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A MONOID OVER WHICH ALL CYCLIC FLAT RIGHT S-ACTS SATISFY CONDITION (E)

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ABSTRACT. Although the properties of flatness and condition (E) for S-acts over a monoid S are incomparable, Liu([10]) showed that necessary and sufficient condition for a monoid S over which all left S-acts that satisfy condition (E) are flat is the regularity of S. But the problem of describing a monoid over which all cyclic flat left S-acts satisfy condition (E) is still open. Thus the purpose of this paper is to characterize monoids over which all cyclic flat right S-acts satisfy condition (E).

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1. Introduction

For many years, a fruitful area of research in semigroup theory has been the investigation of properties connected with projectivity of acts over monoids and we have the following sequence of properties which a given act may or may not possess, arranged in strictly decreasing order of strength:

free ⇒ projective ⇒ strongly flat ⇒ condition (P) ⇒ flat ⇒ weakly flat

Many papers have been written describing classes of monoids over which various of these distinct properties actually coincide. In particular, the problem of when (weak) flatness implies condition (P) has been studied by several authors like Renshaw, Liu, or Bulman-Fleming. By a sequence of properties above, there is no implication between flatness and condition (E). Even if they are incomparable, we investigate the monoids over which these two properties coincide.

Throughout this paper S will denote a monoid. We refer the reader to [1] for basic definitions and terminologies relating to semigroups and acts over monoids,

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and to [2] for definitions and basic results relating to the various flatness properties that we consider in this paper.

A right S-act A is said to satisfy *condition (P)* if whenever $au = a'u'$ with $u,\ u' \in S,\ a, a' \in A$, there exist $a'' \in A,\ s, s' \in S$ with $a = a''s, a' = a''s'$ and $su = s'u'$. A right S-act A is said to satisfy *condition(E)* if whenever $au = au'$ with $a \in A,\ u, u' \in S$, there exist $a'' \in A,\ s \in S$ with $a = a''s$ and $su = su'$. A right S-act that satisfies conditions (P)and (E) is said to be *strongly flat.*

A right S-act A is called *flat* if the functor $-\otimes A$ preserves all monomorphisms and is called *weakly flat* if $-\otimes A$ preserves embeddings of right ideals of S into S.

2. Monoids over which all(cyclic) right S-acts satisfy condition (E)

The following results appeared in [3].

Lemma 1. *Let S be a monoid, and let $p \leq q$ be nonnegative integers and let $x, s, t \in S$. Then $s\rho(x^p, x^q)t$ if and only if $s = t$ or $s = x^p u, t = x^p v, x^m u = x^n v$ for some $u, v \in S$ and some $m, n \geq 0$ integers with $m = n(mod(q-p))$.*

Lemma 2. *Let ρ be a right congruence on a monoid S. Then the following statements are equivalent:*
(1) S/ρ is strongly flat.
(2) S/ρ satisfies condition (E).
(3) For all $s, t \in S$, $s\rho t$ implies $us = ut$ for some $u \in S$ with $u\rho 1$.

Lemma 3. *Let S be a monoid and $x \in S$. Then the cyclic right S-act of the form $S/\rho(x, x^2)$ satisfies condition (E) if and only if $x = x^2$ or x is right invertible.*

Proof. If $x = x^2$, then the cyclic right S-act of the form $S/\rho(x, x^2)$ is isomorphic to S so that it is free. Thus $S/\rho(x, x^2)$ clearly satisfies condition (E). If x is right invertible, then $xy = 1$ for some y in S, hence $x^k\rho(x, x^2)1$ for all positive integers k. If $x\rho 1$, then there are $u, v \in S$ such that $x = xu, 1 = xv$ and $x^r u = x^r v$ for some integer $r \geq 0$. Hence

$$x^{r+1} = x^r x = x^r(xu) = x(x^r u) = x(x^r v) = x^r(xv) = x^r 1 = x^r.$$

If s, t are elements of S such that $s\rho t$, then $x^m s = x^n t$ for some nonnegative integers m, n. Since $x^{r+1} = x^r$, it induces $x^r s = x^r t$, so we take $u = x^r$. Then $us = ut$ with $u\rho 1$. Hence $S/\rho(x, x^2)$ satisfies condition (E).
For the converse, we assume that $S/\rho(x, x^2)$ satisfies condition (E) with $x \neq x^2$. If $S/\rho(x, x^2)$ satisfies condition (E), then there is some u in S such that $ux = ux^2$ with $u\rho 1$. Since if $u\rho 1$, then there are some some s, t in S such that $u = xs, 1 = xt$ and $x^k s = x^k t$ for some $k \geq 0$, x is clearly right invertible. □

