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SECOND ORDER SUFFICIENT CONDITIONS IN OPTIMIZATION IN COMPLEX SPACE

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ABSTRACT. In this paper we give sufficient conditions for a point to be a solution of an optimization problem in complex space.

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1. Introduction

In this paper, we consider the optimization problem in complex space

$$\text{Minimize } \operatorname{Re} f(z) \quad \text{subject to} \quad z \in M, \quad g(z) \in S, \tag{1}$$

where M is a nonempty subset of $\mathbb{C}^n$, S is a nonempty subset of $\mathbb{C}^m$, and $f : M \to \mathbb{C}$ and $g : M \to \mathbb{C}^m$ are two functions.

Even though extremum problems, which contain complex functions with complex variables, have been studied for a long time, the founder of the optimization in complex space is considered to be N. Levinson, who, in a paper appeared in 1966 [22], generalizing Farkas' theorem to the complex space, gave duality theorems for a particular case of the complex linear optimization problem. The results obtained by N. Levinson are analogous to the duality theorems of real linear optimization.

In 1967, M.A. Hanson and B. Mond [21], generalizing P. Wolfe's duality from the optimization in real space to the optimization in complex space and using W.S. Dorn's technique from the real space, proved duality theorems for a particular case of a quadratic optimization problem in complex space. After one year, M. A. Hanson and B. Mond took over their studies of quadratic optimization problems in complex space in two papers [23], [24].

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In 1969, A. Ben-Israel [4] formulated the pair of dual problems for the general linear optimization problem in complex space and gave duality theorems analogous to those from the linear optimization in real space. In the same year, R.A. Abrams, in his doctoral thesis [1], formulated the general optimization problem in complex space. The main results of R.A. Abrams' thesis were published in two papers [3] and [2], the second being written in collaboration with A. Ben-Israel. In [3], a Kuhn-Tucker constraint qualification is given and a Kuhn-Tucker theorem for the general optimization problem in complex space is proved, while in [1] sufficient conditions for a point to be an optimal solution of an optimization problem in complex space, saddle point theorems and duality theorems are presented.

In 1978, D.I. Duca [9] formulated the vectorial optimization problem in complex space and obtained some necessary and sufficient conditions for a point to be the efficient solution of a problem; the same idea is treated in some other papers, too [10], [11], [16], [17], [25], [19].

From 1966 until now, hundreds of papers on optimization in complex space have been written. Among these, we recall some which are very important. Thus, in B.D. Craven and B. Mond's papers [5] and [6] a F. John theorem for the optimization problem in complex space is proved. In D.I. Duca's paper [8], three new complex constraint qualifications are given and a Kuhn-Tucker theorem is proved, the idea being treated again by the author of this paper in [12], where he gave a new proof of the F. John theorem, with richer conclusions than the ones given by B.D. Craven and B. Mond. Three new constraint qualifications, along with several implications among the seven complex constraint qualifications are also given in [12], while at the end of the paper, under the assumption that one of the seven complex constraint qualifications is satisfied, a Kuhn-Tucker theorem is proved. In [13], two saddle point theorems are given. Let us also mention the fact that B.D. Craven and B. Mond in [7] and D.I. Duca in [14] found, under weak enough hypothesis, sufficient conditions for a saddle point to be an optimal solution of the complex optimization problem.

2. Notations and preliminaries

Denote by $\mathbb{C}^n(\mathbb{R}^n)$ n-dimensional complex (real) space, and by $\mathbb{C}^{m\times n}$ the set of $m \times n$ complex matrices. If A is a matrix or a vector, then A^T, $\overline{A}$, A^H denote its transpose, complex conjugate, and conjugate transpose respectively.

For $z, w \in \mathbb{C}^n$; $\langle z, w\rangle = w^H z$ denotes the inner product of z and w.

