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PYTHAGOREAN-HODOGRAPH CURVES IN THE MINKOWSKI PLANE AND SURFACES OF REVOLUTION

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ABSTRACT. In this article, we define Minkowski Pythagorean-hodograph (MPH) curves in the Minkowski plane $\mathbb{R}^{1,1}$ and obtain C^1 Hermite interpolations for MPH quintics in the Minkowski plane $\mathbb{R}^{1,1}$. We also have the envelope curves of MPH curves, and make surfaces of revolution with exact rational offsets. In addition, we present an example of C^1 Hermite interpolations for MPH rational curves in $\mathbb{R}^{2,1}$ from those in $\mathbb{R}^{1,1}$ and a suitable MPH preserving mapping.

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1. Introduction

Offsets of curves and surfaces are widely used in computer aided design and numerically controlled machining. They are mathematically well defined but difficulties in dealing with them arise from the fact that offsets in general are not rational. In other words, it is not guaranteed to have rational offsets even though we start with rational curves or surfaces.

Farouki and Neff have analyzed the properties of offsets ([8], [9]). Farouki, Pham and Pottman have approximated the offsets with rational curves or surfaces ([6], [19], [20]). Finally, Farouki and Sakkalis [11] have developed a notion of Pythagorean-hodograph (PH) curves. They are planar polynomial curves $\mathbf{r}(t)$ such that their hodographs $\mathbf{r}'(t) = (x'(t), y'(t))$ satisfy the Pythagorean equation $x'(t)^2 + y'(t)^2 = \sigma(t)^2$ for some polynomials $\sigma(t)$. The offsets of PH curves are rational because the radical term $\sqrt{x'(t)^2 + y'(t)^2}$ becomes a polynomial $\sigma(t)$. PH curves can effectively be used to approximate curves, and to interpolate given data [10].

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One of the most expensive parts in dealing with offsets is trimming process. So there are amounts of articles about this topic ([1], [2], [3]). Medial axis transform (MAT) [5] can be used to generate offsets of objects. Since offsets can be regarded as envelopes of circles, MAT makes the trimming process almost trivial. When $\gamma(t) = (x(t), y(t), r(t))$ is a segment of the medial axis transform, the envelope formula contain the term $\sqrt{x'(t)^2 + y'(t)^2 - r'(t)^2}$. Motivated by this term, Moon have introduced Minkowski Pythagorean-hodograph (MPH) curves in the Minkowski space $\mathbb{R}^{2,1}$ [12]. Choi et al. have presented G^1 Hermite interpolations of the MAT of a planar domain for MPH cubics [5]. Hermite interpolations for MPH quartic have solved by Kim and Ahn [15]. Recently Kosinka and Jüttler have analyzed G^1 Hermite interpolations for MPH cubics [17]. In higher dimensional space $\mathbb{R}^{3,1}$, Cho et al. [4] define MPH curves and use these curves to parametrize canal surfaces, which has been done by Peternell and Pottmann [18].

In this article, we define MPH curves in the Minkowski plane $\mathbb{R}^{1,1}$. These curves may be considered as particular ones in the Minkowski space $\mathbb{R}^{2,1}$. But considering only curves in $\mathbb{R}^{1,1}$, we could solve C^1 Hermite interpolation problem for MPH quintics in the Minkowski plane $\mathbb{R}^{1,1}$. Here we use the one-to-one correspondence between the PH curves in $\mathbb{R}^2$ and the MPH curves in $\mathbb{R}^{1,1}$. With the advantage of the complex number system we apply the charaterization of PH curves by their complex roots in $\mathbb{R}^2$ and we make the characterization of MPH curves in $\mathbb{R}^{1,1}$. Then with this characterization, we solve C^1 Hermite interpolation problem for MPH quintics in $\mathbb{R}^{1,1}$. For an application of C^1 Hermite interpolations, we obtain the surfaces of revolution, which have exact rational offset surfaces. In addition, appling the interpolations in $\mathbb{R}^{1,1}$ and using some suitable *MPH preserving mappings*, we present an example of C^1 Hertime interpolations with MPH rational curves in $\mathbb{R}^{2,1}$.

2. Minkowski Pythagorean-hodograph curves

In [11], Farouki and Sakkalis have introduced Pythagorean-hodograph curves. A polynomial curve $\mathbf{r}(t) = (x(t), y(t))$ in $\mathbb{R}^2$ is called a *Pythagorean-hodograph (PH)* curve if there is a polynomial $\sigma(t)$ satisfying

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2.$$

PH curves in the Minkowski plane are similarly defined. The *Minkowski plane* $\mathbb{R}^{1,1}$ is the real vector space $\mathbb{R}^2$ whose inner product is defined as follows. For two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^{1,1}$, the inner product $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{R}^{1,1}}$ is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{R}^{1,1}} = a_1 b_1 - a_2 b_2.$$