Lemma 4. *Let I be a proper right ideal of a monoid S and let λ_I be Rees right congruence on S with respect to I. Then the cyclic right S-act S/λ_I satisfies condition (E) if and only if* $\mid I \mid = 1$.

Proof. If λ_I is Rees right congruence on S with respect to I, then the set S/λ_I of equivalence classes of λ_I contains equivalence classes I and $\{x\}$ where $x \notin I$. Hence if $\mid I \mid = 1$, then $S/\lambda_I \cong S$ and so it is trivial. Conversely, we assume that S/λ_I satisfies condition (E) and let $x, y \in I$. Then there is some $u \in S$ such that $ux = uy$ with $u\lambda_I 1$. If $u\lambda_I 1$, then $u = 1$, hence $ux = uy$ implies $x = y$. Thus $\mid I \mid = 1$. □

Liu[7] showed that all cyclic right S-acts satisfy condition (P) if and only if S is either a group or a group adjoined with 0. We now characterize the monoid over which all cyclic right S-acts satisfy condition (E).

Theorem 1. *Let S be a monoid. Then all cyclic right S-acts satisfy condition (E) if and only if S is either* $\{1\}$ *or* $\{1, 0\}$.

Proof. Assume that S is a monoid over which all cyclic right S-acts satisfy condition(E) and for $x \in S$ let ρ be the principal right congruence on S generated by (x, x^2). If S/ρ satisfies condition (E),then $x = x^2$ or x is right invertible.
Let I be the set of all non-right invertible elements of S. If I is empty, then all elements of S are right invertible. Let λ be the principal right congruence on S generated by $(1, x)$ where $x \in S$. Since S/λ also satisfies condition (E), there is some $u \in S$ such that $ux = u$ with $u\lambda 1$. If $u\lambda 1$, then $x^{n+1} = x^n$ for some integer $n \geq 0$ and so it is enough to say that the right invertible element x is 1. Thus $S = \{1\}$.

If I is nonempty, then it is easily seen that I is a right ideal of S. Moreover, I is proper since $1 \notin I$. Let λ_I be Rees right congruence on S with respect to I. If S/λ_I satisfies condition (E), then I is the singleton, and if x is in I, then it is clearly a left zero of S. If J is the set of all left zeros of S, then J is a nonempty set that is actually a right ideal of S. If $1 \in I$, then $S = \{1\}$. If 1 is not in I, then J is a proper right ideal of S. Hence we consider the cyclic right S-act S/λ_J where λ_J is Rees right congruence on S with respect to J. If it satisfies condition (E), then $\mid J \mid = 1$ by the same argument above. If S has only one left zero, then it is actually a zero of S, denoted by 0 and then this fact means that all nonzero elements of S are right invertible. Let y be a nonzero element of S. Since $S/\rho(1, y)$ satisfies condition (E) by assumption, there is some u in S such that $uy = u$ with $u\rho 1$ and, by the same argument above, $y = 1$ so that $S = \{1, 0\}$.

Conversely if $S = \{1\}$, then every right S-act is free so it is trivial that all cyclic right S-acts λ_J satisfy condition (E). Now we assume that $S = \{1, 0\}$. If ρ is any right congruence on S, then ρ is either the universal congruence v or the identity congruence ι. If ρ is the universal congruence v, then $\mid S/\rho \mid = 1$. Hence for any $s, t \in S$, $0s = 0t$ and $0\rho 1$. If ρ is the identity ι, then S/ρ is isomorphic to S and hence it clearly satisfies condition (E). □

Normark[9] proved that all cyclic left S-acts satisfy condition (E) if and only if all left S-acts satisfy condition (E). Thus, from the above theorem, we have the following result.

