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ON SUBSTRUCTURES OF MONOGENIC R-GROUPS

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ABSTRACT. In this paper, we will introduce the noetherian quotients in R-groups, and then investigate the related substructures of the near-ring R and G and the R-group G.

Also, applying the annihilator concept in R-groups and d.g. near-rings, we will survey some properties of the substructures of R and G in monogenic R-groups, and show that R becomes a ring for faithful monogenic R-groups with some condition.

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1. Introduction

Our near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations, say $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b+c) = ab + ac$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*. An element d in R is called *distributive* if $(a+b)d = ad + bd$ for all a and b in R.

An *ideal* of R is a subset I of R such that

(i) $(I, +)$ is a normal subgroup of $(R, +)$,

(ii) $a(I+b) - ab \subset I$ for all a, $b \in R$, that is, $RI \subset I$,

(iii) $(I+a)b - ab \subset I$ for all a, $b \in R$.

If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

On the other hand, an *invariant R-subgroup* of R is a subset H of R such that

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(i) $(H, +)$ is a subgroup of $(R, +)$,
(ii) $RH \subset H$ and
(iii) $HR \subset H$.

If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *(right) R-subgroup* of R.

We consider the following notations: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a, \ for\ all\ r \in R\}$ which is called the *constant part* of R, and $R_d = \{a \in R \mid a\ is\ distributive\}$ which is called the *distributive part* of R.

It is easily checked that R is zero-symmetric if and only if every right ideal of R is an R-subgroup of R [7, p.19, 1.34]. Also, we note that R_0 and R_c are subnear-rings of R, but R_d is not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring.

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, We can define naturally, the sum $f + g$ of any two mappings f, g in $M(G)$ as the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ as the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* on the group G. Also, we define the set

$$M_0(G) = \{f \in M(G) \mid 0f = 0\},$$

then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $(a + b)\theta = a\theta + b\theta$, (ii) $(ab)\theta = a\theta b\theta$. We can replace homomorphism by momomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as ring case [1].

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we write that xr (right scalar multiplication in R) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying

(i) $x(a + b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (If R has a unity 1), for all $x \in G$ and $a, \ b \in R$.

Evidently, every near-ring R can be given the structure of an R-group (unitary if R is unitary) by right multiplication in R. Moreover, every group G has a natural $M(G)$-group structure, from the representation of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

A representation θ of R on G is called *faithful* if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful R-group.*

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, a normal subgroup N of G such that $NR \subset N$ is called a *normal R-subgroup* of G and an *R-ideal* of G is a normal subgroup N of G such that $(N+x)a - xa \subset N$ for all $x \in G$, $a \in R$. Also, we see that normal R-subgroups of G are not equivalent to an R-ideals of R.

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a *monogenic R-group* and the element x is called a *generator* of G [7].

For the remainder concepts and notations on near-rings, we refer to Meldrum [6], and Pilz [7].

2. Results on substructures of monogenic R-groups

A near-ring R is called *distributively generated* (briefly, *d.g.*) if it contains a subsemigroup S of $(R_d, \cdot)$ which generates the additive group $(R, +)$, we denote it by (R, S).

On the other hand, the set of all distributive elements of $M(G)$ are obviously the semigroup $End(G)$ of all endomorphisms of the group G under composition. Here we denote that $E(G)$ is the d.g. near-ring generated by $End(G)$, that is, $E(G)$ is d.g. subnear-ring of $(M_0(G), +, \cdot)$ generated by $End(G)$. It is said to be that $E(G)$ is the *endomorphism near-ring* of the group G.

Lemma 2.1. [7, p.11, 1.13] *Let $e \in R$ be idempotent. Then we get a "Pierce decomposition": for all $a \in R$, $\exists_1 c \in \{x \in R | ex = 0\}$ and $\exists_1 d \in eR$ such that $a = c + d$. Taking $e = 0$, one gets that for all $a \in R$, $\exists_1 c \in R_0$ and $\exists_1 d \in R_c$ such that $a = c + d$. Hence*

$$(R, +) = (R_0, +) \oplus (R_c, +)$$

Now, we will consider the "noetherian quotients" in R-group G, and the relation between the substructures of R and G, also quotient structure relation between them.

Let G be an R-group and K, K_1 and K_2 be subsets of G. Define

$$(K_1 : K_2) := \{a \in R; K_2 a \subset K_1\}.$$

We abbreviate that for $x \in G$

$$(\{x\} : K_2) =: (x : K_2).$$

Similarly for $(K_1 : x)$. $(0 : K)$ is called the *annihilator* of K, denoted it by $A(K)$. We say that G is a *faithful R-group* or that R *acts faithfully* on G if $A(G) = \{0\}$, that is, $(0 : G) = \{0\}$.

Also, we see that from the previous concepts to elementwise, a subgroup H of G such that $xa \in H$ for all $x \in H, a \in R$, is an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that

$$(x + g)a - ga \in N$$

for all $g \in G, x \in N$ *and* $a \in R$ (Meldrum [6]).

