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WEAK AND STRONG CONVERGENCE OF THREE-STEP ITERATIONS WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

JAE UG JEONG

ABSTRACT. In this paper, we prove the weak and strong convergence of the three-step iterative scheme with errors to a common fixed point for two asymptotically nonexpansive mappings in a uniformly convex Banach space under a condition weaker than compactness. Our theorems improve and generalize some previous results.

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1. Introduction

Let K be a nonempty subset of a real normed linear space E. Let T be a self-mapping of K. T is said to be asymptotically nonexpansive with constant t_n if there exists $t_n \in [1, +\infty)$, $\lim_{n\to\infty} t_n = 1$, such that

$$\|T^n x - T^n y\| \leq t_n \|x - y\|, \quad \forall x, y \in K.$$

T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, but the converse does not hold.

It was proved in [1] that if E is uniformly convex and if K is bounded, closed and convex, then the asymptotically nonexpansive mapping has a fixed point.

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Takahasi and Tamuro [6] introduced the following iterative schemes known as Ishikawa iterative schemes for a pair of nonexpansive mappings;

$$\begin{cases} x_1 = x \in K, \\ y_n = b_n T x_n + (1 - b_n) x_n, \\ x_{n+1} = a_n S y_n + (1 - a_n) x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $a_n, b_n \in [0, 1]$.

Khan and Hafiz [3] generalized the scheme (1.1) to the one with errors for a pair of nonexpansive mappings as follows;

$$\begin{cases} x_1 = x \in K, \\ y_n = a_n' T x_n + b_n' x_n + c_n' v_n, \\ x_{n+1} = a_n S y_n + b_n x_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$, $\{c_n'\}$ are sequences in $[0, 1]$ with $0 < \delta \leq a_n, a_n' \leq 1 - \delta < 1$, $a_n + b_n + c_n = 1 = a_n' + b_n' + c_n'$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K.

We further generalize this scheme (1.2) for a pair of asymptotically nonexpansive mappings as follows;

$$\begin{cases} x_1 = x \in K, \\ z_n = a_n S^n x_n + (1 - a_n - \gamma_n) x_n + \gamma_n u_n, \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n) x_n + \mu_n v_n, \\ x_{n+1} = \alpha_n S^n y_n + \beta_n S^n z_n + (1 - \alpha_n - \beta_n - \lambda_n) x_n + \lambda_n w_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ are sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in K.

In this paper, we study the three-step iterative scheme with errors (1.3) for the weak and strong convergence for a pair of asymptotically nonexpansive mappings in a uniformly convex Banach space. Our theorems improve and generalize some previous results.

2. Preliminaries

Let E be a Banach space and let K be a nonempty subset of E. Let T be a mapping of K into itself. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} \,|\, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$. A uniformly convex Banach space is reflexive and strictly convex.

A Banach space E is said to satisfy Opial's condition ([5]) if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.$$

Next we state the following useful lemmas.

Lemma 2.1 [7, Lemma 1]. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be three nonnegative sequences satisfying*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n \quad \textit{for all} \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.2 [2, Lemma 4]. *Let* E *be a uniformly convex Banach space satisfying Opial's condition,* $\phi \neq K \subseteq E$ *closed and convex, and* $T : K \to K$ *asymptotically nonexpansive. Then* $I - T$ *is demiclosed with respect to zero.*

Lemma 2.3 [4, Lemma 1.4]. *Let* E *be a uniformly convex Banach space and* $B_r = \{x \in E \,|\, \|x\| \leq r\}$, $r > 0$. *Then there exists a continuous, strictly increasing and convex function* $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, *such that*

$$\|ax + by + cz + dw\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - abg(\|x-y\|)$$

for all $x, y, z, w \in B_r$ *and all* $a, b, c, d \in [0,1]$ *with* $a + b + c + d = 1$.

3. Main results

In this section, we prove our main theorems. Let K be a nonempty bounded convex subset of a real uniformly convex Banach space E. Let $S, T : K \to K$ be asymptotically nonexpansive mappings.

The following iteration scheme is studied:

$$\begin{cases} x_1 = x \in K, \\ z_n = a_n S^n x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n, \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n, \\ x_{n+1} = \alpha_n S^n y_n + \beta_n S^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in K.

