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ON BOUNDEDNESS OF ϵ-APPROXIMATE SOLUTION SET OF CONVEX OPTIMIZATION PROBLEMS †

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ABSTRACT. Boundedness for the set of all the ϵ-approximate solutions for convex optimization problems are considered. We give necessary and sufficient conditions for the sets of all the ϵ-approximate solutions of a convex optimization problem involving finitely many convex functions and a convex semidefinite problem involving a linear matrix inequality to be bounded. Furthermore, we give examples illustrating our results for the boundedness.

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1. Introduction

Convex optimization problem consists of a convex objective function and convex constraint functions. Recent research works and basic theories for convex optimization problems can be referred in the well-known books [2]. Convex semidefinite optimization problem is to optimize an objective convex function over a linear matrix inequality. When the objective function is linear and the corresponding matrices are diagonal, this problem become a linear optimization problem. So, this problem is an extension of a linear optimization problem. On 1988, Mangasarian [9] presented initially simple and elegant characterizations of the solution set of a convex optimization problem and gave conditions for boundedness of the solution set of a convex quadratic optimization problem. Since then, many authors have tried to extend the results of Mangasarian to

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several kinds of optimization problem ([3],[4],[6],[5],[7]). In particular, boundedness of solution sets for convex quadratic optimization problems [6], linear fractional optimization problems [5] and pseudolinear optimization problems [4], and boundedness of (properly, weakly) efficient solution sets for convex vector optimization problems [3], linear fractional vector optimization problems [9] and quadratic convex vector optimization problems [7] have been investigated. Very recently, Kim et al. [9] studied ϵ-optimality conditions and ϵ-saddle point theorems for ϵ-approximate solutions for convex semidefinite optimization problem which hold under a weakened constraint qualification or which hold without any constraint qualification.

In this paper, boundedness for the set of all the ϵ-approximate solutions for convex optimization problems are considered. We give necessary and sufficient conditions for the sets of all the ϵ-approximate solutions of a convex optimization problem involving finitely many convex functions and a convex semidefinite problem involving a linear matrix inequality to be bounded. Furthermore, we give examples illustrating our results for the boundedness.

2. Preliminaries

Consider the following convex optimization problem (P):

$$\text{(P)} \qquad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad x \in S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \quad i = 1, \cdots, m\}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \cdots, m$, are convex functions.

Definition 2.1. Let $\epsilon \geq 0$. Then $\bar{x} \in S$ is called an *ϵ-approximate solution of (P)* if for any $x \in S$,

$$f(x) + \epsilon \geq f(\bar{x}).$$

Definition 2.2 [1]. Let C be a nonempty set in $\mathbb{R}^n$. Then the *asymptotic cone of the set C*, denoted by C_∞, is

$$C_\infty = \left\{ d \in \mathbb{R}^n \mid \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d \right\}.$$

Proposition 2.1 [1]. *Let C be a nonempty convex set in $\mathbb{R}^n$. Then the asymptotic cone C_∞ is a closed convex cone. Let $x_0 \in C$. Then*

$$C_\infty := \left\{ d \in \mathbb{R}^n \,|\, x_0 + \lambda d \in clC, \forall \lambda > 0 \right\}.$$

Definition 2.3 [1]. For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, there exists a unique function $f_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, associated with f, called the *asymptotic function* such that $epi f_\infty = (epi f)_\infty$.

Proposition 2.2 [1]. *For any proper function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *and any* $\alpha \in \mathbb{R}$ *such that* $lev(f, \alpha) := \{x \mid f(x) \le \alpha\}$, *one has* $(lev(f, \alpha))_\infty \subset lev(f_\infty, d)$, *i.e.,*

$$\{x \mid f(x) \le \alpha\}_\infty \subset \{d \mid f_\infty(d) \le 0\}.$$

Equality holds in the clusion when f *is lower semicontinuous, proper and convex.*

Proposition 2.3 [1]. *Let* $\bigcap_{i \in I} A_i \ne \emptyset$ *and for* $i \in I$, A_i *is a closed convex set in* $\mathbb{R}^n$. *Then*

$$\left(\bigcap_{i \in I} A_i\right)_\infty = \bigcap_{i \in I} (A_i)_\infty.$$

Proposition 2.4 [1]. *A set* $C \subset \mathbb{R}^n$ *is bounded if and only if* $C_\infty = \{0\}$.