Definition 1. In $\mathbb{R}^{1,1}$, a polynomial curve $\mathbf{r}(t) = (s(t), r(t))$ is a *Minkowski Pythagorean-hodograph (MPH)* curve if there is a polynomial $\sigma(t)$ satisfying

$$\langle \mathbf{r}'(t), \mathbf{r}'(t) \rangle_{\mathbf{R}^{1,1}} = r'(t)^2 - s'(t)^2 = \sigma(t)^2.$$

We have the characterization of MPH curves as a corollary to Theorem 3 in [16]:

Theorem 1. *Let* $\mathbf{r}(t) = (s(t), r(t))$ *be a polynomial curve in the Minkowski plane* $\mathbb{R}^{1,1}$. *Then* $\mathbf{r}(t)$ *is a MPH curve with* $s'(t)^2 - r'(t)^2 = \sigma(t)^2$ *for some polynomial* $\sigma(t)$ *if and only if there are polynomials* $h(t)$, $u(t)$, *and* $v(t)$ *with* $\gcd(u(t), v(t)) = 1$ *so that*

$$\begin{aligned} s'(t) &= h(t)[u(t)^2 + v(t)^2], \\ r'(t) &= h(t)[2u(t)v(t)], \\ \sigma(t) &= h(t)[u(t)^2 - v(t)^2]. \end{aligned} \tag{1}$$

Proof. Suppose that $h(t)$, $u(t)$, and $v(t)$ are polynomials with $\gcd(u(t), v(t)) = 1$, satisfying (1). Then we have $s'(t)^2 - r'(t)^2 = \sigma(t)^2$.

Let $\mathbf{r}(t) = (s(t), r(t))$ be a MPH curve in the Minkowski plane $\mathbb{R}^{1,1}$ with

$$s'(t)^2 - r'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$. Then from Theorem 3 in [16], we know that there are polynomials $h(t), u(t), v(t), a(t), b(t), \alpha(t)$ with

$$-\alpha(t)^2 = a(t)b(t), \quad \gcd(u(t), v(t)b(t)) = \gcd(v(t), u(t)a(t)) = \gcd(\alpha(t)) = 1$$

so that

$$s'(t) = h(t)[u(t)^2 a(t) - v(t)^2 b(t)], \quad r'(t) = h(t)[2u(t)v(t)\alpha(t)].$$

But since $\gcd(\alpha(t)) = 1$ and $-\alpha(t) = a(t)b(t)$, we may write $\alpha(t) \equiv 1$, $a(t) \equiv 1$, and $b(t) \equiv -1$. Therefore we obtain (1). □

Suppose that $\mathbf{r}(t) = (s(t), r(t))$ is a MPH curve, which satisfy

$$s'(t)^2 - r'(t)^2 = \sigma(t)^2 \tag{2}$$

for some polynomial $\sigma(t)$. Then the polynomial curve $\mathbf{s}(t) = (A(t), r(t))$ is a PH curve in $\mathbb{R}^2$ where $A(t) = \int_0^t \sigma(\xi)\, d\xi$. Conversely if $\mathbf{s}(t) = (A(t), r(t))$ is a PH curve, which satisfy (2), then $\mathbf{r}(t) = (s(t), r(t))$ be a MPH curve. Therefore from Theorem 1 in [14] we also have the characterization of MPH curves:

Theorem 2. *Let* $\mathbf{r}(t) = (s(t), r(t))$ *be a polynomial curve in the Minkowski plane* $\mathbb{R}^{1,1}$. *Then* $\mathbf{r}(t)$ *is a MPH curve with* $s'(t)^2 - r'(t)^2 = \sigma(t)^2$ *for some polynomial* $\sigma(t)$ *if and only if there exists a polynomial* $p(t)$ *of real variable* t *with complex coefficients, whose roots consist of only real numbers or pairs of complex numbers which are equal up to conjugate, so that*

$$s'(t) = |p(t)|, \quad r'(t) = \mathrm{Im}(p(t)), \quad \sigma(t) = \mathrm{Re}(p(t)). \tag{3}$$

In ([5], [12]), Choi et al. have studied envelope curves of 1-parameter family of circles. With the spine curve $\mathbf{s}(t) = (s_1(t), s_2(t))$ and the radius information $r(t)$, one can write the envelope curve $(x(t), y(t))$ where

$$x(t) = s_1(t) + r(t)\frac{-r'(t)s_1'(t) \mp \sqrt{s_1'(t)^2 + s_2'(t)^2 - r'(t)^2}\, s_2'(t)}{s_1'(t)^2 + s_2'(t)^2},$$

$$y(t) = s_2(t) + r(t)\frac{-r'(t)s_2'(t) \pm \sqrt{s_1'(t)^2 + s_2'(t)^2 - r'(t)^2}\, s_1'(t)}{s_1'(t)^2 + s_2'(t)^2}.$$