Lemma 3.1. *If* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *and* $\{\mu_n\}$ *are sequences in* $[0,1]$ *such that* $\limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ *and* $\{s_n\}$, $\{t_n\}$ *are sequences of real numbers with* $s_n, t_n \geq 1$ *for all* $n \geq 1$ *and* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 1$, *then there exists a positive integer* N_1 *and* $\gamma \in (0,1)$ *such that* $a_n c_n s_n t_n < \gamma$ *for all* $n \geq N_1$.

Proof. By $\limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, there exists a positive integer N_0 and $\eta \in (0,1)$ such that

$$a_n c_n \leq c_n \leq b_n + c_n + \mu_n < \eta, \quad \forall n \geq n_0.$$

Let $\eta' \in (0,1)$ with $\eta' > \eta$. From $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 1$, there exists a positive integer $N_1 \geq N_0$ such that

$$s_n t_n - 1 < \frac{1}{\eta'} - 1, \quad \forall n \geq N_1,$$

from which we have $s_n t_n < \frac{1}{\eta'}$, $\forall n \geq N_1$. Put $\gamma = \frac{\eta}{\eta'}$. Then we have

$$a_n c_n s_n t_n < \frac{\eta}{\eta'} = \gamma$$

for all $n \geq N_1$. □

Lemma 3.2. *Let E be a uniformly convex Banach space and K its nonempty bounded convex subset. Let $S, T : K \to K$ be asymptotically nonexpansive mappings with constants s_n, t_n, respectively and $s_n, t_n \geq 1$, $\sum_{n=1}^{\infty}(s_n - 1) < \infty$, $\sum_{n=1}^{\infty}(t_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be the bounded sequence in K. Let $\{x_n\}$ be the sequence as defined in (3.1). If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.*

Proof. Let $x^* \in F(S) \cap F(T)$. Choose a number $r > 0$ such that $K \subseteq B_r$ and $K - K \subseteq B_r$. By Lemma 2.3, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\|ax + by + cz + dw\|^2 \leq a\|x\| + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - abg(\|x - y\|) \tag{3.2}$$

for all $x, y, z, w \in B_r$ and all $a, b, c, d \in [0,1]$ with $a + b + c + d = 1$. It follows from (3.2) that

