M}(R/\mathbb{M}(R)) = \{0\}$ for all near-rings R.

With a KA-radical class $\mathbb{R}$ we associate its semisimple class $\mathcal{S}\mathbb{R} := \{R \mid \mathbb{R}(R) = \{0\}\}$.

The following properties for a KA-radical class $\mathbb{R}$ are well known.

(i) $\mathbb{R}$ is *hereditary* if and only if $\mathbb{R}(R) \cap I \subseteq \mathbb{R}(\mathrm{I})$ for all $I \lhd R$.

(ii) $\mathcal{S}\mathbb{R}$ is *hereditary* if and only if $\mathbb{R}(\mathrm{I}) \subseteq \mathbb{R}(R) \cap I$ for all $I \lhd R$.

(iii) $\mathbb{R}$ is *c-hereditary* if and only if $\mathbb{R}(R) \cap I \subseteq \mathbb{R}(\mathrm{I})$ for all left invariant ideals I of R.

We say that a class M of near-rings satisfy condition (F_l) if $K \lhd I$ and I is a left invariant ideal of R with $I/K \in \mathbb{M}$, then $K \lhd R$.

Theorem 1.1 (Corollary 2.3 of [5]). *Let* $\mathbb{M}$ *be a class of zero-symmetric near-rings and* $\mathcal{L}$ *be defined by* $\mathcal{L}(R) := (R)\mathbb{M}$ *and* $\mathcal{L}_\circ$ *be the restriction of* $\mathcal{L}$ *to the class of all zero-symmetric near-rings. Then the following are equivalent.*

(1) $\mathcal{L}$ *is a KA-radical in the class of all near-rings with* $\mathcal{L}(I) \subseteq \mathcal{L} \cap I$ *for all* $I \lhd R$ *and equality holds if* I *is left invariant.*

(2) $\mathcal{L}_\circ$ *is an ideal-hereditary* KA*-radical in the class of all zero-symmetric near-rings and* $\mathbb{M}$ *satisfies condition* $(*)$:

$(*)$ *If* $K \lhd I \lhd R$ *with* I *a left invariant ideal of* R *and* $I/K \in \mathbb{M}$*, then* $\overline{R_c} \subseteq K$*, where* $\overline{R_c}$ *is the ideal of* R *generated by the subnear-ring* R_c.

Theorem 1.2 (Theorem 4.2.3 of [5]). *The class of all zero-symmetric 2-primitive near-rings satisfy condition* (F_l).

2. R-groups of type-5/2

Throughout this section R stands for a right near-ring.

Definition 2.1. Let G be an R-group. Then G is called an *R-group of type-5/2* if G is an R-group of type-2 and $Rg = G$ for all $0 \neq g \in \mathrm{G}$.

Remark 2.2. From the definition we have that an R-group of type-5/2 is an R-group of type-2.

Proposition 2.3. *An R-group of type-3 is an R-group of type-5/2.*

Proof. Let G be an R-group of type-3. So, G is an R-group of type-2. Let $0 \neq g \in G$. Since G is an R-group of type-2, it is an R-group of type-1. So, either $Rg = G$ or $Rg = \{0\}$. Suppose that $Rg = 0$. Now $R0 = R_c0 = R_c\, g \subseteq Rg = 0$ and hence $R0 = \{0\}$. So, $rg = r0$ for all $r \in R$. Since G is an R-group of type-3, $g = 0$. This is a contradiction to the fact that $g \neq 0$. Therefore, $Rg = G$. □

Proposition 2.4. *Let R be a zero-symmetric near-ring and $\{0\} \neq G$ be an R-group. Then G is an R-group of type-5/2 if and only if $Rg = G$ for all $0 \neq g \in G$.*

Proof. If G is an R-group of type-5/2, then obviously $Rg = G$ for all $0 \neq g \in G$. Suppose that $Rg = G$ for all $0 \neq g \in G$. Let $\{0\} \neq H$ be an R-subgroup of G. Let $0 \neq h \in H$. Now $G = Rh \subseteq H$ and hence $H = G$. Therefore, G is an R-group of type-2 and hence it is an R-group of type-5/2. □

We present an example of an R-group of type-5/2 which is not an R-group of type-3.