Therefore a polynomial curve $\mathbf{r}(t) = (s_1(t), s_2(t), r(t))$ in the Minkowski space $\mathbb{R}^{2,1}$, which satisfies $s_1'(t)^2 + s_2'(t)^2 - r(t)^2 = \sigma(t)^2$ for some polynomial $\sigma(t)$, produces a trimmed envelope curve. Here if $s_2(t) \equiv 0$, then the curve $\mathbf{r}(t)$ become a MPH curve in $\mathbb{R}^{1,1}$ and produce the envelope curve as

$$x(t) = s(t) - r(t) \cdot \frac{r'(t)}{s'(t)}, \quad y(t) = \pm\, r(t) \cdot \frac{\sqrt{s'(t)^2 - r'(t)^2}}{s'(t)}. \tag{4}$$

Here we write $s(t)$ for $s_1(t)$. The δ-offset curve can be written as

$$x_\delta(t) = s(t) - (r(t) + \delta) \cdot \frac{r'(t)}{s'(t)}, \quad y_\delta(t) = \pm\,(r(t) + \delta) \cdot \frac{\sqrt{s'(t)^2 - r'(t)^2}}{s'(t)}. \tag{5}$$

The formula (4) may come directly from

$$(x(t) - s(t))^2 + y(t)^2 = r(t)^2, \quad (x(t) - s(t)) \cdot s'(t) = -r(t) \cdot r'(t).$$

From the envelope curve $\mathbf{e}(t) = (x(t), y(t))$ of an MPH curve $\mathbf{r}(t) = (s(t), r(t))$ in the Minkowski Plane $\mathbb{R}^{1,1}$, we obtain the surface of revolution

$$\mathbf{S}(t,\theta) = (X(t,\theta), Y(t,\theta), Z(t,\theta)),$$

which is given by

$$X(t,\theta) = x(t), \quad Y(t,\theta) = y(t) \cdot \cos\theta, \quad Z(t,\theta) = y(t) \cdot \sin\theta.$$

The δ-offset $\mathbf{S}_\delta(t,\theta) = (X_\delta(t,\theta), Y_\delta(t,\theta), Z_\delta(t,\theta))$ of $\mathbf{S}(t,\theta)$ can be written by

$$X_\delta(t,\theta) = x_\delta(t), \quad Y_\delta(t,\theta) = y_\delta(t) \cdot \cos\theta, \quad Z_\delta(t,\theta) = y_\delta(t) \cdot \sin\theta.$$

For example, let

$$u(t) = t^2 + t + 1, \quad v(t) = t^2 - t, \quad h(t) = 1.$$

Then we have the MPH curve $\mathbf{r}(t) = (s(t), r(t))$ given by

$$s'(t) = u(t)^2 + v(t)^2 = 2t^4 + 4t^3 + 2t + 1, \quad r'(t) = 2t^4 + -2t.$$

We also have the envelope curve $(x(t), y(t))$ and its δ-offset $(x_\delta(t), y_\delta(t))$ from the formula (4) and (5), and the surface of revolution $\mathbf{S}(t,\theta)$ and its δ-offset $\mathbf{S}_\delta(t,\theta)$. The one on the left in Figure 1 shows the circles of radius $r(0)$ and $r(1)$, centered $(s(0), 0)$ and $(s(1), 0)$, respectively, and the envelope curve and its 1-offset curve. The other in Fig. 1 shows the surfaces of revolution of the envelope curve and its 1-offset.

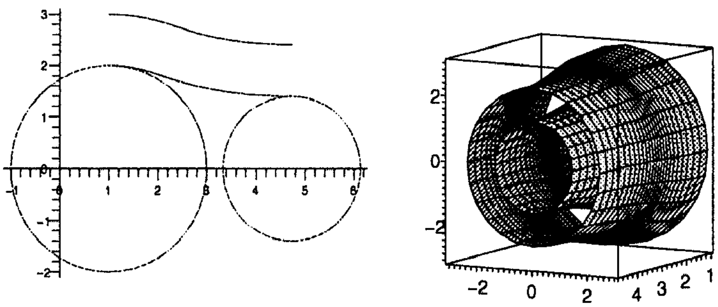

FIGURE 1. The surface of revolution and its 1-offset surface

3. C^1 Hermite interpolations in the Minkowski plane

In this section, we will obtain C^1 Hermite interpolations for Pythagorean-hodograph quintics in the Minkowski plane. In the Euclidean plane, Farouki and Neff have produced C^1 Hermite interpolations [10]. In [13], Moon et al. have analyzed the Hermite interpolations and have given a way of selecting the best among the four interpolations.

We use the complex representation for plane curves, which has been introduced by Farouki [7]: we identify a polynomial curve $\mathbf{r}(t) = (s(t), r(t))$ with a curve $\mathbf{r}(t)^* = s(t) + \sqrt{-1}r(t)$ in the complex plane.