The nonempty set S in $\mathbb{C}^m$ is a polyhedral cone if it is an intersection of closed half-spaces in $\mathbb{C}^m$, each containing 0 in its boundary, i.e.,

$$S = \bigcap_{k=1}^{p} H_{\geq}\left(a^k\right),$$

where

$$H_{\geq}\left(a^k\right) = \left\{v \in \mathbb{C}^m : \operatorname{Re}\left\langle v, a^k\right\rangle \geq 0\right\},\ k = 1, ..., p.$$

If $v \in S = \bigcap_{k=1}^{p} H_{\geq}\left(a^k\right)$, then $S(v)$ is defined to be the intersection of those closed half-space $H_{\geq}\left(a^k\right)$ which include v in their boundaries, i.e.,

$$S(v) = \bigcap_{k \in E} H_{\geq}\left(a^k\right),$$

where

$$E = \{k \in \{1, ..., p\} : \operatorname{Re}\left\langle v, a^k\right\rangle = 0\}.$$

If S is a nonempty subset of $\mathbb{C}^n$, then

$$S^* = \{u \in \mathbb{C}^n : \ \operatorname{Re}\langle u, z\rangle \geq 0, \text{ for all } z \in S\}$$

denotes the polar of the set S.

Definition 1. Let M be a nonempty subset of $\mathbb{C}^n$ and z^0 be an interior point of M.

A function $f : M \to \mathbb{C}^m$ is said to be *differentiable at* z^0 if there exist two matrices $\nabla_z f\left(z^0\right), \nabla_{\overline{z}} f\left(z^0\right) \in \mathbb{C}^{m\times n}$ and a function $\varepsilon : M \to \mathbb{C}^m$, continuous at z^0 and vanishing at this point, i.e.,

$$\lim_{z \to z^0} \varepsilon(z) = \varepsilon\left(z^0\right) = 0,$$

such that, for each $z \in M$, it holds

$$f(z) - f\left(z^0\right) = \left[\nabla_z f\left(z^0\right)\right]\left(z - z^0\right) + \left[\nabla_{\overline{z}} f\left(z^0\right)\right]\left(\overline{z} - \overline{z}^0\right) + \left\|z - z^0\right\| \varepsilon(z).$$

The function f is said to be *differentiable on* $M_0 \subseteq M$ if f is differentiable at each $z \in M_0$.

The function $g : M \to \mathbb{C}$ is said to be *twice differentiable at* z^0 if the function g is differentiable on a neighbourhood $V \subseteq M$ of z^0 and the functions $\nabla_z g(\cdot)$, $\nabla_{\overline{z}} g(\cdot) : V \to \mathbb{C}^n$ are differentiable at z^0. We will denote by:

$$\nabla^2_{zz} g\left(z^0\right) = \nabla_z\left(\nabla_z g\right)\left(z^0\right) \in \mathbb{C}^{n\times n}, \quad \nabla^2_{\overline{z}z} g\left(z^0\right) = \nabla_z\left(\nabla_{\overline{z}} g\right)\left(z^0\right) \in \mathbb{C}^{n\times n},$$

$$\nabla^2_{z\overline{z}} g\left(z^0\right) = \nabla_{\overline{z}}\left(\nabla_z g\right)\left(z^0\right) \in \mathbb{C}^{n\times n}, \quad \nabla^2_{\overline{z}\overline{z}} g\left(z^0\right) = \nabla_{\overline{z}}\left(\nabla_{\overline{z}} g\right)\left(z^0\right) \in \mathbb{C}^{n\times n}.$$

Definition 2. Let M be a nonempty subset of $\mathbb{C}^n$, z be a point from M, S be a subset of $\mathbb{C}^m$ and $f : M \to \mathbb{C}^m$ be a function.

