J. Appl. Math. & Informatics Vol. **26***(2008), No. 1 - 2, pp. 275 - 282*
Website: http://www.kcam.biz

GLOBAL ATTRACTIVITY OF THE RECURSIVE SEQUENCE $x_{n+1} = \frac{\alpha-\beta x_{n-1}}{\gamma+g(x_n)}$

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ABSTRACT. Our aim in this paper is to investigate the global attractivity of the recursive sequence $x_{n+1} = \frac{\alpha-\beta x_{n-1}}{\gamma+g(x_n)}$ under specified conditions. We show that the positive (or zero for $\alpha = 0$) equilibrium point of the equation is a global attractor with a basin that depends on certain conditions posed on the coefficients and the function $g(x)$.

AMS Mathematics Subject Classification : 39A10; 39A11.
Key words and phrases : Difference equations; attractivity; asymptotic stability.

1. Introduction

M.T.Aboutaleb et al [1] studied the rational recursive sequence

$$x_{n+1} = \frac{\alpha-\beta x_n}{\gamma+x_{n-1}} \qquad ,n=0,1,...$$

where the coefficients α, β and γ are non-negative real numbers and obtained sufficient conditions for the global attractivity of the positive equilibrium point.

Li and Sun[6]extended the above results to the following rational recursive sequence

$$x_{n+1} = \frac{\alpha-\beta x_n}{\gamma+x_{n-k}} \qquad ,n=0,1,...$$

H.M.El-Owaidy et al [3]studied the rational recursive sequence

$$x_{n+1} = \frac{\alpha-\beta x_{n-1}}{\gamma+x_n} \qquad ,n=0,1,...$$

Received February 22, 2007.

where the coefficients α, β and$\gamma > 0$ and obtained sufficient conditions for the global attractivity of the positive equilibrium point with basin that depend on certain conditions posed on the coefficients.

Also, El-Owaidy et al [2]studied the rational recursive sequences $x_{n+1} = \frac{-\alpha x_{n-1}}{\beta \pm x_n}$ where the coefficients $\alpha, \beta > 0$ and obtained sufficient conditions for the global attractivity of the zero equilibrium points with basin that depend on certain conditions posed on the coefficients. Other related results refer to [4,5,7].

Our aim in this paper is to investigate the global attractivity of the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_{n-1}}{\gamma + g(x_n)} \qquad , n = 0, 1, ... \tag{1.1}$$

where

$$\alpha \geq 0, \beta, \gamma > 0 \quad \text{and} \quad \gamma > \beta. \tag{1.2}$$

$g(x)$ is a continuous function on $(-\infty, \infty)$ satisfying some conditions which we will explain later. We show that the positive (or zero for $\alpha = 0$) equilibrium point of equation(1.1) is a global attractor with a basin that depends on certain conditions posed on the coefficients and the function $g(x)$.

The study of these equations is quite challenging and rewarding and is still in its infancy.

We believe that the nonlinear rational difference equations are of paramount importance in their own right,and furthermore that results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. We need the following definitions

Definition 1.1. The *equilibrium point* $\bar{x}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}), n = 0, 1, ... \tag{1.3}$$

is the point that satisfies the condition: $\bar{x} = F(\bar{x}, \bar{x})$.

Definition 1.2. Let $\bar{x}$ be an equilibrium point of equation (1.3).

(i) The equilibrium point $\bar{x}$ of equation (1.3)is called *locally stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_{-1}, x_0 \in I$ with $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$,we have $|x_n - \bar{x}| < \epsilon$ for all $n \geq -1$,

(ii)The equilibrium point $\bar{x}$ of equation (1.3)is called *locally asymptotically stable* if it is locally stable, and if there exist $\gamma > 0$ such that for all $x_{-1}, x_0 \in I$ with $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$,we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iii) The equilibrium point $\bar{x}$ of equation (1.3)is called a *global attractor* if for all $x_{-1}, x_0 \in I$,we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iv) The equilibrium point $\bar{x}$ of equation (1.3)is called *globally asymptotically stable* if $\bar{x}$ is locally stable and $\bar{x}$ is also global attractor.

(v)The equilibrium point $\bar{x}$ of equation (1.3)is called *unstable* if $\bar{x}$ is not locally stable.