Example 2.5. Let $(R, +)$ be a group of order ≥ 3. Let $a, b \in R$. Define $ab = a$ if $b \neq 0$ and $ab = 0$ if $b = 0$. Now R is a zero-symmetric near-ring. Moreover, $Ra = R$ for all $0 \neq a \in R$. Therefore, by Proposition 2.4, R is an R-group of type-5/2. Let $0 \neq b$, $0 \neq c \in R$ and $b \neq c$. Now $ab = a = ac$ for all $a \in R$. So, R is not an R-group of type-3.

Now we give an example of an R-group of type-2 which is not an R-group of type-5/2.

Example 2.6. Let $(R, +)$ be a group of order ≥ 3. Let S be a non-empty subset of $R \setminus \{0\}$ such that $R \setminus$ S contains no non-zero subgroup of $(R, +)$. Let $a, b \in R$. Define $ab = a$ if $b \in S$ and $ab = 0$ if $b \notin S$. Now R is a zero-symmetric near-ring. We have that $Rb = \{0\}$ if $b \notin S$ and $Rb = R$ if $b \in S$. Now it is clear that R is an R-group of type-2. But, by Proposition 2.4, R is not an R-group of type-5/2.

Definition 2.7. A modular left ideal L of R is said to be a *5/2-modular left ideal* of R if R/L is an R-group of type-5/2.

Proposition 2.8. *Let G be an R-group of type-5/2 and $0 \neq g \in G$. Then $(0 : g)$ is a 5/2-modular left ideal of R and $R/(0 : g)$ and G are isomorphic R-groups.*

Proof. The mapping $h : R \to G$ defined by $h(r) = rg$ is an R-homomorphism of R onto G with Ker $h = (0 : g)$ which is a modular left ideal of R. Now $R/(0 : g)$ is isomorphic to G as R-groups. So, $(0 : g)$ is a 5/2-modular left ideal of R. □

Definition 2.9. R is called a *5/2-primitive near-ring* if R has a faithful R-group of type-5/2.

Definition 2.10. An ideal I of R is called a *5/2-primitive ideal* of R if R/I is a 5/2-primitive near-ring.

One can easily verify the following.

Proposition 2.11. *Let I be an ideal of R. Then*

(1) *If G is an R-group of type-5/2 and $I \subseteq (0:G)$, then G is also an R/I-group of type-5/2, where $(r+I)g := rg$, $r+I \in R/I$ and $g \in G$. If in addition $I = (0:G)$, then G is a faithful R/I-group.*

(2) *If G is an R/I group of type-5/2, then G is also an R-group of type-5/2, where $rg := (r+I)g$, $r \in R$ and $g \in G$. If in addition G is a faithful R/I-group, then $I = (0:G)_R$.*

An immediate consequence of Propositions 2.8 and 2.11 is the following.

Proposition 2.12. *Let I be an ideal of R. Then the following are equivalent.*

(i) *I is a 5/2-primitive ideal of R.*

(ii) *$I = (0:G)$ for some R-group G of type-5/2.*

(iii) *$I = (L:R)$ for some 5/2-modular left ideal L of R.*

Corollary 2.13. *The following are equivalent*

(i) *$\{0\}$ is a 5/2-primitive ideal of R.*

(ii) *R is 5/2-primitive.*

(iii) *R has a 5/2-modular left ideal L such that $(L:R) = \{0\}$.*

We know that an ideal P of R is a 3-prime ideal of R if $a, b \in R$ and $aRb \subseteq P$ implies $a \in P$ or $b \in P$.

