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VAGUE DEDUCTIVE SYSTEMS OF SUBTRACTION ALGEBRAS

CHUL HWAN PARK

ABSTRACT. The notion of vague deductive systems in subtraction algebras is introduced, and several properties are investigated. Conditions for a vague set to be a vague deductive system are provided. Characterizations of a vague deductive system are established.

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1. Introduction

B. M. Schein [8] have considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). Jun et al. [6] discussed ideal theory of subtraction algebras. In this paper we introduce a notion of a vague deductive system in a subtraction algebra, and study some properties of them. We give conditions for a vague set to be a vague deductive system, and establish characterizations of a vague deductive system.

2. Basic results on subtraction algebras

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

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The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [6]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Proposition 2.1. [6] *Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

3. Basic results on vague sets

Defitinition 3.1. [3] A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(1) A truth membership function $t_A : U \to [0, 1]$ and
(2) A false membership function $f_A : U \to [0, 1]$

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the "evidence for u", and $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \Big\{ \langle u, [t_A(u), f_A(u)] \rangle \mid u \in U \Big\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A and is denoted by $V_A(u)$.

Defitinition 3.2. [3] A vague set A of a set U is called

(1) the *zero vague set* of U if $t_A(u) = 0$ and $f_A(u) = 1$ for all $u \in U$,
(2) the *unit vague set* of U if $t_A(u) = 1$ and $f_A(u) = 0$ for all $u \in U$.
(3) the *α-vague set* of U if $t_A(u) = \alpha$ and $f_A(u) = 1 - \alpha$ for all $u \in U$, where $\alpha \in (0, 1)$.

For $\alpha, \beta \in [0, 1]$ we now define (α, β)-cut and α-cut of a vague set.

Defitinition 3.3. [3] Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The *(α, β)-cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \geq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts are also called *vague-cuts* of the vague set A.

Defitinition 3.4. [3] The *α-cut* of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\alpha,\beta)} = A_\alpha$. Equivalently, we can define the α-cut as $A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}$. For our discussion, we shall use the following notations, which are given in [3], on interval arithmetic.

Notation 3.5. Let $I[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. If $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ be two elements of $I[0, 1]$, we call $I_1 \geq I_2$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly we understand the relations $I_1 \leq I_2$ and $I_1 = I_2$. Clearly the relation $I_1 \geq I_2$ does not necessarily imply that $I_1 \supseteq I_2$ and conversely. We define the term "*imax*" to mean the maximum of two intervals as

$$imax(I_1, I_2) = [\max(a_1, a_2), \max(b_1, b_2)].$$

Similarly we define "*imin*". The concept of "*imax*" and "*imin*" could be extended to define "*isup*" and "*iinf*" of infinite number of elements of $I[0, 1]$.

It is obvious that $L = \{I[0, 1], isup, iinf, \leq\}$ is a lattice with universal bounds $[0, 0]$ and $[1, 1]$ (see [3]).

4. Vague deductive systems

In what follows let X be a subtraction algebra unless otherwise specified.

Defitinition 4.1. A nonempty subset D of X is called a *deductive system* of X (it is called an *ideal* of X in [5]) if it satisfies:

(1) $0 \in D$,
(2) $(\forall x \in X)(\forall y \in D)(x - y \in D \Rightarrow x \in D)$.

Defitinition 4.2. A vague set A of X is called a *vague deductive system* of X if the following conditions are true:

(c1) $(\forall x \in X)\ (V_A(0) \geq V_A(x))$,
(c2) $(\forall x, y \in X)\ (V_A(x) \geq \min\{V_A(x-y), V_A(y)\})$,

that is,

$$t_A(0) \geq t_A(x),\ 1 - f_A(0) \geq 1 - f_A(x), \tag{1}$$

and

$$\begin{aligned} &t_A(x) \geq \min\{t_A(x-y), t_A(y)\}, \\ &1 - f_A(x) \geq \min\left\{1 - f_A(x-y), 1 - f_A(y)\right\} \end{aligned} \tag{2}$$

for all $x, y \in X$.

Example 4.3. Consider a subtraction algebra $X = \{0, a, b\}$ with the following Cayley table:

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Let A be a vague set in X defined as follows:

$$A = \{\langle 0, [0.6, 0.2]\rangle, \langle a, [0.3, 0.6]\rangle, \langle b, [0.5, 0.3]\rangle\}.$$

It is routine to verify that A is a vague deductive system of X.