(vi) The equilibrium point $\bar{x}$ of equation (1.3)is called a *repeller* if there exists $r > 0$ such that if $x_{-1}, x_0 \in I$ and $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < r$,then there exists $N \geq -1$ such that

$$|x_N - \bar{x}| \geq r.$$

Clearly,a repeller is an unstable equilibrium.

Let

$$z_{n+1} + pz_n + qz_{n-1} = 0, \qquad ,n = 0, 1, ... \tag{1.4}$$

be the linearized equation associated with equation(1.3)about the equilibrium point $\bar{x}$. Then Its characteristic equation is

$$\lambda^2 + p\lambda + q = 0. \tag{1.5}$$

We need the following theorem:

Theorem A[4].

(i) *If both roots of equation(1.5)have absolute value less than one,then the equilibrium point $\bar{x}$ of equation (1.3)is locally asymptotically stable.*

(ii) *If at least one of the roots of equation(1.5)has absolute value greater than one,then $\bar{x}$ is unstable.*

(iii) *Both roots of equation(1.5)have absolute value less than one if and only if*

$$|p| < 1 + q < 2.$$

In this case, $\bar{x}$ is locally asymptotically stable.

(iv) *Both roots of equation(1.5)have absolute value greater than one if and only if*

$$|q| > 1 \quad \textit{and} \quad |p| < |1 + q|.$$

In this case, $\bar{x}$ is a repeller.

(v) *One root of equation(1.5)has absolute value greater than one while the other root has absolute value less than one if and only if*

$$p^2 - 4q > 0 \quad \textit{and} \quad |p| > |1 + q|.$$

In this case $\bar{x}$, is a saddle point.

2. THE CASE $\alpha > 0$

In this section, we study the behavior of the difference equation

$$x_{n+1} = \frac{\alpha - \beta x_{n-1}}{\gamma + g(x_n)} \qquad ,n = 0, 1, ... \tag{2.1}$$

where

$$\alpha, \beta, \gamma > 0 \quad \text{and} \quad \gamma > \beta. \tag{2.2}$$

$g(x)$ is a continuous function on $(-\infty,\infty)$ satisfying

$$\left.\begin{array}{ll}\text{(i)} & g(x)>0 \quad \text{for} \quad x>0. \\ \text{(ii)} & g(x) \quad \text{is increasing on} \quad (-\infty,\infty). \\ \text{(iii)} & \dfrac{x}{g(x)} \quad \text{is nondecreasing on} \quad (0,\infty).\end{array}\right\} \tag{2.3}$$

Lemma 2.1. *Assume that (2.2) and (2.3) hold. Then equation (2.1) has the unique positive equilibrium point.*

Proof. Let $F(x)=x-\dfrac{\alpha-\beta x}{\gamma+g(x)}$. It is clear that $F(x)$ is continuous function on $[0,\infty)$. Since $F(0)=-\dfrac{\alpha}{\gamma+g(0)}<0$, and $\lim\limits_{x\to\infty}F(x)=\infty$ then there exists an $\bar{x}\in(0,\infty)$ such that $F(\bar{x})=0$. On the other hand, if $x>y$ we can simply show that $F(x)>F(y)$. So, $F(x)$ is increasing on$[0,\infty)$. Hence, $\bar{x}$ is the unique positive equilibrium point of equation (2.1).
This completes the proof. □

Lemma 2.2. *Assume that conditions (2.2) and (2.3) hold. Let $\{x_n\}$ be a solution of equation (2.1). If $x_m, x_{m+1}\in[0,\alpha/\beta]$ for some $m\geq -1$,then*

$$x_{m+i}\in[C,D], \qquad \forall i\geq 4$$

where

$$C=\frac{\alpha-\beta\alpha/\gamma}{\gamma+g(\alpha/\beta)} \quad \textit{and} \quad D=\frac{\alpha}{\gamma} \tag{2.3}$$

Proof. We can see that $x_{m+2}, x_{m+3}\in[0,\alpha/\gamma]$.Then

$$C=\frac{\alpha-\beta\alpha/\gamma}{\gamma+g(\alpha/\beta)}\leq\frac{\alpha-\beta\alpha/\gamma}{\gamma+g(\alpha/\gamma)}\leq x_{m+4}\leq\frac{\alpha}{\gamma}=D.$$