i) We say that the function f is *convex* at z with respect to S if, for each $v \in M \setminus \{z\}$ and each $\alpha \in]0, 1[$ with the property that

$$(1 - \alpha) z + \alpha v \in M,$$

we have

$$(1 - \alpha) f(z) + \alpha f(v) - f((1 - \alpha)z + \alpha v) \in S.$$

ii) We say that the function f is *quasiconvex* at z with respect to S if, for each $v \in M \setminus \{z\}$ with the property that

$$f(z) - f(v) \in S$$

and, for each $\alpha \in]0,1[$ such that $(1-\alpha)z + \alpha v \in M$, we have

$$f(z) - f((1-\alpha)z + \alpha v) \in S.$$

iii) We say that the function f is *pseudoconvex* at z with respect to S if z is an interior point of M, f is differentiable at z and, for each $v \in M \setminus \{z\}$ such that

$$[\nabla_z f(z)](v-z) + [\nabla_{\overline{z}} f(z)](\overline{v} - \overline{z}) \in S,$$

we have

$$f(v) - f(z) \in S.$$

iv) We say that the function f is *concave* (respectively quasiconcave, pseudoconcave) at z with respect to S if the function $(-f)$ is convex (respectively quasiconvex, pseudoconvex) at z with respect to S.

v) We say that the function f has convex (respectively concave, quasiconvex, quasiconcave, pseudoconvex, pseudoconcave) real part at z with respect to $\mathbb{R}^m_+$ if f is convex (respectively concave, quasiconvex, quasiconcave, pseudoconvex, pseudoconcave) at z with respect to $S = \{w \in \mathbb{C}^m : \operatorname{Re} w \in \mathbb{R}^m_+\}$.

Theorem 1. *Let M be a nonempty subset of $\mathbb{C}^n$, z be an interior point of M, S be a closed convex cone in $\mathbb{C}^m$ and $f : M \to \mathbb{C}^m$ be a differentiable function at z.*

1^0. *If the function f is convex at z with respect to S, then*

$$f(v) - f(z) - [\nabla_z f(z)](v-z) - [\nabla_{\overline{z}} f(z)](\overline{v} - \overline{z}) \in S \text{ for all } v \in M.$$

2^0. *If the function f is concave at z with respect to S, then*

$$f(z) - f(v) - [\nabla_z f(z)](z-v) - [\nabla_{\overline{z}} f(z)](\overline{z} - \overline{v}) \in S \text{ for all } v \in M.$$

3^0. *If the function f is quasiconvex at z with respect to S, then, for each $z \in M$ with the property that $f(z) - f(v) \in S$, we have*

$$[\nabla_z f(z)](z-v) + [\nabla_{\overline{z}} f(z)](\overline{z} - \overline{v}) \in S.$$

4^0. *If the function f is quasiconcave at z with respect to S, then, for each $z \in M$ with the property that $f(v) - f(z) \in S$, we have*

$$[\nabla_z f(z)](v-z) + [\nabla_{\overline{z}} f(z)](\overline{v} - \overline{z}) \in S.$$

In [14] one proves the following sufficient conditions for optimal solutions of the optimization in complex space.

Theorem 2. *Let M be a nonempty subset of $\mathbb{C}^n$ and z^0 be an interior point of M. Let also $f : M \to \mathbb{C}$ and $g : M \to \mathbb{C}^m$ be two differentiable functions at z^0 and S be a closed convex cone in $\mathbb{C}^m$.*

A sufficient condition for

$$z^0 \in X = \{z \in M : g(z) \in S\}$$

to be a solution of Problem (1) is that there exists a point $v^0 \in \mathbb{C}^m$ such that the function $L : M \to \mathbb{C}$ defined by

$$L(z) = f(z) - \langle g(z), v^0 \rangle \quad \text{for all } z \in M,$$

has pseudoconvex real part at z^0 with respect to $\mathbb{R}_+$ and

$$v^0 \in S^*, \quad \mathrm{Re}\langle g(z^0), v^0 \rangle = 0$$

and

$$\mathrm{Re}\Big\langle z - z^0, \overline{\nabla_z f(z^0)} + \nabla_{\overline{z}} f(z^0)$$

$$-[\nabla_z g(z^0)]^H (v^0) - [\nabla_{\overline{z}} g(z^0)]^T (\overline{v}^0) \Big\rangle \geq 0$$

for all $z \in X$.