We want to find C^1 Hermite interpolations for MPH quintics $\mathbf{r}(t) = (s(t), r(t))$. That is, for given data $\mathbf{p}_i = (p_{i1}, p_{i2}), \mathbf{p}_f = (p_{f1}, p_{f2}), \mathbf{d}_i = (d_{i1}, d_{i2})$, and $\mathbf{d}_f = (d_{f1}, d_{f2})$, we are going to find MPH quintics $\mathbf{r}(t)$, which satisfy

$$\mathbf{r}(0) = \mathbf{p}_i, \quad \mathbf{r}(1) = \mathbf{p}_f, \quad \mathbf{r}'(0) = \mathbf{d}_i, \quad \mathbf{r}'(1) = \mathbf{d}_f. \tag{6}$$

Here by changing the initial and terminal points, we may assume that $d_{i1} > 0$, $d_{f1} > 0$, and $p_{f1} - p_{i1} > 0$.

Theorem 3. *Let* $\mathbf{p}_i = (p_{i1}, p_{i2})$, $\mathbf{p}_f = (p_{f1}, p_{f2})$, $\mathbf{d}_i = (d_{i1}, d_{i2})$ *and* $\mathbf{d}_f = (d_{f1}, d_{f2})$ *be vectors in* $\mathbb{R}^2$ *with*

$$d_{i1} > 0, \quad d_{f1} > 0, \quad p_{f1} - p_{i1} > 0,$$
$$d_{i1}^2 \geq d_{i2}^2, \quad d_{f1}^2 \geq d_{f2}^2, \quad (p_{f1} - p_{i1})^2 \geq (p_{f2} - p_{i2})^2.$$

We set

$$A = \sqrt{d_{i1}^2 - d_{i2}^2} + \sqrt{-1}d_{i2}, \quad B = \pm\sqrt{d_{f1}^2 - d_{f2}^2} + \sqrt{-1}d_{f2},$$

$$C = \pm\sqrt{\frac{B}{A}}, \qquad D = \frac{A}{8}\left(C - \frac{1}{3}\right)^2,$$
$$\alpha = p_{f1} - p_{i1} - \frac{|A|}{9} - |D|, \quad \beta = p_{f2} - p_{i2} - \operatorname{Im}\left(\frac{A}{9} + D\right),$$
$$E = \pm\sqrt{\alpha^2 - \beta^2} + \sqrt{-1}\beta.$$

If $\alpha \geq 0$ *and* $\alpha^2 - \beta^2 \geq 0$, *then* $\mathbf{r}(t) = (s(t), r(t))$ *with*

$$s(t) = p_{i1} + |K|\left(\frac{t^5}{5} - \frac{\operatorname{Re} X}{2}t^4 + \frac{|X|^2 + 2\operatorname{Re} Y}{3}t^3 - \operatorname{Re}(X\overline{Y})t^2 + |Y|^2 t\right),$$
$$r(t) = p_{i2} + \frac{\operatorname{Im} K}{5}t^5 - \frac{\operatorname{Im}(KX)}{2}t^4 + \frac{\operatorname{Im}(K(X^2 + 2Y))}{3}t^3 - \operatorname{Im}(KXY)t^2 + \operatorname{Im}(KY^2)t,$$

are MPH quintics, which satisfy (6), *where*

$$Y = \frac{1}{\left(\pm\sqrt{\frac{30E}{A}} + \frac{5}{2}(C+1)\right)}, \quad X = 1 + (1 - C)Y, \quad K = \frac{A}{Y^2}.$$

Proof. We want to find MPH quintics $\mathbf{r}(t) = (s(t), r(t))$ with $s'(t)^2 - r'(t)^2 = \sigma(t)^2$ for some polynomial $\sigma(t)$, which satisfy (6). Moreover we want that $s'(t)$ and $r'(t)$ are relatively prime. Then from Theorem 2 we have

$$s'(t) = |K(t - \lambda_1)^2(t - \lambda_2)^2|, \quad r'(t) = \operatorname{Im}(K(t - \lambda_1)^2(t - \lambda_2)^2|) \tag{7}$$

for some complex constant K, λ_1, λ_2. We set $X = \lambda_1 + \lambda_2$ and $Y = \lambda_1\lambda_2$. Then we have

$$\begin{aligned} s'(t) &= |K(t^2 - Xt + Y)^2| \\ &= |K|(t^2 - Xt + Y)(t^2 - \overline{X}t + \overline{Y}) \\ &= |K|[t^4 - (X + \overline{X})t^3 + (|X|^2 + Y + \overline{Y})t^2 - (X\overline{Y} + \overline{X}Y)t + |Y|^2], \quad (8) \\ r'(t) &= \operatorname{Im}(K(t^2 - Xt + Y)^2) \\ &= \operatorname{Im}(K[t^4 - 2Xt^3 + (X^2 + 2Y)t^2 - 2XYt + Y^2]). \end{aligned}$$