Lemma 5. *Let S be a monoid. Then all right S-acts satisfy condition (E) if and only if S is either* $\{1\}$ *or* $\{1, 0\}$.

3. Monoids over which all (cyclic) flat right S-acts satisfy condition (E)

Although the properties of flatness and condition (E) are incomparable, Liu showed that all right S-acts that satisfy condition (E) are flat if and only if S is regular. However it remains an open question to characterize the monoids over which all cyclic flat right S-acts satisfy condition (E).

Liu[7] proved that if S is right reversible and if all flat cyclic right S-acts satisfy condition (P), then $E(S) \subseteq \{0, 1\}$ where $E(S)$ is the set of idempotent elements of S. He showed that, under the additional hypothesis that S is left PP(all principal left ideals of S are projective), all flat cyclic right S-acts have property (P) if and only if S is either a right cancellative monoid, or a right cancellative monoid with zero adjoined. Bulman-Fleming and Normak[4] showed that, over left PP monoids S, every flat cyclic right act satisfies condition (P) if and only if every element of S is either right cancellative or right zero. They also showed that all flat cyclic S-acts are strongly flat if and only if S is right nil. Golchin and Renshaw, in recent paper, showed that if $S = G \cup N$ where G is a group and either $N = \emptyset$ or every element of N is right nil, then every flat cyclic right S-act satisfies condition (P). Moreover, among periodic monoid S, they proved that exactly those of the form described above have this feature.

Lemma 6. *Let S be a right reversible monoid. If every cyclic flat right S-act satisfies condition (E), then the set* $E(S)$ *of idempotents of S is either* $\{1\}$ *or* $\{1, 0\}$.

Proof. It is proved by similar arguments that Liu did in [7]. □

Remark. Bulman-Fleming studied flatness properties of cyclic acts S/ρ where $\rho = \rho(s, t)$ is the principal right congruence on S generated by (s, t) and determined conditions on S under which all flat or weakly flat acts of this type are actually strongly flat or projective.

We frequently deal in this paper with cyclic right S-acts (of the form S/ρ where ρ is a right congruence on S) and so we give characterizations of the various flatness concepts for such right S-acts which will be used often in this paper. Note that if λ is a left congruence on S and if $u \in S$, then $u\lambda$ denotes the left congruence on S defined by $x(u\lambda)y$ if and only if $(xu)\lambda(yu)$. By $\triangle$ we denote the equality relation on S.

Lemma 7. *Let S be a monoid and let ρ be a right congruence on S. Then*

(1) *S/ρ is free if and only if there exist $u, v \in S$ such that $vu = 1$ and for all $x, y \in S, vx = vy \Leftrightarrow x\rho y$.*

(2) *S/ρ is projective if and only if there exists $e \in S$ such that $e\rho 1$, and $x\rho y \Rightarrow ex = ey$, for $x, y \in S$.*

(3) *S/ρ is strongly flat if and only if for all $x, y \in S$ with $x\rho y$ there exists $u \in S$ such that $u\rho 1$ and $ux = uy$.*

(4) *S/ρ satisfies condition (P) if and only if for all $x, y \in S$ with $x\rho y$ there exist $u, v \in S$ such that $u\rho 1\rho v$ and $ux = uy$.*

(5) *S/ρ is flat if and only if for any left congruence λ on S and any $x, y \in S$, if $x(\rho \vee \lambda)y$ then there exist $u, v \in S$ such that $ux\lambda vy, u(\rho \vee x\lambda)1$, and $(\rho \vee y\lambda)1$.*

(6) *S/ρ is weakly flat if and only if the condition of (5) above holds when $\lambda = \triangle$.*

(7) *S/ρ is principally weakly flat if and only if whenever $u, v, x \in S$ and $ux\rho vx$ then $u(\rho \vee x\triangle)v$*

(8) *S/ρ is torsion-free if and only if whenever $u, v, c \in S$, c is cancellable, and $uc\rho vc$, then $u\rho v$.*

Lemma 8. *([6]) If $e^2 = e \in S$ and if $\rho = \rho(we, e)$ where $w \in S$, then S/ρ is flat.*

Theorem 2. *Let S be a monoid. If all cyclic flat right S-acts satisfy condition (E), then S is aperiodic. Conversely if S is aperiodic, then the cyclic flat right S-acts of the form $S/\rho(x, 1)$ where $x \in S$ satisfy condition (E).*