So,the result follows by induction. □

Assume that there exists $k\geq 2$ such that the following conditions hold

$$\gamma\geq k\alpha/\beta \quad \text{and} \quad \alpha\geq k\beta^2. \tag{2.4}$$

Lemma 2.3. *Assume that the conditions(2.4)hold for some $k\geq 2$. Let $\{x_n\}$ be a solution of equation(2.1). If $x_m\in[-(k-1)\alpha/\beta,\alpha/\beta]$ and $x_{m+1}\in[\max\{-(k-1)\alpha/\beta, g^{-1}(-(k-1)\alpha/\beta)\},\alpha/\beta]$ for some $m\geq -1$, then*

$$x_{m+i}\in[C,D], \qquad \forall i\geq 6$$

Proof. We can see that $x_{m+2}, x_{m+3} \in [0, \alpha/\beta]$. Then the proof follows immediately from Lemma 2.2 . □

Lemma 2.4. *Assume that the conditions(2.4)hold for some $k \geq 2$. Let $\{x_n\}$ be a solution of equation(2.1). Suppose that $g(-\frac{\alpha}{\beta}) \geq -(k-1)\alpha/\beta$, $x_m \in [-(k-1)\alpha/\beta, \infty)$ and $x_{m+1} \in [\max\{-(k-1)\alpha/\beta, g^{-1}(-(k-1)\alpha/\beta)\}, \alpha/\beta]$ for some $m \geq -1$ such that $|x_m - g(x_{m+1})| \leq \gamma + k\alpha/\beta$. Then*

$$x_{m+i} \in [C, D], \qquad \forall i \geq 8.$$

Proof. If $\alpha - \beta x_m \geq 0$, then $x_{m+2} \in [0, \alpha/\beta]$. If $\alpha - \beta x_m < 0$, then $x_{m+2} \in [-\alpha/\beta, 0]$. In both cases $x_{m+2} \in [-\alpha/\beta, \alpha/\beta]$. $x_{m+3} \in [0, \alpha/\beta]$. Then by using Lemma 2.3, we have $x_{m+i} \in [C, D]$, $\forall i \geq 8$. and then the proof is complete. □

Theorem 2.5. *If there exists $k \geq 2$ such that conditions(2.4)hold, then the positive equilibrium point $\bar{x}$ of equation (2.1)is a global attractor with a basin S such that if $g(-\frac{\alpha}{\beta}) \geq -(k-1)\alpha/\beta$, and $|x_{-1} - g(x_0)| \leq \gamma + k\alpha/\beta$, then*

$$S = \Big[-(k-1)\alpha/\beta, \infty\Big) \times \Big[\max\{-(k-1)\alpha/\beta, g^{-1}(-(k-1)\alpha/\beta)\}, \alpha/\beta\Big].$$

If else, then

$$S = \Big[-(k-1)\alpha/\beta, \alpha/\beta\Big) \times \Big[\max\{-(k-1)\alpha/\beta, g^{-1}(-(k-1)\alpha/\beta)\}, \alpha/\beta\Big].$$

Proof. Let $\{x_n\}$ be a solution of equation(2.1)with initial conditions $(x_{-1}, x_0) \in S$. Then by Lemma 2.3 and Lemma 2.4, $x_m \in [0, \alpha/\beta], \forall m \geq 3$. By Lemma 2.2

$$x_m \in [C, D] \quad \forall m \geq 7,$$

where C and D are defined in (2.3). Set

$$\lambda = \lim_{n\to\infty} \inf x_n \quad \text{and} \quad \Lambda = \lim_{n\to\infty} \sup x_n.$$

Let $\epsilon > 0$ be such that $\epsilon < \min\{(\alpha/\beta) - \Lambda, \lambda\}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\lambda - \epsilon < x_n < \Lambda + \epsilon, \forall n \geq n_0.$$

Hence

$$\frac{\alpha - \beta(\Lambda + \epsilon)}{\gamma + g(\Lambda + \epsilon)} < x_{n+1} < \frac{\alpha - \beta(\lambda - \epsilon)}{\gamma + g(\lambda - \epsilon)} \quad \forall n \geq n_0 + 1.$$

Then we get the inequality

$$\frac{\alpha - \beta(\Lambda + \epsilon)}{\gamma + g(\Lambda + \epsilon)} < \lambda < \Lambda < \frac{\alpha - \beta(\lambda - \epsilon)}{\gamma + g(\lambda - \epsilon)}.$$