These yield

$$s(t) - s(0) = |K|\left(\frac{t^5}{5} - \frac{(X + \overline{X})t^4}{4} + \frac{(|X|^2 + Y + \overline{Y})t^3}{3} - \frac{(X\overline{Y} + \overline{X}Y)t^2}{2} + |Y|^2 t\right), \tag{9}$$
$$r(t) - r(0) = \operatorname{Im}\left[K\left(\frac{t^5}{5} - \frac{Xt^4}{2} + \frac{(X^2 + 2Y)t^3}{3} - XYt^2 + Y^2 t\right)\right].$$

From (6), (8), and (9), we have

$$\mathbf{d}_i^* = |KY^2| + \sqrt{-1}\,\mathrm{Im}(KY^2), \tag{10}$$

$$\mathbf{d}_f^* = |K(1-X+Y)^2| + \sqrt{-1}\,\mathrm{Im}(K(1-X+Y)^2), \tag{11}$$

$$\mathbf{p}_f^* - \mathbf{p}_i^* = |K|\left(\frac{1}{5} - \frac{X+\overline{X}}{4} + \frac{|X|^2+Y+\overline{Y}}{3} - \frac{X\overline{Y}+\overline{X}Y}{2} + |Y|^2\right) \tag{12}$$

$$+\sqrt{-1}\,\mathrm{Im}\left[K\left(\frac{1}{5} - \frac{X}{2} + \frac{2Y+X^2}{3} - XY + Y^2\right)\right].$$

Since $\mathbf{d}_i^* \neq 0$, from (10) we have $KY^2 \neq 0$. We set $A = KY^2$, $B = K(1-X+Y)^2$, and $C = \dfrac{1-X+Y}{Y}$. From (10) and (11), we get

$$A = \sqrt{d_{i1}^2 - d_{i2}^2} + \sqrt{-1}\,d_{i2},$$

$$B = \pm\sqrt{d_{f1}^2 - d_{f2}^2} + \sqrt{-1}\,d_{f2},$$

$$C = \pm\sqrt{\frac{A}{B}}.$$

Now we observe that

$$\left|\frac{A}{9}\right| + |D| + |E| = |K|\left(\frac{1}{5} - \frac{X+\overline{X}}{4} + \frac{|X|^2+Y+\overline{Y}}{3} - \frac{X\overline{Y}+\overline{X}Y}{2} + |Y|^2\right),$$

and

$$\mathrm{Im}\left(\frac{A}{9} + D + E\right) = \mathrm{Im}\left[K\left(\frac{1}{5} - \frac{X}{2} + \frac{2Y+X^2}{3} - XY + Y^2\right)\right],$$

where

$$D = \frac{KY^2}{8}\left(\frac{1-X+Y}{Y} - \frac{1}{3}\right)^2,$$

$$E = \frac{KY^2}{30}\left[\frac{1}{Y} - \frac{5}{2}\left(\frac{1-X+Y}{Y} + 1\right)\right]^2.$$

Therefore from (12) we have

$$\mathbf{p}_f^* - \mathbf{p}_i^* = \left(\left|\frac{A}{9}\right| + |D| + |E|\right) + \sqrt{-1}\,\mathrm{Im}\left(\frac{A}{9} + D + E\right).$$

In other words, we have $E = \pm\sqrt{\alpha^2 - \beta^2} + \sqrt{-1}\beta$, where

$$\alpha = p_{1x} - p_{0x} - \left|\frac{A}{9}\right| - |D|, \qquad \beta = p_{1y} - p_{0y} - \mathrm{Im}\left(\frac{A}{9} + D\right).$$

Here we must have $\alpha \geq 0$ and $\alpha^2 - \beta^2 \geq 0$. Since $E = \dfrac{A}{30}\left(\dfrac{1}{Y} - \dfrac{5}{2}(C+1)\right)^2$, we have

$$Y = \frac{1}{\left(\pm\sqrt{\dfrac{30F}{A}} + \dfrac{5}{2}(C+1)\right)}.$$

Since $C = (1 - X + Y)/Y$ and $A = KY^2$, we have

$$X = 1 + (1 - C)Y, \qquad K = \frac{A}{Y^2}.$$

Therefore from (9), the proof is done. □

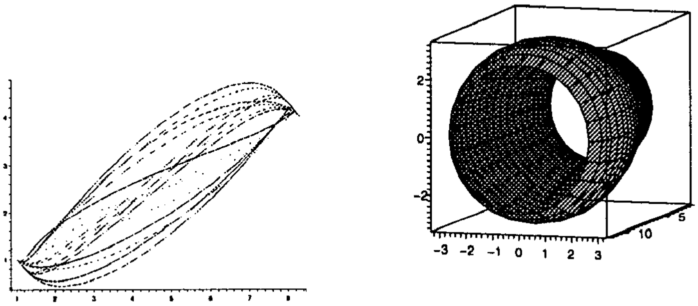

FIGURE 2. Hermite interpolations in $\mathbb{R}^{1,1}$ and the surface of revolution