Proposition 2.14. *Let P be a 5/2-primitive ideal of R. Then P is a 3-prime ideal of R.*

Proof. Let P be a 5/2-primitive ideal of R. We get an R-group G such that $P = (0:G)$. Let $a, b \in R$ and $aRb \subseteq P = (0:G)$. Suppose that $b \notin P$. Now $bg \neq 0$ for some $g \in G$. So $R(bg) = G$ as G is an R-group of type-5/2. Therefore, $aG = aR(bg) = (aRb)g = \{0\}$. So $a \in (0:G) = P$. Hence P is 3-prime. □

We know that a 3-primitive ideal of a zero-symmetric near-ring is equiprime and 3-prime. So with the introduction of 5/2-primitive ideals, we have primitive ideals which are 3-prime but not equiprime.

3. The Jacobson radical of type-5/2

Definition 3.1. *The Jacobson radical of R of type-5/2*, denoted by $J_{5/2}(R)$, is defined as the intersection of all 5/2-primitive ideals of R and if R has no such ideals, then $J_{5/2}(R)$ is defined as R.

Remark 3.2. By Proposition 2.12, $J_{5/2}(R) = \cap\ \{(0:G) \mid G$ is an R-group of type-5/2$\} = \cap\ \{(L:R) \mid L$ is a 5/2-modular left ideal of $R\}$.

The following proposition is immediate.

Proposition 3.3. $J_{5/2}(R) = \cap\ \{P \mid R/P$ *is a 5/2-primitive near-ring*$\}$.

Proposition 3.4. $J_{5/2}(R) = \cap\ \{L \mid L$ *is a 5/2-modular left ideal of* $R\}$.

Proof. If R has no 5/2-primitive ideals, then by Proposition 2.12, R has no 5/2-modular left ideals. So, if $J_{5/2}(R) = R$, then the result follows. Now suppose that R has a 5/2-primitive ideal. So there is an R-group of type-5/2. We have $J_{5/2}(R) = \cap \{(0 : G) \mid G \text{ is an } R\text{-group of type-5/2}\}$. Let G be an R-group of type-5/2. Let $0 \neq g \in G$. Since $Rg = G$, we get that $r \to$ rg is an R-homomorphism of R onto G with Kernel $(0 : g)$. So $R/(0 : g)$ and G are isomorphic R-groups and hence $(0 : g)$ is a 5/2-modular left ideal of R. Therefore $(0 : G)$ is an intersection of 5/2-modular left ideals of R. This shows that $J_{5/2}(R)$ is an intersection of 5/2-modular left ideals of R. Let T be a 5/2-modular left ideal of R. Now R/T is an R-group of type-5/2. Since T is modular, by Corollary 3.24 of [3], we get that $(T : R) \subseteq T$. So $J_{5/2}(R) \subseteq (T : R) \subseteq T$. Hence $J_{5/2}(R)$ is the intersection of all 5/2-modular left ideals of R. □

Lemma 3.5. *Let R be a zero-symmetric near-ring and S be an invariant subnearring of R. If L is a 5/2-modular left ideal of S, then L is an ideal of the R-group S and S/L is an R-group of type-5/2.*

Proof. Let L be a 5/2-modular left ideal of S. Since an R-group of type-5/2 is an R-group of type-2, L is a 2-modular left ideal of S. Therefore, by Theorem 3.34 of [3], L is an ideal of the R-group S and S/L is an R-group of type-2. Let $0 \neq s + L \in S/L$. Since S/L is an S-group of type-5/2, $S(s + L) = S/L$. Therefore $S/L = S(s + L) \subseteq R(s + L) \subseteq S/L$. So $R(s + L) = S/L$ and hence S/L is an R-group of type-5/2. □

Theorem 3.6. *Let S be an invariant subnear-ring of a zero-symmetric near-ring R. Then $J_{5/2}(S) \subseteq J_{5/2}(R) \cap S$.*