Proposition 2.2. *Let G be an R-group and K_1 and K_2 subsets of G. Then we have the following conditions:*

(1) *If K_1 is a normal R-subgroup of G, then $(K_1 : K_2)$ is a normal right R-subgroup of the near-ring R.*
(2) *If K_1 is an R-subgroup of G, then $(K_1 : K_2)$ is a right R-subgroup.*
(3) *If K_1 is an R-ideal of G and K_2 is an R-subgroup of G, then $(K_1 : K_2)$ is a two-sided ideal of R.*

Proof. (1) and (2) are proved by Meldrum [6]. Now, we prove only (3) : Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$K_2(ra) = (K_2r)a \subset K_2a \subset K_1,$$

so that $ra \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a left ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$k\{(a + r_1)r_2 - r_1r_2\} = (ka + kr_1)r_2 - kr_1r_2 \in K_1$$

for all $k \in K_2$, since $K_2a \subset K_1$ and K_1 is an ideal of G. Thus $(K_1 : K_2)$ is a right ideal of R. Therefore $(K_1 : K_2)$ is a two-sided ideal of R. □

Corollary 2.3[6]. *Let R be a near-ring and G an R-group.*

(1) *For any $x \in G$, $(0 : x)$ is a right ideal of R.*
(2) *For any R-subgroup K of G, $(0 : K)$ is a two-sided ideal of R.*
(3) *For any subset K of G, $(0 : K) = \bigcap_{x \in K} (0 : x)$.*

An idempotent $e \in R$is called *central* if it is in the center of $(R, \cdot)$, that is, if for all $a \in R$, $ea = ae$. Using this notion, we have the following important proposition with Baer like condition.

Proposition 2.4. *Let e be a central idempotent of R with eR is a two-sided ideal of R. Then $A(eR) = A(e)$, and*

$$R = eR \oplus A(e)$$

Proof. From the Corollary 2.3 (1) and (2), $A(eR) = (0 : eR)$ is a two-sided ideal of R. Since e is a central idempotent of R, obviously we see that $A(eR) = A(e)$. Finally, by the Lemma 2.1, we get $R = eR \oplus A(e)$. □

Proposition 2.5. *Let R be a near-ring and G an R-group. Then we have the following conditions:*

(1) *$A(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/A(G)$-group.*
(2) *For any $x \in G$, we get $xR \cong R/(0 : x)$ as R-groups.*

Proof. (1) By Proposition 2.2 and Corollary 2.3, $A(G)$ is a two-sided ideal of R. We now make G an $R/A(G)$-group by defining, for $r \in R, r + A(G) \in R/A(G)$, the action $x(r + A(G)) = xr$. If $r + A(G) = r' + A(G)$, then $-r' + r \in A(G)$ hence $x(-r' + r) = 0$ for all x in G, that is to say, $xr = xr'$. This tells us that

$$x(r + A(G)) = xr = xr' = x(r' + A(G));$$

thus the action of $R/A(G)$ on G has been shown to be well defined. The verification of the structure of an $R/A(G)$-group is routine. Finally, to see that G is a faithful $R/A(G)$-group, we note that if $x(r + A(G)) = 0$ for all $x \in G$, then by the definition of $R/A(G)$-group structure, we have $xr = 0$. Hence $r \in A(G)$. This says that only the zero element of $R/A(G)$ annihilates all of G. Thus G is a faithful $R/A(G)$-group.

(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi : R \longrightarrow xR$ defined by $\phi(r) = xr$ is an R-ephimorphism, so that from the isomorphism theorem, since the kernel of ϕ is $(0 : x)$, we deduce that

$$xR \cong R/(0 : x)$$

as R-groups. □

Corollary 2.6. [7] *Let G be a monogenic R-group with x as a generator. Then we have the following isomorphic relation.*

$$G \cong R/(0 : x).$$

Lemma 2.7. [2] *If R is a zero symmetric near-ring and A, B, K are R-ideals of an R-group G, then we have the following two conditions:*

(1) *We get an additive abelian group:*

$$G' = [(A + K) \cap (B + K)]/[(A \cap B) + K]$$

and for any x, $y \in G'$, and $r \in R$, we have $(x + y)r = xr + yr$.

(2) *We obtain a quotient ring $R/(0 : G')$.*

Proposition 2.8. *Let G be a faithful monogenic R-group with generator x, where R is a zero symmetric near-ring. If I and J are right ideals of R and $I \cap J \subseteq (0:x)$, then R is a ring.*

Proof. From the Proposition 2.5 (2), we have that

$$G = xR \cong R/(0:x) = [(I + (0:x) \cap J + (0:x)]/[(I \cap J) + (0:x)] = G'$$

On the other hand, since G is faithful, by the definition, we see that

$$(0:G') \cong (0:G) = A(G) = 0$$

Consequently, Lemma 2.7 implies that R is a ring. □

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