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
&= \|a_n(S^n x_n - x^*) + (1 - a_n - \gamma_n)(x_n - x^*) + \gamma_n(u_n - x^*)\|^2 \\
&\leq a_n\|S^n x_n - x^*\|^2 + (1 - a_n - \gamma_n)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g(\|S^n x_n - x_n\|) \\
&\leq (a_n s_n^2 + (1 - a_n - \gamma_n))\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2, \\
&\|y_n - x^*\|^2 \\
&\leq \|b_n(T^n z_n - x^*) + (1 - b_n - c_n - \mu_n)(x_n - x^*) + c_n(T^n x_n - x^*) \\
&\quad + \mu_n(v_n - x^*)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq b_n\|T^n z_n - x^*\|^2 + (1-b_n-c_n-\mu_n)\|x_n - x^*\|^2 + c_n\|T^n x_n - x^*\|^2 \\
&\quad + \mu_n\|v_n - x^*\|^2 - b_n(1-b_n-c_n-\mu_n)g(\|T^n z_n - x_n\|) \\
&= b_n t_n^2\|z_n - x^*\|^2 + (1-b_n-c_n-\mu_n)\|x_n - x^*\|^2 + c_n t_n^2\|x_n - x^*\|^2 \\
&\quad + \mu_n\|v_n - x^*\|^2 - b_n(1-b_n-c_n-\mu_n)g(\|T^n z_n - x_n\|)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(S^n y_n - x^*) + (1-\alpha_n-\beta_n-\lambda_n)(x_n - x^*) + \beta_n(S^n z_n - x^*) \\
&\quad + \lambda_n(w_n - x^*)\|^2 \\
&\leq \alpha_n\|S^n y_n - x^*\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - x^*\|^2 + \beta_n\|S^n z_n - x^*\|^2 \\
&\quad + \lambda_n\|w_n - x^*\|^2 - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|S^n y_n - x_n\|) \\
&\leq \alpha_n s_n^2\|y_n - x^*\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - x^*\|^2 + \beta_n s_n^2\|z_n - x^*\|^2 \\
&\quad + \lambda_n\|w_n - x^*\|^2 - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|S^n y_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + (\alpha_n c_n s_n^2 t_n^2 + \alpha_n s_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n \\
&\quad - \lambda_n)\|x_n - x^*\|^2 + \alpha_n\mu_n s_n^2\|v_n - x^*\|^2 + (\alpha_n b_n s_n^2 t_n^2 + \beta_n s_n^2)\|z_n - x^*\|^2 \\
&\quad - \alpha_n b_n s_n^2(1-b_n-c_n-\mu_n)g(\|T^n z_n - x_n\|) + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|S^n y_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + (\alpha_n c_n s_n^2 t_n^2 + \alpha_n s_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n \\
&\quad - \lambda_n)\|x_n - x^*\|^2 + \alpha_n\mu_n s_n^2\|v_n - x^*\|^2 \\
&\quad + (\alpha_n b_n s_n^2 t_n^2 + \beta_n s_n^2)[(a_n s_n^2 + (1-a_n-\gamma_n))\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2] \\
&\quad - \alpha_n b_n s_n^2(1-b_n-c_n-\mu_n)g(\|T^n z_n - x_n\|) + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|S^n y_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + [\alpha_n c_n s_n^2(t_n^2 - 1) + \alpha_n(s_n^2 - 1) + \alpha_n b_n s_n^2(t_n^2 - 1) \\
&\quad + a_n\beta_n s_n^2(s_n^2 - 1) + a_n\alpha_n b_n s_n^2 t_n^2(s_n^2 - 1) + \beta_n(s_n^2 - 1)]\|x_n - x^*\|^2 \\
&\quad + \alpha_n\mu_n s_n^2\|v_n - x^*\|^2 + (\alpha_n b_n s_n^2 t_n^2 + \beta_n s_n^2)\gamma_n\|u_n - x^*\|^2 \\
&\quad - \alpha_n b_n s_n^2(1-b_n-c_n-\mu_n)g(\|T^n z_n - x_n\|) + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|S^n y_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + (\alpha_n c_n s_n^2 + \alpha_n b_n s_n^2)(t_n^2 - 1)\|x_n - x^*\|^2 \\
&\quad + (\alpha_n + a_n\beta_n s_n^2 + a_n\alpha_n b_n s_n^2 t_n^2 + \beta_n)(s_n^2 - 1)\|x_n - x^*\|^2 \\
&\quad + \mu_n s_n^2\|v_n - x^*\|^2 + s_n^2(t_n^2 + 1)\gamma_n\|u_n - x^*\|^2 \\
&\quad - \alpha_n b_n s_n^2(1-b_n-c_n-\mu_n)g(\|T^n z_n - x_n\|) + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|S^n y_n - x_n\|). \qquad (3.3)
\end{aligned}
$$

Since $\{s_n\}$, $\{t_n\}$ and K are bounded, there exists a constant $M > 0$ such that

$$(\alpha_n c_n s_n^2 + \alpha_n b_n s_n^2)\|x_n - x^*\|^2 \le M$$

and

$$(\alpha_n + a_n\beta_n s_n^2 + a_n\alpha_n b_n s_n^2 t_n^2 + \beta_n)\|x_n - x^*\|^2 \le M$$

for all $n \ge 1$. Put

$$L = \sup\left\{s_n^2(t_n^2+1)\|u_n - x^*\|^2 : n \ge 1\right\},$$

$$A = \sup\left\{s_n^2\|v_n - x^*\|^2 : n \ge 1\right\}$$

and

$$C = \max\left\{M, L, A, r^2\right\}.$$

By (3.3), we have

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 + C\{(t_n^2 - 1) + (s_n^2 - 1) + \mu_n + \gamma_n + \lambda_n\}.$$

Since $0 \le t^2 - 1 \le 2t(t-1)$ for all $t \ge 1$, the assumptions $\sum_{n=1}^{\infty}(t_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(s_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(t_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(s_n^2 - 1) < \infty$. It follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. □