This inequality yields

$$\frac{\alpha - \beta\Lambda}{\gamma + g(\Lambda)} \leq \lambda \leq \Lambda \leq \frac{\alpha - \beta\lambda}{\gamma + g(\lambda)},$$

which implies that $\alpha - \beta\Lambda - \gamma\lambda \leq \lambda g(\Lambda) \leq \Lambda g(\lambda) \leq \alpha - \beta\lambda - \gamma\Lambda$. Therefore $(\gamma - \beta)(\Lambda - \lambda) \leq 0$. Hence $\lambda = \Lambda = \bar{x}$. Then, the proof is complete. □

Next, we give some analysis on the semi-cycles of any solution $\{x_n\}$ of equation (2.1) about $\bar{x}$ with initial conditions $x_{-1}, x_0 \in [0, \alpha/\beta]$.

Theorem 2.6. *Assume that the initial conditions $x_{-1}, x_0 \in [0, \alpha/\beta]$ such that they are not equal to $\bar{x}$. Then the following statements are true*
(i) *$\{x_n\}$cannot have two consecutive terms equal to $\bar{x}$.*
(ii) *Every semicycle of $\{x_n\}$ has at most two terms.*
(iii) *$\{x_n\}$ is strictly oscillatory.*

Proof. (i) If $x_l = x_{l+1} = \bar{x}$ for some $l \in \mathbb{N}$, then $x_{l-1} = \bar{x}$,which implies that $x_{l-1} = x_{l-2} = ... = x_0 = x_{-1} = \bar{x}$, which is a contradiction.
(ii) Assume that C is a negative semicycle starts with two terms x_l, x_{l+1}. Then $0 \leq x_l, x_{l+1} < \bar{x}$ which implies that $x_{l+2} > \bar{x}$. Now Suppose that C is a positive semicycle starts with two terms x_l, x_{l+1}. If $x_l \geq \bar{x}$ and $x_{l+1} > \bar{x}$,then we have $x_{l+2} < \bar{x}$, also,if $x_l > \bar{x}$ and $x_{l+1} \geq \bar{x}$,then we have $x_{l+2} < \bar{x}$.
(iii) From(i)and(ii),we get $\{x_n\}$ is strictly oscillatory. This completes the proof. □

Remark. When $g(x) = x$, we get the results due to El-Owaidy et-al[3].

3. THE CASE $\alpha = 0$

In this section, we study the behavior of the difference equation

$$x_{n+1} = \frac{-\beta x_{n-1}}{\gamma + g(x_n)} \qquad ,n = 0, 1, ... \tag{3.1}$$

where $\beta, \gamma > 0$, $g(x)$ satisfies (2.3) and is differentiable on $(-\infty, \infty)$. The change of variables $x_n = \beta y_n$ reduces equation(3.1)to the difference equation

$$y_{n+1} = \frac{-y_{n-1}}{\frac{\gamma}{\beta} + \frac{1}{\beta}g(\beta y_n)} \qquad ,n = 0, 1, ... \tag{3.2}$$

Equation(3.2)has two equilibrium points $\bar{y}_1 = 0$ and $\bar{y}_2 = \frac{1}{\beta}g^{-1}(-\beta - \gamma)$.

Theorem 3.1. (i) *If $\gamma + g(0) > \beta$,then $\bar{y}_1 = 0$ is locally asymptotically stable.*
(ii) *If $\gamma + g(0) < \beta$,then $\bar{y}_1 = 0$ is unstable(repeller).*

(iii) *If* $\gamma + g(0) = \beta$, *then the linearized stability analysis fails.*
(iv) *The equilibrium point* $\bar{y}_2 = \frac{1}{\beta}g^{-1}(-\beta - \gamma)$ *is unstable (saddle point).*

Proof. The linearized equation associated with equation (3.2) about $\bar{y}_1 = 0$ has the form

$$z_{n+1} + \frac{\beta}{\gamma + g(0)} z_{n-1} = 0 \qquad ,n = 0, 1, ... \tag{3.3}$$