Example 4.4. Consider a subtraction algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Let A be a vague set in X defined as follows:

$$A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.5, 0.3]\rangle, \langle d, [0.7, 0.2]\rangle\}.$$

It is routine to verify that A is a vague deductive system of X.

Proposition 4.5. *Every vague deductive system A of X satisfies:*

$$(\forall x, y \in X)\,(x \le y \Rightarrow V_A(x) \ge V_A(y)). \tag{3}$$

Proof. Let $x, y \in X$ be such that $x \le y$. Then $x - y = 0$, and so

$$t_A(x) \ge \min\{t_A(x-y), t_A(y)\} = \min\{t_A(0), t_A(y)\} = t_A(y),$$

$$\begin{aligned} 1 - f_A(x) &\ge \min\{1 - f_A(x-y), 1 - f_A(y)\} \\ &= \min\{1 - f_A(0), 1 - f_A(y)\} \\ &= 1 - f_A(y). \end{aligned}$$

This shows that $V_A(x) \ge V_A(y)$. □

Proposition 4.6. *Every vague deductive system A of X satisfies:*

$$(\forall x, y, z \in X)\left(V_A(x-z) \ge \min\{V_A((x-y)-z), V_A(y)\}\right). \tag{4}$$

Proof. Using (c2) and (S3), we have

$$\begin{aligned} V_A(x-z) &\ge \min\{V_A((x-z)-y), V_A(y)\} \\ &= \min\{V_A((x-y)-z), V_A(y)\} \end{aligned}$$

for all $x, y, z \in X$. □

We give conditions for a vague set to be a vague deductive system .

Theorem 4.7. *If A is a vague set in X satisfying* (c1) *and* (4), *then A is a vague deductive system of X.*

Proof. Taking $z = 0$ in (4) and using (a2), we have

$$\begin{aligned} V_A(x) &= V_A(x-0) \\ &\ge \min\{V_A((x-y)-0), V_A(y)\} \\ &= \min\{V_A(x-y), V_A(y)\} \end{aligned}$$

for all $x, y \in X$. Hence A is a vague deductive system of X. □

Corollary 4.8. *Let A be a vague set in X. Then A is a vague deductive system of X if and only if it satisfies conditions* (c1) *and* (4).

The following is a characterization of a vague deductive system of X.

Theorem 4.9. *Let A be a vague set in X. Then A is a vague deductive system of X if and only if it satisfies the following conditions:*

$$(\forall x, y \in X)(V_A(x - y) \geq V_A(x)), \tag{5}$$

$$(\forall x, a, b \in X)\Big(V_A(x - ((x - a) - b)) \geq \min\{V_A(a), V_A(b)\}\Big). \tag{6}$$

Proof. Assume that A is a vague deductive system of X. Using (a3), (c1) and (c2), we get

$$V_A(x - y) \geq \min\{V_A((x - y) - x), V_A(x)\} = \min\{V_A(0), V_A(x)\} = V_A(x)$$

for all $x, y \in X$. Since

$$(x - ((x - a) - b)) - a = (x - a) - ((x - a) - b) \leq b,$$

it follows from (3) that $V_A((x - ((x - a) - b)) - a) \geq V_A(b)$ so from (c2) that

$$\begin{aligned} V_A(x - ((x - a)) - b)) &\geq \min\{V_A((x - ((x - a) - b)) - a), V_A(a)\} \\ &\geq \min\{V_A(a), V_A(b)\}. \end{aligned}$$

Conversely let A be a vague set in X satisfying conditions (5) and (6). If we take $y = x$ in (5), then $V_A(0) = V_A(x - x) \geq V_A(x)$ for all $x \in X$. Using (6), we obtain

$$\begin{aligned} V_A(x) &= V_A(x - 0) \\ &= V_A(x - ((x - y) - (x - y))) \\ &= V_A(x - ((x - (x - y)) - y)) \\ &\geq \min\{V_A(x - y), V_A(y)\} \end{aligned}$$

for all $x, y \in X$. Hence A is a vague deductive system of X. □

Proposition 4.10. *Every vague deductive system A of X satisfies the following assertion:*

$$(\forall x, y \in X)(\exists x \vee y \Rightarrow V_A(x \vee y) \geq \min\{V_A(x), V_A(y)\}). \tag{7}$$