Lemma 3.3. *Let E be a uniformly convex Banach space and K its nonempty bounded convex subset. Let $S, T : K \to K$ be asymptotically nonexpansive mappings with constants s_n, t_n, respectively and $s_n, t_n \ge 1$, $\sum_{n=1}^{\infty}(s_n - 1) < \infty$, $\sum_{n=1}^{\infty}(t_n - 1) < \infty$. Let $\{x_n\}$ be the sequence as defined in (3.1), and $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \ge 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$. Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be the bounded sequences in K. If $F(S) \cap F(T) \ne \phi$ and*

$$0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1,$$

$$0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1,$$

then

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0 = \lim_{n\to\infty}\|Tx_n - x_n\|.$$

Proof. Assume that $\liminf_{n\to\infty} \alpha_n > 0$ and $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n+c_n+\mu_n) < 1$. Then there exist a positive integer n_0 and $\nu, \eta, \eta' \in (0,1)$ such that

$$0 < \eta < b_n, \quad 0 < \nu < \alpha_n \quad \text{and} \quad b_n + c_n + \mu_n < \eta' < 1$$

for all $n \ge n_0$. By (3.3), we have

$$\begin{aligned}\nu\eta(1-\eta')g(\|T^n z_n - x_n\|) \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ + C\{(t_n^2 - 1) + (s_n^2 - 1) + \mu_n + \gamma_n + \lambda_n\}\end{aligned} \tag{3.4}$$

for all $n \ge n_0$, where

$$\begin{aligned} x^* &\in F(S) \cap F(T), \\ C &= \max\{M, L, A, r^2\}, \\ L &= \sup\{s_n^2(t_n^2+1)\|u_n - x^*\|^2 : n \ge 1\}, \\ A &= \sup\{s_n^2\|v_n - x^*\|^2 : n \ge 1\}. \end{aligned}$$

It follows from inequality (3.4) that for $m \ge n_0$

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|T^n z_n - x_n\|) &\le \frac{1}{\nu\eta(1-\eta')}\Big\{\sum_{n=n_0}^{m}(\|x_n - x^*\| - \|x_{m+1} - x^*\|^2) \\ &\quad + C\sum_{n=n_0}^{m}((t_n^2 - 1) + (s_n^2 - 1) + \mu_n + \gamma_n + \lambda_n)\Big\} \\ &\le \frac{1}{\nu\eta(1-\eta')}\Big\{\|x_{n_0} - x^*\|^2 \\ &\quad + C\sum_{n=n_0}^{m}((t_n^2 - 1) + (s_n^2 - 1) + \mu_n + \gamma_n + \lambda_n)\Big\}.\end{aligned} \tag{3.5}$$

Since $0 \le t^2 - 1 \le 2t(t-1)$ for all $t \ge 1$, the assumptions $\sum_{n=1}^{\infty}(s_n - 1) < \infty$, $\sum_{n=1}^{\infty}(t_n - 1) < \infty$ imply that $\sum_{n=1}^{\infty}(s_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(t_n^2 - 1) < \infty$. Let $m \to \infty$ in inequality (3.5). Then we get $\sum_{n=n_0}^{\infty} g(\|T^n z_n - x_n\|) < \infty$ and therefore

$$\lim_{n\to\infty} g(\|T^n z_n - x_n\|) = 0.$$

Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{3.6}$$

Since $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$, then by using a similar method, together with inequality (3.3), it can be shown that

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{3.7}$$

From $z_n = a_n S^n x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n$ and $y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n$ we have

$$\|x_n - z_n\| \le a_n\|x_n - S^n x_n\| + \gamma_n\|x_n - u_n\|,$$

$$\|y_n - x_n\| \le b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| + \mu_n\|v_n - x_n\|$$

and

$$\begin{aligned}\|T^n x_n - x_n\| &\le \|T^n x_n - T^n z_n\| + \|T^n z_n - x_n\|\\ &\le t_n\|x_n - z_n\| + \|T^n z_n - x_n\|\\ &\le a_n t_n\|x_n - S^n x_n\| + t_n\gamma_n\|x_n - u_n\| + \|T^n z_n - x_n\|.\end{aligned}$$