An important particular case of Theorem 2 is the following corollary.

Corollary 1. *Let M be a nonempty subset of $\mathbb{C}^n$, z^0 be an interior point of M and S be a closed convex cone in $\mathbb{C}^m$. Let also $f : M \to \mathbb{C}$ be a function differentiable at z^0, with convex real part at z^0 with respect to $\mathbb{R}_+$ and $g : M \to \mathbb{C}^m$ be a function differentiable at z^0 and concave at z^0 with respect to S.*

A sufficient condition for

$$z^0 \in X = \{z \in M : g(z) \in S\}$$

to be a solution of Problem (1) is that there exists a point $v^0 \in \mathbb{C}^m$ such that

$$v^0 \in S^*, \quad \mathrm{Re}\langle g(z^0), v^0 \rangle = 0$$

and

$$\overline{\nabla_z f(z^0)} + \nabla_{\overline{z}} f(z^0) - [\nabla_z g(z^0)]^H (v^0) - [\nabla_{\overline{z}} g(z^0)]^T (\overline{v}^0) = 0.$$

Theorem 3. *Let M be a nonempty subset of $\mathbb{C}^n$, z^0 be an interior point of M and S be a polyhedral cone in $\mathbb{C}^m$. Let also $f : M \to \mathbb{C}$ be a function differentiable at z^0 with pseudoconvex real part at z^0 with respect to $\mathbb{R}_+$ and $g : M \to \mathbb{C}^m$ be a function differentiable at z^0 and quasiconcave at z^0 with respect to $S(g(z^0))$.*

A sufficient condition for

$$z^0 \in X = \{z \in M : g(z) \in S\}$$

to be a solution of Problem (1) is that there exists a point $v^0 \in \mathbb{C}^m$ *such that*

$$v^0 \in \left(S\left(g\left(z^0\right)\right)\right)^*$$

and

$$\operatorname{Re}\langle z - z^0, \overline{\nabla_z f(z^0)} + \nabla_{\overline{z}} f(z^0) -$$
$$- [\nabla_z g(z^0)]^H \left(v^0\right) - [\nabla_{\overline{z}} g(z^0)]^T \left(\overline{v}^0\right)\rangle \geq 0 \text{ for all } z \in X.$$

In this paper, we give sufficient conditions for the solutions of optimization problems in complex space without hypotheses of convexity about functions.

3. Main Results

Theorem 4. *Let* M *be a nonempty subset of* $\mathbb{C}^n$, z^0 *be an interior point of* M, $f : M \to \mathbb{C}$ *and* $g : M \to \mathbb{C}^m$ *be twice differentiable functions at* z^0 *and*

$$S = \bigcap \{H_{\geq}(a^k) : k \in \{1, ..., q\}\}$$

be a polyhedral cone in $\mathbb{C}^m$.

Let

$$(\gamma_k) \in \mathbb{R}^q_+ \tag{2}$$

be such that

$$v^0 = \sum_{k=1}^{q} \gamma_k a^k \in S^* \tag{3}$$

satisfies

$$\operatorname{Re}\left\langle g\left(z^0\right), v^0\right\rangle = 0 \tag{4}$$

and

$$\overline{\nabla_z f(z^0)} + \nabla_{\overline{z}} f(z^0) - [\nabla_z g(z^0)]^H \left(v^0\right) - [\nabla_{\overline{z}} g(z^0)]^T \left(\overline{v}^0\right) = 0. \tag{5}$$

Let also

$$E = \left\{k \in \{1, ...q\} : \operatorname{Re}\langle g\left(z^0\right), a^k\rangle = 0\right\}$$

and $E_+ = \{k \in E : \gamma_k > 0\}$.