Proof. Suppose there exists $x \vee y$ for $x, y \in X$. Let w be an upper bound of x and y. Then $x \vee y = w - ((w - y) - x)$ is the least upper bound for x and y (see Proposition 2.1), and so

$$V_A(x \vee y) = V_A(w - ((w - y) - x)) \geq \min\{V_A(x), V_A(y)\}$$

by (6). This completes the proof. □

Proposition 4.11. *Let A be a vague set in X. Then A is a vague deductive system of X if and only if it satisfies:*

$$(\forall x, y, z \in X)\,(x - y \leq z \Rightarrow V_A(x) \geq \min\{V_A(y), V_A(z)\}). \tag{8}$$

Proof. Assume that A is a vague deductive system of X and let $x, y, z \in X$ be such that $x - y \leq z$. Then $V_A(z) \leq V_A(x - y)$ by (3). It follows from (c2) that $V_A(x) \geq \min\{V_A(x - y), V_A(y)\} \geq \min\{V_A(y), V_A(z)\}$. Conversely suppose that A satisfies (8). Since $0 - y \leq y$ for all $y \in X$, we have

$$V_A(0) \geq \min\{V_A(y), V_A(y)\} = V_A(y)$$

by (8). Thus (c1) is valid. Since $x - (x - y) \leq y$ for all $x, y \in X$ by (a4), it follows from (8) that $V_A(x) \geq \min\{V_A(x - y), V_A(y)\}$. Hence A is a vague deductive system of X. □

As a generalization of Proposition 4.11, we have the following results.

Theorem 4.12. *If a vague set A in X is a vague deductive system of X, then*

$$\prod_{i=1}^{n} x - w_i = 0 \Rightarrow V_A(x) \geq \min\{V_A(w_i) \mid i = 1, 2, \cdots, n\} \tag{9}$$

for all $x, w_1, w_2, \cdots, w_n \in X$, where

$$\prod_{i=1}^{n} x - w_i = (\cdots((x - w_1) - w_2) - \cdots) - w_n.$$

Proof. The proof is by induction on n. Let A be a vague deductive system of X. By (3) and (8), we know that the condition (9) is valid for $n = 1, 2$. Assume that A satisfies the condition (9) for $n = k$, that is,

$$\prod_{i=1}^{k} x - w_i = 0 \Rightarrow V_A(x) \geq \min\{V_A(w_i) \mid i = 1, 2, \cdots, k\}$$

for all $x, w_1, w_2, \cdots, w_k \in X$. Let $x, w_1, w_2, \cdots, w_k, w_{k+1} \in X$ be such that $\prod_{i=1}^{k+1} x - w_i = 0$. Then

$$V_A(x - w_1) \geq \min\{V_A(w_j) \mid j = 2, 3, \cdots, k + 1\}.$$

Since A is a vague deductive system of X, it follows from (c2) that

$$\begin{aligned} V_A(x) &\geq \min\{V_A(x - w_1), V_A(w_1)\} \\ &\geq \min\{V_A(w_1), \min\{V_A(w_j) \mid j = 2, 3, \cdots, k + 1\}\} \\ &= \min\{V_A(w_i) \mid i = 1, 2, \cdots, k + 1\}. \end{aligned}$$

This completes the proof. □

Now we consider the converse of Theorem 4.12.

Theorem 4.13. Let A be a vague set in X satisfying the condition (9). Then A is a vague deductive system of X.

Proof. Note that $\big(\cdots((0 - \underbrace{x) - x) - \cdots\big) - x}_{n \text{ times}} = 0$ for all $x \in X$. It follows from (9) that $V_A(0) \geq V_A(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $x - y \leq z$. Then

$$0 = (x - y) - z = \big(\cdots(((x - y) - z) - \underbrace{0) - \cdots\big) - 0}_{n-2 \text{ times}},$$

and so

$$V_A(x) \geq \min\{V_A(y),\, V_A(z),\, V_A(0)\} = \min\{V_A(y),\, V_A(z)\}.$$

Hence, by Proposition 4.11, we conclude that A is a vague deductive system of X. □

Theorem 4.14. *Let A be a vague deductive system of X. Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ is a crisp deductive system of X.*

Proof. Obviously, $0 \in A_{(\alpha,\beta)}$. Let $x, y \in X$ be such that $y \in A_{(\alpha,\beta)}$ and $x - y \in A_{(\alpha,\beta)}$. Then $V_A(y) \geq [\alpha, \beta]$, i.e., $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$; and $V_A(x - y) \geq [\alpha, \beta]$, i.e., $t_A(x - y) \geq \alpha$ and $1 - f_A(x - y) \geq \beta$. It follows from (2) that

$$t_A(x) \geq \min\{t_A(x - y), t_A(y)\} \geq \alpha,$$
$$1 - f_A(x) \geq \min\{1 - f_A(x - y), 1 - f_A(y)\} \geq \beta$$

so that $V_A(x) \geq [\alpha, \beta]$. Hence $x \in A_{(\alpha,\beta)}$, and so $A_{(\alpha,\beta)}$ is a deductive system of X. □

The deductive systems like $A_{(\alpha,\beta)}$ are also called *vague-cut deductive systems* of X. Clearly we have the following result.