Proposition 2.5 [1]. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lower semicontinuous, convex function. The asymptotic function is a positively homogeneous, lsc, proper convex function, and for any* $d \in \mathbb{R}^n$ *one has*

$$f_\infty(d) = \sup \left\{ f(x + d) - f(x) \mid x \in domf \right\}$$

and for all $x \in domf$,

$$\begin{aligned} f_\infty(d) &= \lim_{t \to +\infty} \frac{f(x + td) - f(x)}{t} \\ &= \sup_{t > 0} \frac{f(x + td) - f(x)}{t}. \end{aligned}$$

3. ϵ-approximate solution set of convex optimization problems

Now we give necessary and sufficient conditions for the set of all the ϵ-approximate solutions of (P) to be bounded.

Theorem 3.1. *Let* $\epsilon \ge 0$. *Assume that* $\inf_{x \in S} f(x)$ *is finite, i.e.,* f *is bounded below. Then the following are equivalent:*

(1) $\left\{ d \in \mathbb{R}^n \mid f(x + d) \le f(x),\ g_i(x + d) \le g_i(x), \forall x \in \mathbb{R}^n,\ i = 1, \cdots, m \right\} = \{0\}$;

(2) $\left\{ d \in \mathbb{R}^n \mid f(x_0 + \lambda d) \le f(x_0),\ g_i(x_0 + \lambda d) \le g_i(x_0), \forall \lambda > 0,\ i = 1, \cdots, m \right\}$ $= \{0\}$, *where* x_0 *is any given point in* $\mathbb{R}^n$;

(3) $S_\infty \cap \left\{ d \in \mathbb{R}^n \mid f(x_0 + \lambda d) \le f(x_0), \forall \lambda > 0 \right\} = \{0\}$, *where* x_0 *is any given point in* $\mathbb{R}^n$;

(4) *The set of all* $\epsilon -$ *approximate solutions of* (P) *is compact.*

Proof. Let E be the set of all the ϵ-approximate solutions of (P). Then, since $\inf_{x \in S} f(x)$ is finite, $E \neq \emptyset$. Moreover,

$$\begin{aligned} E &= S \cap \Big\{ x \mid f(x) \leq f(y) + \epsilon, \forall y \in S \Big\} \\ &= S \cap \bigcap_{y \in S} \{ x \mid f(x) \leq f(y) + \epsilon \}. \end{aligned}$$

So, E is a nonempty closed and convex. Thus it follows from Proposition 2.4 that (4) holds if and only if $E_\infty = \{0\}$. From Propositions 2.2 and 2.3, we get

$$\begin{aligned} E_\infty &= S_\infty \cap \Big\{ d \in \mathbb{R}^n \mid f_\infty(d) \leq 0 \Big\} \\ &= \Big\{ d \in \mathbb{R}^n \mid f_\infty(d) \leq 0,\ (g_i)_\infty(d) \leq 0,\ i = 1, \cdots, m \Big\}. \end{aligned}$$

By Proposition 2.5, we have,

$f_\infty(d) \leq 0,\ (g_i)_\infty(d) \leq 0,\ i = 1, \cdots, m$

$\Longleftrightarrow$ $f(x+d) \leq f(x),\ g_i(x+d) \leq g_i(x),\ i = 1, \cdots, m$, for any $x \in \mathbb{R}^n$.

$\Longleftrightarrow$ for any given point $x_0 \in \mathbb{R}^n$,

$$f(x_0 + \lambda d) \leq f(x_0),\ g_i(x_0 + \lambda d) \leq g_i(x_0), \text{ for any } \lambda > 0.$$

So we have the conclusion. □

Now we give examples to illustrate Theorem 3.1.

Example 3.1. Consider the following convex optimization problem:

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & f(x) = -x \\ & \text{subject to} & g(x) := [\max\{0, x\}]^2 \leq 0. \end{array}$$

The set of all ϵ-approximate solutions of (P) is $[-\epsilon, 0]$. Moreover, $S := \{x \in \mathbb{R} \mid g(x) \leq 0\} = (-\infty, 0]$ and $S_\infty = (-\infty, 0]$. Thus

$$S_\infty \cap \{d \in \mathbb{R} \mid f(0 + \lambda d) \leq f(0), \forall \lambda > 0\} = \{0\}.$$

We give an example to which Theorem 3.1 can not be applied.