If, for each solution $d \in \mathbb{C}^n \setminus \{0\}$ *of the system*

$$\begin{cases} [\nabla_z g(z^0)]\,(d) + [\nabla_{\overline{z}} g(z^0)](\overline{d}) \in S(g(z^0)), \\ \operatorname{Re}\langle [\nabla_z g(z^0)]\,(d) + [\nabla_{\overline{z}} g(z^0)]\,(\overline{d}), a^k\rangle = 0, \ k \in E_+ \end{cases} \tag{6}$$

we have

$$\Big\langle [\overline{\nabla^2_{zz} f(z^0)} + \overline{\nabla^2_{z\bar z} f(z^0)} + \nabla^2_{\bar z z} f(z^0) + \nabla^2_{\bar z\bar z} f(z^0) \\ - \left[\nabla^2_{zz} g(z^0)\right]^H (v^0) - \left[\nabla^2_{z\bar z} g(z^0)\right]^H (v^0) \\ - \left[\nabla^2_{\bar z z} g(z^0)\right]^T (\overline{v}^0) - \left[\nabla^2_{\bar z\bar z} g(z^0)\right]^T (\overline{v}^0)] (d)\,, d\Big\rangle > 0, \tag{7}$$

where

$$\left[\nabla^2_{zz} g(z^0)\right]^H (v^0) = \sum_{j=1}^m v_j^0 \; \overline{\nabla^2_{zz} g_j(z^0)},$$

$$\left[\nabla^2_{z\bar z} g(z^0)\right]^H (v^0) = \sum_{j=1}^m v_j^0 \; \overline{\nabla^2_{z\bar z} g_j(z^0)},$$

$$\left[\nabla^2_{\bar z z} g(z^0)\right]^T (\overline{v}^0) = \sum_{j=1}^m \overline{v}_j^0 \; \nabla^2_{\bar z z} g_j(z^0),$$

$$\left[\nabla^2_{\bar z\bar z} g(z^0)\right]^T (\overline{v}^0) = \sum_{j=1}^m \overline{v}_j^0 \; \nabla^2_{\bar z\bar z} g_j(z^0),$$

then z^0 is a local solution of Problem (1).

Proof. Let us assume that z^0 is not a local solution of Problem (1). Then there exists a sequence $(z^j)_{j\in\mathbb{N}}$ from $X \setminus \{z^0\}$ which converges to z^0 such that

$$\operatorname{Re} f(z^j) < \operatorname{Re} f(z^0), \text{ for all } j \in \mathbb{N}. \tag{8}$$

For each $j \in \mathbb{N}$ we make the following denotations:

$$t_j = \left\|z^j - z^0\right\| \quad \text{and} \quad d^j = \frac{1}{t_j}(z^j - z^0).$$

Then we have

$$t_j > 0 \quad \text{and} \quad z^j = z^0 + t_j d^j, \text{ for all } j \in \mathbb{N}$$

and $\lim\limits_{j\to\infty} t_j = 0$. Since $\left\|d^j\right\| = 1$ for all $j \in \mathbb{N}$, it follows that the sequence $(d^j)_{j\in\mathbb{N}}$ contains a convergent subsequence. Without loss of generality, we can assume that the sequence itself is convergent. Let $d = \lim\limits_{j\to\infty} d^j$. Then $\|d\| = 1$. From (8) we deduce that

$$\operatorname{Re} \frac{1}{t_j}[f(z^0 + t_j d^j) - f(z^0)] < 0, \text{ for all } j \in \mathbb{N}.$$

From this, by passing to limit, we obtain that

$$\operatorname{Re}\left\{[\nabla_z f(z^0)]\,(d) + [\nabla_{\bar z} f(z^0)]\,(\overline{d})\right\} \le 0. \tag{9}$$