Proposition 4.15. *Let A be a vague deductive system of X. Two vague-cut deductive systems $A_{(\alpha,\beta)}$ and $A_{(\omega,\gamma)}$ with $[\alpha, \beta] < [\omega, \gamma]$ are equal if and only if there is no $x \in X$ such that*

$$[\alpha, \beta] \leq V_A(x) \leq [\omega, \gamma].$$

Theorem 4.16. *Let X be finite and let A be a vague deductive system of X. Consider the set $V(A)$ given by*

$$V(A) := \{V_A(x) \mid x \in X\}.$$

Then A_i are the only vague-cut deductive systems of X, where $i \in V(A)$.

Proof. Consider $[a_1, a_2] \in I[0,1]$ where $[a_1, a_2] \notin V(A)$. If

$$[\alpha, \beta] < [a_1, a_2] < [\omega, \gamma]$$

where $[\alpha, \beta], [\omega, \gamma] \in V(A)$, then

$$A_{(\alpha,\beta)} = A_{(a_1,a_2)} = A_{(\omega,\gamma)}.$$

If

$$[a_1, a_2] < [a_1, a_3]$$

where $[a_1, a_3] = imin\{x \mid x \in V(A)\}$, then

$$A_{(a_1,a_3)} = X = A_{(a_1,a_2)}.$$

Hence for any $[a_1, a_2] \in I[0,1]$, the vague-cut deductive system $A_{(a_1,a_2)}$ is one of A_i for $i \in V(A)$. This completes the proof. □

Theorem 4.17. *Any deductive system D of X is a vague-cut deductive system of some vague deductive system of X.*

Proof. Consider the vague set A of X given by

$$V_A(x) = \begin{cases} [\alpha, \alpha], & \text{if } x \in D, \\ [0,0], & \text{if } x \notin D, \end{cases} \tag{10}$$

where $\alpha \in (0,1)$. Since $0 \in D$, we have

$$V_A(0) = [\alpha, \alpha] \geq V_A(x)$$

for all $x \in X$. Let $x, y \in X$. If $x \in D$, then

$$V_A(x) = [\alpha, \alpha] \geq \min\{V_A(x-y), V_A(y)\}.$$

Assume that $x \notin D$. Then $y \notin D$ or $x - y \notin D$. It follows that

$$V_A(x) = [0,0] = \min\{V_A(x-y), V_A(y)\}.$$

Thus A is a vague deductive system of X. Clearly $D = A_{(\alpha,\alpha)}$. □

Theorem 4.18. *Let A be a vague deductive system of X. Then the set*

$$D := \{x \in X \mid V_A(x) = V_A(0)\}$$

is a crisp deductive system of X.

Proof. Obviously $0 \in D$. Let $x, y \in X$ be such that $x - y \in D$ and $y \in D$. Then $V_A(x-y) = V_A(0) = V_A(y)$, and so

$$V_A(x) \geq \min\{V_A(x-y), V_A(y)\} = V_A(0)$$

by (c2). Since $V_A(0) \geq V_A(x)$ for all $x \in X$, it follows that $V_A(x) = V_A(0)$ so that $x \in D$. Therefore D is a crisp deductive system of X. □

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Chul Hwan Park received his B.S., M.S. and Ph.D. degree from the Department of Mathematics of University of Ulsan,Korea, in 1986,1988 and 1997 respectively. From 1997 to 1998, he was a researcher at the Institute of Basic Science,The Kyungpook National Universty,Korea(supported by KOSEF). He is currently a full time lecture at the Department of Mathematics in University of Ulsan,Korea since 2005. His research interests are in the areas of Fuzzy Algebraic Structure,BCK-algebra,semigroup and Commutative ring.

Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea
e-mail: chpark@ulsan.ac.kr or skyrosemary@gmail.com