The characteristic equation of equation (3.3) is

$$\lambda^2 + \frac{\beta}{\gamma + g(0)} = 0.$$

Then the proof of (i),(ii) and (iii)follows immediately from Theorem A. The linearized equation associated with equation (3.2) about $\bar{y}_2 = \frac{1}{\beta}g^{-1}(-\beta - \gamma)$ has the form

$$z_{n+1} - \frac{1}{\beta}g^{-1}(-\beta - \gamma)g'(g^{-1}(-\beta - \gamma))z_n - z_{n-1} = 0 \tag{3.4}$$

The characteristic equation of equation (3.4) is

$$\lambda^2 - \frac{1}{\beta}g^{-1}(-\beta - \gamma)g'(g^{-1}(-\beta - \gamma))\lambda - 1 = 0. \tag{3.5}$$

Equation(3.5)has two roots λ_1, λ_2 such that $|\lambda_1\lambda_2| = 1$. From (2.3) $g^{-1}(-\beta - \gamma) \neq 0$, and $g'(g^{-1}(-\beta - \gamma)) \neq 0$. Hence $|\lambda_1| > 1$, and $|\lambda_2| < 1$. Then the proof is complete. □

In the following, we assume that $\gamma + g(-\beta) > \beta$.

Lemma 3.2. *Assume that the initial values* $y_{-1}, y_0 \in [-1, 0]$. *Then* $\{y_{4n-1}\}$, $\{y_{4n}\}$*is monotonically increasing to zero while* $\{y_{4n+1}\}$, $\{y_{4n+2}\}$*is monotonically decreasing to zero.*

Proof. Let $y_{-1}, y_0 \in [-1, 0]$,then $y_1, y_2 \in [0, 1]$ and $y_3, y_4 \in [-1, 0]$. By induction we can see that $\{y_{4n-1}\}, \{y_{4n}\} \in [-1, 0]$ and $\{y_{4n+1}\}, \{y_{4n+2}\} \in [0, 1]$ $\quad ,n = 0, 1, ...$ Since

$$\frac{y_{4n+2}}{y_{4n-2}} = \frac{\beta^2}{(\gamma + g(\beta y_{4n-1}))(\gamma + g(\beta y_{4n+1}))} < 1,$$

then $y_{4n+2} < y_{4n-2}$ $\quad ,n = 0, 1, ...$ Similarly we can see that $y_{4n+5} < y_{4n+1}$, $y_{4n+4} > y_{4n}$ and $y_{4n+3} > y_{4n-1}$ $\quad ,n = 0, 1, ...$ and the result follows. □

Lemma 3.3. (i) *Assume that* $y_{-1} \in [-1, 0]$ *and* $y_0 \in [0, 1]$. *Then* $\{y_{4n-1}\}$, $\{y_{4n+2}\}$*is monotonically increasing to zero while* $\{y_{4n}\}, \{y_{4n+1}\}$*is monotonically decreasing to zero.*

(ii) *Assume that* $y_{-1}, y_0 \in [0,1]$. *Then* $\{y_{4n+1}\}, \{y_{4n+2}\}$*is monotonically increasing to zero while* $\{y_{4n-1}\}, \{y_{4n}\}$*is monotonically decreasing to zero.*

(iii) *Assume that* $y_{-1} \in [0,1]$ *and* $y_0 \in [-1,0]$. *Then* $\{y_{4n}\}, \{y_{4n+1}\}$*is monotonically increasing to zero while* $\{y_{4n-1}\}, \{y_{4n+2}\}$*is monotonically decreasing to zero.*

Proof. The proof is similar to that of Lemma 3.2 and will be omitted . □

Theorem 3.4. *The equilibrium point* $\bar{y}_1 = 0$ *of equation (2.1)is a global attractor with a basin*

$$S = [-1,1]^2.$$

Proof. The proof follows immediately from Lemma 3.2 and Lemma 3.3 . □

Theorem 3.5. *The equilibrium point* $\bar{y}_1 = 0$ *of equation (2.1)is a global attractor with a basin*

$$S = \left\{ (y_{-1}, y_0); y_{-1} > 0, 0 < y_0 < 1 \quad and \quad \left| y_{-1} - \frac{g(\beta y_0)}{\beta} \right| < \frac{\gamma}{\beta} \right\}.$$

Proof. Suppose $y_{-1}, y_0 \in S$.Then $y_1, y_2 \in [-1,0]$. Then the proof follows immediately from Theorem 3.4 . □

Remark. When $g(x) = x$, we get the results due to El-Owaidy et-al[2].

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