Thus

$$\begin{aligned}\|S^n x_n - x_n\| &\le \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\|\\ &\le s_n\|x_n - y_n\| + \|S^n y_n - x_n\|\\ &\le s_n b_n\|T^n z_n - x_n\| + s_n c_n\|T^n x_n - x_n\| + s_n\mu_n\|v_n - x_n\|\\ &\quad + \|S^n y_n - x_n\|\\ &\le s_n(b_n + c_n)\|T^n z_n - x_n\| + a_n c_n s_n t_n\|x_n - S^n x_n\|\\ &\quad + c_n s_n t_n\gamma_n\|x_n - u_n\| + s_n\mu_n\|v_n - x_n\| + \|S^n y_n - x_n\|.\end{aligned} \tag{3.8}$$

By Lemma 3.1, there exist a positive integer N_1 and $\gamma \in (0,1)$ such that $a_n c_n s_n t_n < \gamma$ for all $n \ge N_1$. This together with (3.8) implies that for $n \ge N_1$

$$\begin{aligned}(1-\gamma)\|S^n x_n - x_n\| &\le (1 - a_n c_n s_n t_n)\|S^n x_n - x_n\|\\ &\le s_n(b_n + c_n)\|T^n z_n - x_n\| + c_n s_n t_n\gamma_n\|x_n - u_n\|\\ &\quad + s_n\mu_n\|v_n - x_n\| + \|S^n y_n - x_n\|.\end{aligned}$$

It follows from (3.6) and (3.7) that

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0. \tag{3.9}$$

Moreover, since

$$\begin{aligned}\|S^n z_n - x_n\| &\le \|S^n z_n - S^n x_n\| + \|S^n x_n - x_n\|\\ &\le s_n\|z_n - x_n\| + \|S^n x_n - x_n\|\\ &\le (1 + a_n s_n)\|S^n x_n - x_n\| + s_n\gamma_n\|x_n - u_n\|,\end{aligned}$$

it follows from (3.9) that

$$\lim_{n\to\infty} \|S^n z_n - x_n\| = 0. \tag{3.10}$$

Since

$$x_{n+1} - x_n = \alpha_n(S^n y_n - x_n) + \beta_n(S^n z_n - x_n) + \lambda_n(w_n - x_n),$$

we have

$$\begin{aligned}\|x_{n+1} - S^n x_{n+1}\| &\le \|x_{n+1} - x_n\| + \|S^n x_{n+1} - S^n x_n\| + \|S^n x_n - x_n\| \\ &\le (1+s_n)\|x_{n+1} - x_n\| + \|S^n x_n - x_n\| \\ &\le (1+s_n)\alpha_n\|S^n y_n - x_n\| + (1+s_n)\beta_n\|S^n z_n - x_n\| \\ &\quad + (1+s_n)\lambda_n\|w_n - x_n\| + \|S^n x_n - x_n\|,\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - T^n x_{n+1}\| &\le \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\ &\le (1+t_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\ &\le (1+t_n)\alpha_n\|S^n y_n - x_n\| + (1+t_n)\beta_n\|S^n z_n - x_n\| \\ &\quad + \lambda_n(1+t_n)\|w_n - x_n\| + a_n t_n\|x_n - S^n x_n\| \\ &\quad + t_n\gamma_n\|x_n - u_n\| + \|T^n z_n - x_n\|.\end{aligned}$$

These together with (3.6), (3.7), (3.9), (3.10), imply that

$$\|x_{n+1} - S^n x_{n+1}\| \to 0 \quad \text{as} \quad n \to \infty,$$

$$\|x_{n+1} - T^n x_{n+1}\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\begin{aligned}\|x_{n+1} - Sx_{n+1}\| &\le \|x_{n+1} - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - Sx_{n+1}\| \\ &\le \|x_{n+1} - S^{n+1}x_{n+1}\| + s_1\|x_{n+1} - S^n x_{n+1}\|,\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\ &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + t_1\|x_{n+1} - T^n x_{n+1}\|,\end{aligned}$$

which imply

$$\lim_{n\to\infty} \|x_{n+1} - Sx_{n+1}\| = 0,$$

$$\lim_{n\to\infty} \|x_{n+1} - Tx_{n+1}\| = 0.$$

□

Theorem 3.1. *Let E be a uniformly convex Banach space satisfying Opial's condition, and $K, S, T, \{x_n\}, \{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be as taken in Lemma 3.3. If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $x^* \in F(S) \cap F(T)$. Then, as in Lemma 3.2, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and $\{x_n\}$ is bounded.

Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S)\cap F(T)$. Let z_1 and z_2 be two weak limits of the sequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3.3, we have

$$\lim_{i\to\infty} \|x_{n_i} - Sx_{n_i}\| = 0.$$

And Lemma 2.2 guarantees that $(I - S)z_1 = 0$, i.e., $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$. Again in the same way, we can prove that $z_2 \in F(S)\cap F(T)$.

Next, we prove the uniqueness. Now we are allowed to apply Lemma 3.2, which provides us with the existence of $a = \lim_{n\to\infty} \|x_n - z_1\|$ and $b = \lim_{n\to\infty} \|x_n - z_2\|$. Assuming that $z_1 \neq z_2$ and taking into account the fact that $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$, it follows from Opial's condition that

$$\begin{aligned} a = \limsup_{i\to\infty} \|x_{n_i} - z_1\| &< \limsup_{i\to\infty} \|x_{n_i} - z_2\| = b \\ &= \limsup_{j\to\infty} \|x_{n_j} - z_2\| \\ &< \limsup_{j\to\infty} \|x_{n_j} - z_1\| = a, \end{aligned}$$

which is a contradiction. Hence $z_1 = z_2$. This completes the proof. □

Theorem 3.2. *Let E be a uniformly convex Banach space and K its non-empty bounded convex subset. Let $S, T : K \to K$ be completely continuous asymptotically nonexpansive mappings with s_n, t_n, respectively and $s_n, t_n \geq 1$, $\sum_{n=1}^{\infty}(s_n-1) < \infty$, $\sum_{n=1}^{\infty}(t_n-1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n+\gamma_n$, $b_n+c_n+\mu_n$ and $\alpha_n+\beta_n+\lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$. Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be the bounded sequences in K. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined in (3.1). If $F(S)\cap F(T) \neq \phi$ and*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1,$$

$$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1,$$

then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of S and T.

Proof. In the proof of Lemma 3.3, we have

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|x_{n+1} - S^n x_{n+1}\| = 0,$$

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{3.11}$$

Thus

$$\begin{aligned}\|x_{n+1}-Sx_{n+1}\| &\leq \|x_{n+1}-S^{n+1}x_{n+1}\|+\|Sx_{n+1}-S^{n+1}x_{n+1}\|\\ &\leq \|x_{n+1}-S^{n+1}x_{n+1}\|+s_1\|x_{n+1}-S^{n}x_{n+1}\|\\ &\to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

which implies

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0. \tag{3.12}$$

Since S is completely continuous and $\{x_n\}\subseteq K$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Sx_{n_k}\}$ converges. Therefore from (3.12) $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k}=q$. By the continuity of S and (3.12), we have that $Sq=q$. So, q is a fixed point of S. By Lemma 3.2, $\lim_{n\to\infty}\|x_n-q\|$ exists. But $\lim_{k\to\infty}\|x_{n_k}-q\|=0$. Thus

$$\lim_{n\to\infty}\|x_n-q\|=0.$$

Since

$$\begin{aligned}\|y_n-x_n\| &\leq b_n\|T^nz_n-x_n\|+c_n\|T^nx_n-x_n\|+\mu_n\|v_n-x_n\|\\ &\to 0 \quad \text{as} \quad n\to\infty\end{aligned}$$

and

$$\begin{aligned}\|z_n-x_n\| &\leq a_n\|S^nx_n-x_n\|+\gamma_n\|x_n-u_n\|\\ &\to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

it follows that $\lim_{n\to\infty} y_n=q$ and $\lim_{n\to\infty} z_n=q$. This completes the proof. □

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Jae Ug Jeong studied Mathematics at Pusan National University. After having lectureship at Pusan University, he became instructor at Dongeui University in 1982 and promoted to assistant Professor in 1984. He received his Ph.D from Gyeongsang National University in 1991 and became Professor in 1991. He taught analysis, differential equations, nonlinear analysis and measure theory. His main research interests include nonlinear analysis, fixed point and variational inequality.

Department of Mathematics, Dong Eui University, Pusan 614-714, South Korea
e-mail: `jujeong@deu.ac.kr`