From $z^j \in X$ for all $j \in \mathbb{N}$, we have $g(z^j) \in S$ for all $j \in \mathbb{N}$ and hence $g(z^j) - g(z^0) \in S(g(z^0))$ for all $j \in \mathbb{N}$. It follows that

$$\frac{1}{t_j}\left\{g\left(z^0+t_jd^j\right)-g\left(z^0\right)\right\}\in S\left(g\left(z^0\right)\right)\ \text{ for all } j\in\mathbb{N}.$$

From this, by passing to limit, we obtain that

$$[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right)\in S(g(z^0)),\tag{10}$$

or, equivalently

$$\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),a^k\rangle\geq 0\ \text{ for all } k\in E.\tag{11}$$

We will show, by contradiction, that

$$\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),a^k\rangle=0\ \text{ for all }\ k\in E_+.\tag{12}$$

Let us assume that there exists $k_0\in E_+$ such that

$$\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),a^{k_0}\rangle>0.\tag{13}$$

Then, from (2), (3), (4), (11) and (13), we obtain

$$\begin{aligned}&\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),v^0\rangle\\&=\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),\sum_{k=1}^{q}\gamma_k a^k\rangle\\&=\sum_{k=1}^{q}\gamma_k\,\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),a^k\rangle\\&=\sum_{k\in E}\gamma_k\,\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),a^k\rangle\\&\geq\gamma_{k_0}\,\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),a^{k_0}\rangle>0.\end{aligned}$$

Hence

$$\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),v^0\rangle>0.\tag{14}$$

On the other hand, from (5) and (9), we have

$$\begin{aligned}&\mathrm{Re}\langle[\nabla_z g(z^0)]\,(d)+[\nabla_{\overline{z}} g(z^0)]\,\left(\overline{d}\right),v^0\rangle\\&=\mathrm{Re}\langle d,[\nabla_z g(z^0)]^{\mathrm{H}}\left(v^0\right)+[\nabla_{\overline{z}} g(z^0)]^{\mathrm{T}}\left(\overline{v}^0\right)\rangle\\&=\mathrm{Re}\langle d,\overline{\nabla_z f(z^0)}+\nabla_{\overline{z}} f(z^0)\rangle\\&=\mathrm{Re}\left\{[\nabla_z f(z^0)]\,(d)+[\nabla_{\overline{z}} f(z^0)]\,\left(\overline{d}\right)\right\}\leq 0,\end{aligned}$$

which contradicts (14).

From (12) and (10), it follows that d is a solution of system (6). Then we have

$$\begin{aligned}\Big\langle&[\overline{\nabla^2_{zz}f(z^0)}+\overline{\nabla^2_{z\overline{z}}f(z^0)}+\nabla^2_{\overline{z}z}f(z^0)+\nabla^2_{\overline{z}\overline{z}}f(z^0)\\&-\left[\nabla^2_{zz}g(z^0)\right]^{\mathrm{H}}\left(v^0\right)-\left[\nabla^2_{z\overline{z}}g(z^0)\right]^{\mathrm{H}}\left(v^0\right)\\&-\left[\nabla^2_{\overline{z}z}g(z^0)\right]^{\mathrm{T}}\left(\overline{v}^0\right)-\left[\nabla^2_{\overline{z}\overline{z}}g(z^0)\right]^{\mathrm{T}}\left(\overline{v}^0\right)]\,(d),d\Big\rangle>0.\end{aligned}\tag{15}$$