Proof. If S has no 5/2-primitive ideals then $J_{5/2}(S) = S \subseteq J_{5/2}(R) \cap S$. So, suppose that S has 5/2-primitive ideals. Let P be a 5/2-primitive ideal of S. We get an S-group G of type-5/2 such that $P = (0 : G)_S$. Let $0 \neq g \in G$. Now $S/(0 : g)_S$ and G are isomorphic as S-groups and that $L := (0 : g)_S$ is a 5/2-modular left ideal of S and $P = (0 : G)_S = (0 : S/L)_S = (L : S)_S$. By Lemma 3.5, S/L is an R-group of type-5/2. So $Q := (0 : S/L)_R = (L : S)_R$ is a 5/2-primitive ideal of R. Therefore $P = (L : S)_S = (L : S)_R \cap S = Q \cap S$. Hence $J_{5/2}(S) \subseteq J_{5/2}(R) \cap S$. □

Lemma 3.7. *Let S be an invariant subnear-ring of a zero-symmetric near-ring R. Let L be a 5/2-modular left ideal of R and $S \not\subseteq L$. Then $L \cap S$ is a 5/2-modular left ideal of S.*

Proof. We have that L is a 5/2-modular left ideal of R and $S \not\subseteq L$. Now $R = S + L$. So $R/L = (S + L)/L \simeq_R S/(S \cap L)$ and that $S/(S \cap L)$ is an R-group of type-5/2. Let L be modular by e. Now $r - re \in L$ for all $r \in R$. Let $s \in S - (S \cap L)$. Since $0 \neq s + L \in R/L$, $R(s + L) = R/L$ and that $Rs + L = R$.

Now $e = rs+l$, $r \in R$, $l \in L$. $S \cap L$ is a left ideal of S modular by rs. Let $t \in S$. Now $te - t \in L$. So $te - t = t(rs+l) - t = (t(rs+l) - t(rs)) + (t(rs) - t) \in L$ and that $t(rs) - t \in L \cap S$. Therefore $t + (L \cap S) = t(rs) + (L \cap S) \in (Ss + L \cap S)/(L \cap S)$ and that $S/(L \cap S) = (Ss + L \cap S)/(L \cap S) = S(s + (L \cap S))$. Hence $S/(L \cap S)$ is an S-group of type-5/2. Since $L \cap S$ is a modular left ideal of $S, L \cap S$ is a 5/2-modular left ideal of S. □

Theorem 3.8. *Let R be a zero-symmetric near-ring and S be an invariant subnearring of R. Then $\mathrm{J}_{5/2}(S) \subseteq \mathrm{J}_{5/2}(R) \cap S$.*

Proof. If $\mathrm{J}_{5/2}(R) = R$, then $\mathrm{J}_{5/2}(S) \subseteq R \cap S = \mathrm{J}_{5/2}(R) \cap S$. Suppose that $\mathrm{J}_{5/2}(R) \neq R$. So R has 5/2-modular left ideals. Let L be a 5/2-modular left ideal of R. If $S \subseteq L$, then $\mathrm{J}_{5/2}(S) \subseteq S \cap L$. Now suppose that $S \nsubseteq L$. By Lemma 3.7, $S \cap L$ is a 5/2-modular left ideal of S. So $\mathrm{J}_{5/2}(\mathrm{S}) \subseteq S \cap L$. Therefore, by Proposition 3.4, $\mathrm{J}_{5/2}(S) \subseteq \mathrm{J}_{5/2}(R) \cap S$. □

Theorem 3.9. *Let R be a zero-symmetric near-ring and S be an invariant subnearring of R. Then $\mathrm{J}_{5/2}(S) = \mathrm{J}_{5/2}(R) \cap S$.*

Theorem 3.10. *$\mathrm{J}_{5/2}$ is an ideal-hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

We show now that $\mathrm{J}_{5/2}$ is not a KA-radical in the class of all near-rings.