Proof. Assume that S is a monoid over which all flat cyclic right S-acts satisfy condition (E) and let $x \in S$. Since the cyclic right S-act $S/\rho(x, 1)$ is clearly flat, it satisfies condition (E) by assumption and then there is some u in S such that $ux = u$ with $u\rho 1$. If $u\rho 1$, then there are some nonnegative integers m, n such that $x^m u = x^n$. Hence $x^{n+1} = x^n x = x^m(ux) = x^m u = x^n$. Therefore S is aperiodic. Conversely we assume that S is aperiodic and for $s, t \in S, s\rho(x, 1)t$. If $s\rho(x, 1)t$, then $x^m s = x^n t$ for some integers $m, n \geq 0$ and then $x^m s = x^n t$ implies $x^r s = x^r t$ for some integer $r > 0$. Also since $1\rho x^r$, we take $u = x^r$. Then $us = ut$ with $u\rho 1$ Thus $S/\rho(x, 1)$ satisfies condition (E). □

Theorem 3. *S is a right reversible monoid over which all cyclic flat right S-acts satisfy condition (E) if and only if $S = \{1\}$ or $S \setminus \{1\}$ is a nilsemigroup.*

Proof. Assume that S is right reversible and let $x \in S$. If all cyclic flat right S-acts satisfy condition (E), then $x^{n+1} = x^n$ for some $n \in N$, and x^n is idempotent that belongs to the subsemigroup $< x >$. Also if S is right reversible, then the set $E(S)$ of idempotents of S is either $\{1\}$ or $\{1,0\}$ by lemma 6. Hence the fact that $E(S) = \{1\}$ implies that $x^n = 1$ with $1x = 1 = x1$. If $E(S) = \{1,0\}$, then $x^n = 1$ implies $x = 1$ and $x^n = 0$ induces that x is nilpotent with $x \neq 1$ since x^n is in $E(S)$. Thus $S = \{1\}$ or $S \setminus \{1\}$ is a nilsemigroup.
If we assume that $S = \{1\}$ or $S \setminus \{1\}$ is a nilsemigroup, then S is clearly right reversible. Also if $S = \{1\}$, then it is trivial that all cyclic flat right S-acts satisfy condition (E). If $S \setminus \{1\}$ is a nilsemigroup, then for every $x \in S$ with $x \neq 1$, x is nilpotent. Thus x^n for some $n \in N$ is a (right) zero of S so that S is right nil. Therefore, by Bulman-Fleming and Normak[4], every cyclic flat right S-act is strongly flat(clearly it satisfies condition (E)). □

References

1. A.H.Clifford and G.B.Preston, *Algebraic theory of semigroups*, Math. Surveys, No.7, A.M.S., Vol.1
2. J.M.Howie, *An introduction to semigroup theory*, Academic Press, 1976.
3. Bulman-Fleming, S., *Flat and Strongly flat S-systems*, Communications in Algebra, 20(1992), 2553-2567.
4. Bulman-Fleming, S. and P. Normak, *Monoids over which all flat cyclic right acts are strongly flat*, Semigroup Forum, 50(1995), 223-241.
5. Akbar Golchin and James Renshaw, *Periodic Monoids over which all flat cyclic right acts satisfy condition (P)*, Semigroup Forum, Version of November(1997), 1-3.
6. Bulman-Fleming, S. and P. Normak,*Flatness properties of monocyclic acts*, Mh. Math., 122(1996), 307-323.
7. Liu Zhongkui, *Characterization of Monoids by Condition (P) of Cyclic Left Acts*, Semigroup Forum, 49(1994), 31-39
8. Liu Zhongkui, *Monoids over which all flat left acts are regular*, J. of Pure and Applied Algebra, 111(1996), 199-203
9. Normak, P., *On equalizer-flat and pullback-flat acts*, Semigroup Forum, 36(1987), 293-313
10. Liu Zhongkui, *A characterization of regular monoids by flatness of left acts*, Semigroup Forum, 46(1993), 85-89
11. Liu Zhongkui, *Monoids over which all regular left acts are flat*, Semigroup Forum, 50(1995), 135-139

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