Example 3.2. Consider the following convex optimization problem:

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & f(x,y) = 2^{-x-y} \\ & \text{subject to} & g_1(x,y) = |x| - y \leq 0, \\ & & g_2(x,y) = -x + y \leq 0. \end{array}$$

The set of all ϵ- approximate solution of (P) is $\{(x,y) \mid -log_2\epsilon \leq x+y,\ x = y,\ x \geq 0\}$. Moreover $S := \{(x,y) \in \mathbb{R}^2 |\ g_1(x,y) \leq 0,\ g_2(x,y) \leq 0\} = \{(x,y) \,|\, x = y, x \geq 0\}$ and $S_\infty = \{(x,y) \,|\, x = y, x \geq 0\}$. Thus

$$S_\infty \cap \{d \in \mathbb{R}^2 \,|\, f(0 + \lambda d) \leq f(0), \forall \lambda > 0\} \neq \{0\}.$$

4. ϵ-approximate solution set of convex semidefinite optimization problems

Consider the following convex semidefinite programming model problem:

$$\text{(SDP)} \qquad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0,$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function, and for $i = 0, 1, \cdots, m$, $F_i \in S_n$, the space of $n \times n$ real symmetric matrices. The space S_n is partially ordered by the Löwner order; that is, for $M, N \in S_n$, $M \succeq N$ if and only if $M - N$ is positive semidefinite. The inner product in S_n is defined by $(M, N) = Tr[MN]$, where $Tr[\cdot]$ is the trace operation. Let $S := \{M \in S_n \mid M \succeq 0\}$. Then S is self-dual, that is,

$$S^+ = \{\theta \in S_n \mid (\theta, Z) \geq 0 \; \forall Z \in S\} = S.$$

Clearly, $A := \left\{ x \in \mathbb{R}^m \,\middle|\, F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0 \right\}$ is the feasible set of (SDP).

Proposition 4.1. $A_\infty = \{(d_1, \cdots, d_m) \in \mathbb{R}^m \mid d_1 F_1 + \cdots + d_m F_m \succeq 0\}$.

Proof. Let $B = \{(d_1, \cdots, d_m) \in \mathbb{R}^m \mid d_1 F_1 + \cdots + d_m F_m \succeq 0\}$. Clearly $0 \in B$. Let $d := (d_1, \cdots, d_m) \in A_\infty$ be such that $d \neq 0$. Then for any $x := (x_1, \cdots, x_m) \in A$ and any $\alpha \geq 0$ and any $w \in \mathbb{R}^n$,

$$\begin{aligned} w^T \left[F_0 + \sum_{i=1}^{m} (x_i + \alpha d_i) F_i \right] w &= w^T \left(F_0 + \sum_{i=1}^{m} x_i F_i \right) w + \alpha w^T \left(\sum_{i=1}^{m} d_i F_i \right) w \\ &\geq 0. \end{aligned}$$

Thus $w^T \left(\sum_{i=1}^{m} x_i F_i \right) w \geq 0$ for any $w \in \mathbb{R}^n$ and hence $(d_1, \cdots, d_m) \in B$. Hence $A_\infty \subset B$.

Conversely, let $d \in B$. Then for any $x \in A$ and any $\alpha \geq 0$,

$$\begin{aligned} F_0 + \sum_{i=1}^{m} (x_i + \alpha d_i) F_i &= F_0 + \sum_{i=1}^{m} x_i F_i + \alpha \sum_{i=1}^{m} d_i F_i \\ &\in S + S = S. \end{aligned}$$

Thus $x + \alpha d \in A_\infty$ and hence $B \subset A_\infty$ □

Theorem 4.1. *Suppose that* $\inf_{x \in A} f(x)$ *is finite. The set of all* ϵ*-approximate solutions of (SDP) is bounded if and only if*

$$\left\{ d \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^{m} d_i F_i \succeq 0 \right\} \cap \{d \in \mathbb{R}^m \mid f_\infty(d) \leq 0\} = \{0\}.$$