Consider the dihedral group $\mathrm{D}_8 = \{0, a, 2a, 3a, b, a+b, 2a+b, 3a+b\}$. Let T be the near-ring given in Example 11 of [3], (p.418) whose additive group is D_8. As mentioned in [4], $\{0\}$, $J = \{0, a, 2a, 3a\}$ and T are the ideals of T. Moreover, these are the only left ideals of T. Now T/J is the constant near-ring on Z_2. Since T/J is a T-group of type-5/2, J is a 5/2-primitive ideal and is the only 5/2-primitive ideal of T. So $\mathrm{J}_{5/2}(T) = J$.

We need the following result.

Proposition 3.11 (Proposition 3.3 of [4]). *Let Q be an ideal-mapping which satisfies* (H1) *and for which $Q(T) = J$ and $F \in S_Q$, where F is the field of order 2. Then Q is not idempotent and hence not a* KA*-radical mapping.*

Theorem 3.12. *$\mathrm{J}_{5/2}$ is not a* KA*-radical in the class of all near-rings.*

Proof. By Proposition 3.3, we have that $\mathrm{J}_{5/2}$ is the H-radical corresponding to the class of all 5/2-primitive near-rings. As seen above $\mathrm{J}_{5/2}(T) = J$. Moreover, the two element field is in $S_{\mathrm{J}_{5/2}}$. So, by Proposition 3.11, $\mathrm{J}_{5/2}$ is not a KA-radical in the class of all near-rings. □

4. The Jacobson radical of type-(5/2)(0)

It is known that Jacobson radicals of type-2 and 3 are ideal-hereditary KA-radicals in the class of all zero-symmetric near-rings and the Jacobson radical of type-2 is not even a KA-radical in the class of all near-rings. S. Veldsman [5] introduced R-groups of type-2(0) and 3(0) and the corresponding Jacobson radicals of type-2(0) and 3(0) which are extensions of the Jacobson radicals of type-2 and 3 respectively of zero-symmetric near-rings to the class of all near-rings and has shown that these two new radicals are KA-radicals in the class of all near-rings.

In this section we introduce R-groups of type-$(5/2)(0)$ and the corresponding Jacobson radical of type-$(5/2)(0)$. We show that it is a KA-radical in the class of all near-rings.

Definition 4.1. Let G be an R-group of type-5/2. G is called an *R-group of type*-$(5/2)(0)$ if $R0 = \{0\}$, where 0 is the additive identity in G.

Proposition 4.2. *Let G be an R-group of type-5/2. Then G is an R-group of type-$(5/2)(0)$ if and only if $R_c \subseteq (0 : G)$, where R_c is the constant part of R.*

Proof. Let G be an R-group of type-$(5/2)(0)$. $R_c g = (Ro)g = R(og) = R0 = \{0\}$ for all $g \in R$. So, $R_c \subseteq (0 : G)$. Suppose now that $R_c \subseteq (0 : G)$. Now $R_c 0 = \{0\}$, where 0 is the additive identity in G. So $R0 = \{0\}$ and hence G is an R-group of type-$(5/2)(0)$. □

Corollary 4.3. *Let R is a zero-symmetric near-ring and G be an R-group. Then G is type-$(5/2)(0)$ if and only if it is of type-$5/2$.*

Definition 4.4. A near-ring R is said to be $(5/2)(0)$-primitive if it has a faithful R-group of type-$(5/2)(0)$. An ideal I of R is called $(5/2)(0)$-primitive if R/I is a $(5/2)(0)$-primitive near-ring.