If we denote by $F : M \to \mathbb{C}$ the function$F(z) = f(z) - \langle g(z), v^0 \rangle$, for all $z \in M$, then

$$\begin{aligned} \nabla_z F(z^0) &= \nabla_z f(z^0) - \left[\nabla_z g(z^0)\right](\overline{v}^0), \\ \nabla_{\overline{z}} F(z^0) &= \nabla_{\overline{z}} f(z^0) - \left[\nabla_{\overline{z}} g(z^0)\right](\overline{v}^0), \\ \nabla^2_{zz} F(z^0) &= \nabla^2_{zz} f(z^0) - \sum_{k=1}^{m} \overline{v}^0_k \, \nabla^2_{zz} g_k(z^0), \\ \nabla^2_{z\overline{z}} F(z^0) &= \nabla^2_{z\overline{z}} f(z^0) - \sum_{k=1}^{m} \overline{v}^0_k \, \nabla^2_{z\overline{z}} g_k(z^0), \\ \nabla^2_{\overline{z}z} F(z^0) &= \nabla^2_{\overline{z}z} f(z^0) - \sum_{k=1}^{m} \overline{v}^0_k \, \nabla^2_{\overline{z}z} g_k(z^0), \\ \nabla^2_{\overline{z}\overline{z}} F(z^0) &= \nabla^2_{\overline{z}\overline{z}} f(z^0) - \sum_{k=1}^{m} \overline{v}^0_k \, \nabla^2_{\overline{z}\overline{z}} g_k(z^0). \end{aligned}$$

It follows that inequality (15) can be written as

$$\left\langle \left[\nabla^2_{zz} F(z^0) + \nabla^2_{z\overline{z}} F(z^0) + \overline{\nabla^2_{\overline{z}z} F(z^0)} + \overline{\nabla^2_{\overline{z}\overline{z}} F(z^0)}\right](d), d \right\rangle > 0. \qquad (16)$$

On one hand, we have

$$\begin{aligned} \operatorname{Re} F(z^j) &= \operatorname{Re} f(z^j) - \operatorname{Re}\langle g(z^j), v^0 \rangle \le \operatorname{Re} f(z^0) \\ &= \operatorname{Re} f(z^0) - \operatorname{Re}\langle g(z^0), v^0 \rangle \\ &= \operatorname{Re} F(z^0) \quad \text{for all } j \in \mathbb{N}, \end{aligned}$$

because $g(z^j) \in S$ for all $j \in \mathbb{N}$, $v^0 \in S^*$ and $\operatorname{Re}\langle g(z^0), v^0 \rangle = 0$. On the other hand, for each $j \in \mathbb{N}$

$$\begin{aligned} &\operatorname{Re}\left\{F(z^j) - F(z^0) - t_j \left\langle \nabla_z F(z^0) + \overline{\nabla_{\overline{z}} F(z^0)}, d^j \right\rangle\right\} \\ &= \operatorname{Re}\left\{f(z^j) - \langle g(z^j), v^0 \rangle - f(z^0) + \langle g(z^0), v^0 \rangle\right\} \\ &-t_j \operatorname{Re}\left\langle d^j, \overline{\nabla_z f(z^0)} - \left[\nabla_z g(z^0)\right]^{\mathrm{H}}(v^0) + \nabla_{\overline{z}} f(z^0) - \left[\nabla_{\overline{z}} g(z^0)\right]^{\mathrm{T}}(\overline{v}^0)\right\rangle \\ &= \operatorname{Re}\left\{f(z^j) - f(z^0) - \langle g(z^j), v^0 \rangle\right\} \\ &\le \operatorname{Re}\{f(z^j) - f(z^0)\} < 0. \end{aligned}$$

Hence, for each $j \in \mathbb{N}$

$$\operatorname{Re} \frac{1}{t_j^2} \left\{F(z^0 + t_j d^j) - F(z^0) - t_j \langle \nabla_z F(z^0) + \nabla_{\overline{z}} F(z^0), d^j \rangle\right\} < 0.$$

From this, by passing to limit, we obtain

$$\operatorname{Re}\left\langle \left[\nabla^2_{zz} F(z^0) + \nabla^2_{z\overline{z}} F(z^0) + \overline{\nabla^2_{\overline{z}z} F(z^0)} + \overline{\nabla^2_{\overline{z}\overline{z}} F(z^0)}\right](d), d \right\rangle \le 0$$

which contradicts (16).

Therefore z^0 is a local solution of Problem (1). □

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