Proof. Let W be the set of all ϵ-approximate solutions of (SDP). Then by assumption, $W \neq \emptyset$. Also, from Propositions 2.2, 2.3 and 4.1, we have,

$$\begin{aligned} W_\infty &= \Big(A \cap \{\bar{x} \in \mathbb{R}^m \mid f(x) + \epsilon \geq f(\bar{x}),\ \forall x \in A\}\Big)_\infty \\ &= A_\infty \cap \{d \in \mathbb{R}^m \mid f_\infty(d) \leq 0\} \\ &= \left\{ d \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^m d_i F_i \succeq 0\} \cap \{d \in \mathbb{R}^m \mid f_\infty(d) \leq 0 \right\}. \end{aligned}$$

Hence W is bounded if and only if

$$\left\{ d \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^m d_i F_i \succeq 0 \right\} \cap \{d \in \mathbb{R}^m \mid f_\infty(d) \leq 0\} = \{0\}.$$

□

Now we will give examples illustrating Theorem 4.1.

Example 4.1. Consider the following convex semidefinite programming problem:

$$\text{(SDP)} \qquad \text{minimize} \quad \max\{|x_1 + 1|,\ |x_2|\}$$
$$\text{subject to} \quad \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1 + 1 \end{pmatrix} \succeq 0.$$

Let

$$F_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathrm{F}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} d_1 F_1 + d_2 F_2 \succeq 0 \quad &\Longleftrightarrow \quad \begin{pmatrix} 0 & d_1 & 0 \\ d_1 & d_2 & 0 \\ 0 & 0 & d_1 \end{pmatrix} \succeq 0 \\ &\Longleftrightarrow \quad d_1 \geq 0,\ d_2 \geq 0, -d_1^2 \leq 0,\ -d_1^3 \geq 0 \\ &\Longleftrightarrow \quad d_1 = 0,\ d_2 \geq 0. \end{aligned}$$

Let $f(x_1, x_2) = max\{\,|x_1 + 1|, |x_2|\,\}$. In fact,the feasible set for (SDP) is

$$A = \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}.$$

We have

$(\bar{x}_1, \bar{x}_2)$ is an ϵ-approximate solution of (SDP).

$\Longleftrightarrow (\bar{x}_1, \bar{x}_2) \in A$ and for any $(x_1, x_2) \in A, f(x_1, x_2) + \epsilon \geq f(\bar{x}_1, \bar{x}_2)$.

$\Longleftrightarrow \bar{x}_1 = 0,\ \bar{x}_2 \geq 0$ and $max\{1, x_2\} + \epsilon \geq max\{1, \bar{x}_2\}$ for any $x_2 \geq 0$.

$\Longleftrightarrow \bar{x}_1 = 0, 0 \leq \bar{x}_2 \leq 1 + \epsilon$.

Thus the set of all ϵ-approximate solutions of (SDP) is $\{(0, \bar{x}_2) \mid 0 \leq \bar{x}_2 \leq 1 + \epsilon\}$. Hence the set of all ϵ-approximate solution of (SDP) is bounded.

Now we will show that using Theorem 4.1, the set of all ϵ-approximate solutions of (SDP) is bounded. If $d_2 > 0$, then we have, for any $(x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned} f_\infty(0, d_2) &= \lim_{t \to \infty} \frac{f(x_1, x_2 + td_2) - f(x_1, x_2)}{t} \\ &= \lim_{t \to \infty} \frac{max\{|x_1 + 1|, |x_2 + td_2|\} - max\{|x_1 + 1|, |x_2|\}}{t} \\ &= \lim_{t \to \infty} \frac{x_2 + td_2 - max\{|x_1 + 1|, |x_2|\}}{t} \\ &= d_2. \end{aligned}$$

If $d_2 = 0$, $f_\infty(0, d_2) = 0$. Hence we have

$$\begin{aligned} &\{(d_1, d_2) \in \mathbb{R}^2 \,|\, d_1F_1 + d_2F_2 \geq 0\} \cap \{(d_1, d_2) \in \mathbb{R}^2 \,|\, f_\infty(0, d_2) \leq 0\} \\ &= \{(0, d_2) \in \mathbb{R}^2 \,|\, d_2 \geq 0\} \cap \{(0, d_2) \in \mathbb{R}^2 \,|\, f_\infty(0, d_2) \leq 0\} \\ &= \{(0, 0)\}. \end{aligned}$$

Thus by Theorem 4.1, the set of all ϵ-approximate solutions of (SDP) is bounded.

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