Proposition 4.5. *Let I be an ideal of R. Then the following are equivalent.*

(i) *I is $(5/2)(0)$-primitive ideal of R.*

(ii) *$I = (0 : G)$ for some R-group G of type-$(5/2)(0)$.*

Proof. Suppose that I is a $(5/2)(0)$-primitive ideal of R. R/I is a $(5/2)(0)$-primitive on some R/I-group G of type-$(5/2)(0)$. Since G is a faithful R/I-group of type-$(5/2)(0)$, G is an R-group of type-5/2 and $I = (0 : G)$. Also, since R/I is zero-symmetric, $R_c \subseteq I = (0 : G)$ and hence G is an R-group of type-$(5/2)(0)$. Conversely, suppose that $I = (0 : G)$ for an R-group G of type-$(5/2)(0)$. Since G is an R-group of type-$(5/2)(0)$ and $I = (0 : G)$, G is a faithful R/I-group of type-5/2. Also since $\mathrm{R}_c \subseteq (0 : G) = I$, R/I is a zero-symmetric near-ring and hence G is a faithful R/I-group of type-$(5/2)(0)$. So R/I is a $(5/2)(0)$-primitive near-ring and hence I is a $(5/2)(0)$-primitive ideal of R. □

Corollary 4.6. *The following are equivalent*

(i) *$\{0\}$ is a $(5/2)(0)$-primitive ideal of R.*

(ii) *R is $(5/2)(0)$-primitive.*

Corollary 4.7. *R is $(5/2)(0)$-primitive if and only if R is a zero-symmetric and $(5/2)$-primitive.*

Remark 4.8. It is clear that a $(5/2)(0)$-primitive ideal of R contains R_c, the constat part of R.

Definition 4.9. Let R be a near-ring. $\mathrm{J}_{(5/2)(0)}(R)$ is defined as the intersection of all $(5/2)(0)$-primitive ideal of R and $\mathrm{J}_{(5/2)(0)}(R) = R$ if R has no $(5/2)(0)$-primitive ideals. $\mathrm{J}_{(5/2)(0)}$ is called the *Jacobson radical of type*-$(5/2)(0)$.

Remark 4.10. If R is a ring, then $\mathrm{J}_{(5/2)(0)}(R)$ is the Jacobson radical of R.

We show now that $\mathrm{J}_{(5/2)(0)}$ is a KA-radical in the class of all near-rings, its semisimple class is hereditary and radical class is c-hereditary.

Theorem 4.11. *The class of all zero-symmetric 5/2-primitive near-rings satisfy condition* (F_l).

Proof. Since a zero-symmetric 5/2-primitive near-ring is a 2-primitive near-ring, by Theorem 1.2, we get that the class of all zero-symmetric 5/2-primitive near-rings also satisfy condition (F_l). □

Theorem 4.12. *Let R be a near-ring.* $\mathrm{J}_{(5/2)(0)}$ *is a* KA-*radical in the class of all near-rings,* $\mathrm{J}_{(5/2)(0)}(I) \subseteq \mathrm{J}_{(5/2)(0)}(R) \cap I$ *for all* $I \lhd R$ *and the equality holds if I is a left invariant ideal.*

Proof. Let $\mathbb{M}$ be the class of all zero-symmetric 5/2-primitive near-rings. Now by Corollary 4.7, $\mathrm{J}_{(5/2)(0)}(R) = (R)\mathbb{M}$ for all near-rings R. By Theorem 3.10, $\mathrm{J}_{5/2}$ is an ideal-hereditary KA-radical in the class of all zero-symmetric near-rings. In view of Theorem 1.1, it is enough to show that M satisfies condition $(*)$ of Theorem 1.1. Let $K \lhd I \lhd R$ and I be a left invariant ideal of R with $I/K \in \mathbb{M}$. By Theorem 4.11, $\mathbb{M}$ satisfies condition (F_l). So $K \lhd R$. Since I is a left invariant ideal of R, $R_c \subseteq I$. Also since I/K is a zero-symmetric near-ring, $R_c = I_c \subseteq K$. Since $R_c \subseteq K$ and $K \lhd R$, we get that $\overline{R_c} \subseteq K$. This completes the proof. □

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