Let $\mathbf{p}_i = (1,2)$, $\mathbf{p}_f = (10,3)$, $\mathbf{d}_i = (3,-1)$ and $\mathbf{d}_f = (4,-2)$ be Hermite data in $\mathbb{R}^{1,1}$. Theorem 3 gives sixteen MPH quintics as Hermite interpolants for the data (Fig. 2). Among those MPH quintics, the one, which is chosen from

$$A = \sqrt{d_{i1}^2 - d_{i2}^2} + \sqrt{-1}d_{i2}, \qquad B = \sqrt{d_{f1}^2 - d_{f2}^2} + \sqrt{-1}d_{f2},$$

$$C = \sqrt{\frac{B}{A}}, \qquad E = \sqrt{\alpha^2 - \beta^2} + \sqrt{-1}\beta,$$

$$Y = \frac{1}{\left(-\sqrt{\dfrac{30E}{A}} + \dfrac{5}{2}(C+1)\right)},$$

is the only regular one, i.e., $s'(t)^2 - r'(t)^2 \neq 0$ for all $t \in [0,1]$. The one on the right in Fig. 2 shows the surface of revolution from this regular one.

Now let $\mathbf{q}_i = (q_{i1}, q_{i2})$, $\mathbf{q}_f = (q_{f1}, q_{f2})$, b_i, b_f be G^1 data for envelope curves $\mathbf{e}(t) = (x(t), y(t))$ in (4) with $q_{i2} > 0$, $q_{f2} > 0$. We want to find envelope curves

$\mathbf{e}(t)$ satisfying

$$\mathbf{q}_i = \mathbf{e}(0), \quad \mathbf{q}_f = \mathbf{e}(1), \quad b_i = \frac{y'(0)}{x'(0)}, \quad b_f = \frac{y'(1)}{x'(1)}.$$

From the data we have G^1 data $\mathbf{p}_i = (p_{i1}, p_{i2}), \mathbf{p}_f = (p_{f1}, p_{f2}), d_i, d_f$ for MPH curves $\mathbf{r}(t) = (s(t), r(t))$ in $\mathbb{R}^{1,1}$ satisfying (4). From the facts:

(a) $(p_{i1}, 0)$ is the intersection point of x-axis and the line $(x - q_{i1}) + b_i(y - q_{i2}) = 0$;

(b) p_{i2} is the distance between $\mathbf{q}_i$ and $(p_{i1}, 0)$,

and from (4), we obtain

$$p_{i1} = q_{i1} + q_{i2}b_i, \quad p_{i2} = q_{i2}\sqrt{1+b_i^2}, \quad d_i = \frac{b_i}{\sqrt{1+b_i^2}}$$

and

$$p_{f1} = q_{f1} + q_{f2}b_f, \quad p_{f2} = q_{f2}\sqrt{1+b_f^2}, \quad d_f = \frac{b_f}{\sqrt{1+b_f^2}}.$$

For example, let $\mathbf{q}_i = (1,1)$, $\mathbf{q}_f = (10,5)$, and $b_i = 0 = b_f$. Then we have $\mathbf{p}_i = (1,1)$, $\mathbf{p}_f = (10,5)$, and $d_i = 0 = d_f$. With these data we apply Theorem 3 to solve Hermite interpolations for MPH quintics in the Minkowski plane and obtain the corresponding envelope curves and their offsets. We have the envelope curve $\mathbf{e}(t) = (x(t), y(t))$ and its 1-offset with $\mathbf{d}_i = (8,0)$ and $\mathbf{d}_f = (4,0)$ in Fig. 3.

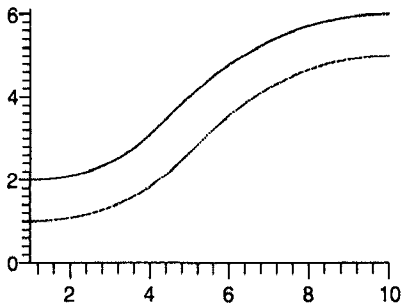

FIGURE 3. Envelope curve and its 1-offset

4. An application to Hermite interpolation in $\mathbb{R}^{2,1}$

In this section, we give an example of MPH rational curves as Hermite interpolants in $\mathbb{R}^{2,1}$. Here we apply Theorem 3 and *MPH preserving mappings.*

Consider the stereographic projection $\Psi : \mathbb{R}^3\backslash\{(x,y,z) \in \mathbb{R}^3 \ : \ z = 1\} \to \mathbb{R}^2$, which is defined by

$$\Psi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Then Ψ maps a point (x,y,z) to the point, which is the intersection point of xy-plane and the line passing $(0,0,1)$ and (x,y,z).

Now consider a curve $\gamma\colon [a,b] \to \mathbb{R}^3\backslash\{(x,y,z) \in \mathbb{R}^3 \ : \ z = 1\}$ and the curve $\lambda(t) = \Psi(\gamma(t))$. We set $\gamma(t) = (x(t), y(t), z(t))$. Then we have

$$\lambda(t) = \Psi(\gamma(t)) = \left(\frac{x(t)}{1-z(t)}, \frac{y(t)}{1-z(t)}\right)$$

and

$$\lambda'(t) = \left(\frac{x'(t)(1-z(t)) + x(t)z'(t)}{(1-z(t))^2}, \frac{y'(t)(1-z(t)) + y(t)z'(t)}{(1-z(t))^2}\right).$$

Let $\Psi(x,y,z) = (U(x,y,z), V(x,y,z))$, $u(t) = U(\lambda(t))$, and $v(t) = V(\lambda(t))$. Then since

$$u'(t)^2 = \frac{(1-z(t))^2x'(t)^2 + 2x(t)(1-z(t))x'(t)z'(t) + x(t)^2z'(t)^2}{(1-z(t))^4}$$

and

$$v'(t)^2 = \frac{(1-z(t))^2y'(t)^2 + 2y(t)(1-z(t))y'(t)z'(t) + y(t)^2z'(t)^2}{(1-z(t))^4},$$

we have

$$\begin{aligned} u'(t)^2 - v'(t)^2 = {} & \frac{1}{(1-z(t))^4}[(1-z(t))^2(x'(t)^2 - y'(t)^2) \\ & + 2(1-z(t))z'(t)(x(t)x'(t) - y(t)y'(t)) + z'(t)^2(x(t)^2 - y(t)^2)]. \quad (13) \end{aligned}$$

Let S and T be the surfaces, which are defined by $\{(x,y,z) \in \mathbb{R}^3 \ : \ z^2 + x^2 - y^2 = 1, \ z \neq 1\}$ and $\mathbb{R}^{1,1}\backslash\{(u,v) \in \mathbb{R}^{1,1} \ : \ u^2 - v^2 = -1\}$, respectively. Then $\Psi\colon S \to T$ is a one-to-one correspondence with the inverse mapping

$$\Psi^{-1}(u,v) = \left(\frac{2u}{u^2 - v^2 + 1}, \frac{2v}{u^2 - v^2 + 1}, \frac{u^2 - v^2 - 1}{u^2 - v^2 + 1}\right).$$

Suppose $\gamma(t) = (x(t), y(t), z(t))$ are on S. Then we have

$$z(t)z'(t) + x(t)x'(t) - y(t)y'(t) = 0.$$

Therefor from (13), we have

$$u'(t)^2 - v'(t)^2 = \left(\frac{1}{1-z(t)}\right)^2 \left(x'(t)^2 - y'(t)^2 + z'(t)^2\right).$$

This equation implies that $\gamma(t)$ is a MPH curve if and only if $\lambda(t) = \Psi(\gamma(t))$ is a MPH curve. In this sense Ψ and Ψ^{-1} are *MPH preserving* mappings.

Let $\mathbf{P}_i = (0,0,-1)$ and $\mathbf{P}_f = (1,0,0)$ be points on S and $\mathbf{D}_i = (1,0,0)$ and $\mathbf{D}_f = (0,0,1)$ be tangent vectors to S on $\mathbf{P}_i$ and $\mathbf{P}_f$, respectively. Then we can find MPH *rational* curves on S, as Hermite interpolants for the above data. First let

$$\mathbf{p}_i = \Psi(\mathbf{P}_i) = (0,0), \mathbf{p}_f = \Psi(\mathbf{P}_f) = (1,0), \mathbf{d}_i = d\Psi|_{\mathbf{P}_i}(\mathbf{D}_i) = (1/2, 0),$$

and $\mathbf{d}_f = d\Psi|_{\mathbf{P}_f}(\mathbf{D}_f) = (1,0)$. Now apply Theorem 3 in order to obtain Hermite interpolants $\gamma(t)$ for the data $\mathbf{p}_i$, $\mathbf{p}_f$, $\mathbf{d}_i$, and $\mathbf{d}_f$. Then we have Hermite interpolants $\gamma(t) = \Psi^{-1}(\lambda(t))$ on the surface S for the data $\mathbf{P}_i$, $\mathbf{P}_f$, $\mathbf{D}_i$, and $\mathbf{D}_f$. See Fig. 4.

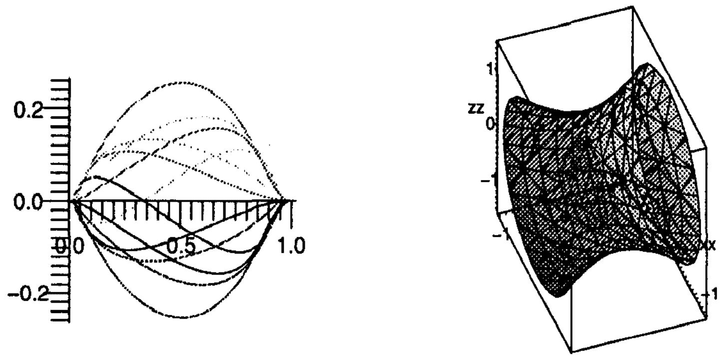

FIGURE 4. Hermite interpolations in $\mathbb{R}^{1,1}$ and on S

Remark 1. Hermite interpolants $\gamma(t) = \Psi(\gamma(t))$ are MPH *rational* curves on S. It means that these curves are sometimes unbounded and not connected. This property comes from the singularity of Ψ^{-1}. Therefore in order to obtain suitable Hermite interpolants, we must further explore general MPH preserving mappings. These subject are our current research topic.

5. Conclusion

In this paper we introduce MPH curves in the Minkowski plane $\mathbb{R}^{1,1}$. Using the one-to-one correspondence between the PH curves in $\mathbb{R}^2$ and the MPH curves in $\mathbb{R}^{1,1}$, and the complex representation for plane curves, we solve C^1 Hermite interpolation problem for MPH quintics. These Hermite interpolations induce an important application in Computer Graphics and Industry; the surfaces of revolution with exact rational offset surfaces. In addition, using some suitable MPH preserving mappings, we show an example where a C^1 Hermite interpolation problem for MPH rational curves in $\mathbb{R}^{2,1}$ is to be reduced to a C^1 Hermite interpolation problem for those in $\mathbb{R}^{1,1}$. This implies that the theory for MPH curves in $\mathbb{R}^{1,1}$ might be an essential basis for the theory for those in $\mathbb{R}^{2,1}$.

Finally, we introduce some research themes for further studies. Related to the results in this paper, there still remain several problems about the generalization and theoretical completion of the reducing method using MPH preserving mappings. Moreover, studies for the general theories for MPH preserving mappings and their applications are also in need. Now, we are tackling one of them.

REFERENCES

1. S. Aomura and T. Uehara, *Self-intersection of an offset surface*, Comput. Aided Design **22** (1990), 417–421.
2. C. Asteasu, *Intersection of arbitrary surfaces*, Comput. Aided Design **20** (1988), 533–538.
3. C. L. Bajaj, C. M. Hoffmann, R. E. Lynch and J. E. H. Hopcroft, *Tracing surface intersections*, Comput. Aided Geom. Design **5** (1988), 285–307.
4. H. C. Cho, H. I. Choi, S.-H. Kwon, D. S. Lee and N.-S. Wee, *Clifford algebra, Lorentzian geometry, and rational parametrization of canal surfaces*, Comput. Aided Geom. Design **21** (2004), 327–339.
5. H. I. Choi, C. Y. Han, H. P. Moon, K. H. Roh and N.-S. Wee, *Medial axis transform and offset curves by Minkowski Pythagorean hodograph curves*, Comput. Aided Design **31** (1999), 59–72.
6. R. T. Farouki, *The approximation of non-degenerate offset surfaces*, Comput. Aided Geom. Design **3** (1986), 15–43.
7. R. T. Farouki, *The conformal map $z \to z^2$ of the hodograph plane*, Comput. Aided Geom. Design **11** (1994), 363–390.
8. R. T. Farouki and C. A. Neff, *Algebraic properties of plane offset curves*, Comput. Aided Geom. Design **7** (1990), 101–127.
9. R. T. Farouki and C. A. Neff, *Analytic properties of plane offset curves*, Comput. Aided Geom. Design **7** (1990), 83–99.
10. R. T. Farouki and C. A. Neff, *Hermite interpolation by Pythagorean hodograph quintics*, Math. Comput. **64** (1995), 1589–1609.
11. R. T. Farouki and T. Sakkalis, *Pythagorean hodographs*, IBM Journal of Research and Development **34** (1990), 736–752.
12. H. P. Moon, *Minkowski Pythagorean hodographs*, Comput. Aided Geom. Design **16** (1999), 739–753.
13. H. P. Moon, R. T. Farouki and H. I. Choi, *Construction and shape analysis of PH quintic Hermite interpolants*, Comput. Aided Geom. Design **18** (2001), 93–115.
14. G.-I. Kim, *Algebraic characterization of generic strongly semi-regular rational PH plane curves*, J. Appl. Math. & Computing **19** (2005), 241–251.
15. G.-I. Kim and M.-H. Ahn, *C^1 Hermite interpolation using MPH quartic*, Comput. Aided Geom. Design **20** (2003), 469–492.
16. G.-I. Kim and S. Lee, *Higher dimensional Minkowski Pythagorean hodograph curves*, J. Appl. Math. & Computing **14** (2004), 405–413.
17. J. Kosinka and B. Jüttler, *G^1 Hermite interpolation by Minkowski Pythagorean hodograph cubics*, Comput. Aided Geom. Design **23** (2006), 401–418.
18. H. Pottman, *Rational curves and surfaces with rational offsets*, Comput. Aided Geom. Design **12** (1995), 175–192.
19. B. Pham, *Offset approximation of uniform B-splines*, Comput. Aided Design **20** (1988), 471–474.
20. H. Pottman, *Rational curves and surfaces with rational offsets*, Comput. Aided Geom. Design **12** (1995), 175–192.

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