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MACWILLIAMS IDENTITIES OVER $M_{n\times s}(Z_4)$ WITH RESPECT TO THE RT METRIC

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ABSTRACT. There has been a recent growth of interest in codes with respect to a newly defined non-Hamming metric grown as the Rosenbloom-Tsfasman metric (RT, or ρ, in short). In this paper, the definitions of the Lee complete ρ weight enumerator and the exact complete ρ weight enumerator of a code over $M_{n\times s}(Z_4)$ are given, and the MacWilliams identities with respect to this RT metric for the two weight enumerators of a linear code over $M_{n\times s}(Z_4)$ are proven too. At last, we also prove that the MacWilliams identities for the Lee and exact complete ρ weight enumerators of a linear code over $M_{n\times s}(Z_4)$ are the generalizations of the MacWilliams identities for the Lee and complete weight enumerators of the corresponding code over Z_4.

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1. Introduction

The weight distribution of codes, in a classical coding theory, is an important investigation field. Whether in fields or in rings, using the weight distribution of linear codes to explore the weight distribution of their dual codes is very significant. In [5], the weight distribution of codes over fields is elaborated systematically. Also later, codes over Z_4 and their weight distributions are investigated in [1, 3, 4, 12]. Recently, there has been a growth of interest in codes with respect to a newly defined non-Hamming metric grown as the Rosenbloom-Tsfasman metric (RT, or ρ, in short) in [8]. The structure of linear codes over $F_q[u]/(u^s)$ and Galois rings with respect to this RT metric have been investigated in [6, 7]. About the weight distribution of codes with respect to the RT metric,

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S. T. Dougherty and others have established some results in [2, 9, 10, 11]. But we can't get efficient information from these weight enumerators of codes in the above paper, in spite of these weight enumerators have given some better weight distributions of codes.

In this paper, associating the complete ρ weight enumerator with the Lee weight and the exact weight, the definitions of the Lee complete ρ weight enumerator and the exact complete ρ weight enumerator of a code over $M_{n\times s}(Z_4)$ are given, and the MacWilliams identities for the two weight enumerators with respect to the RT metric are proven. These two weight enumerators are superior to the complete ρ weight enumerator, and they include more information of the codeword. Especially, we can determine a code from its exact complete ρ weight enumerator. Moreover, the definitions are the generalizations of the Lee weight enumerator $Lee_C(X,Y)$ and the complete weight enumerator $cwe_C(x_0,x_1,x_2,x_3)$ of a code over Z_4 defined in [12]. At last, we also prove that the MacWilliams identities for the Lee and exact complete ρ weight enumerators of a linear code over $M_{n\times s}(Z_4)$ are the generalizations of the MacWilliams identities for the Lee and complete weight enumerators of the corresponding code over Z_4. All these results, whether in the determination of the minimal distance of codes or in encoding and decoding, will play important roles in coding theory.

2. Preliminaries

Let $M_{n\times s}(Z_4)$ denote the set of all $n\times s$ matrices over Z_4. Let $p=(p_0,p_1,\cdots,p_{s-1})$. Then, the RT (or ρ) weight of p is defined by

$$w_N(p)=\begin{cases}\max\{i|p_i\neq 0\}+1, & p\neq 0\\ 0, & p=0\end{cases}.$$

Let $\rho(p,q)=w_N(p-q)$, where $p,q\in M_{1\times s}(Z_4)$, and this is extended to the RT (or ρ) weight of P as $w_N(P)=\sum_{i=1}^{n}w_N(P_i)$, where $P=(P_1,P_2,\cdots,P_n)^T\in M_{n\times s}(Z_4)$ and $P_i=(p_{i0},p_{i1},\cdots,p_{i,s-1})\in M_{1\times s}(Z_4), 1\le i\le n$. Let $\rho(P,Q)=w_N(P-Q)$, where $P,Q\in M_{n\times s}(Z_4)$. Note that ρ is a metric on $M_{n\times s}(Z_4)$.

For $s=1$, the ρ metric is just the usual Hamming metric.

Definition 1. A Z_4−submodule of C is called a *linear code.*

Let $C\subset M_{n\times s}(Z_4)$ be a linear code, the set $w_r(C)=|\{P\in C|w_N(P)=r\}|$, where $0\le r\le ns$, is called *the weight spectrum of* C, and the ρ *weight enumerator of* C is defined by

$$W_C(z)=\sum_{r=0}^{ns}w_r(C)z^r=\sum_{P\in C}z^{w_N(P)}.$$

Let $p=(p_0,p_1,\cdots,p_{s-1})$ and $q=(q_0,q_1,\cdots,q_{s-1})\in M_{1\times s}(Z_4)$. Then the inner product of p and q is defined by $\langle p,q\rangle=\sum_{i=0}^{s-1}p_iq_{s-1-i}$, and this is extended to the inner product of P and Q as

$$\langle P,Q\rangle=\sum_{i=1}^{n}\langle P_i,Q_i\rangle,$$

where $P=(P_1,P_2,\cdots,P_n)^T$, $Q=(Q_1,Q_2,\cdots,Q_n)^T\in M_{n\times s}(Z_4)$ and $P_i=(p_{i0},p_{i1},\cdots,p_{i,s-1})$, $Q_i=(q_{i0},q_{i1},\cdots,q_{i,s-1})\in M_{1\times s}(Z_4), 1\le i\le n$.

For $s=1$, the inner product defined above becomes

$$\langle P,Q\rangle=\sum_{i=1}^{n}\langle P_i,Q_i\rangle=\sum_{i=1}^{n}p_{i0}q_{i0},$$

and it coincides with the usual inner product $\langle p,q\rangle=\sum_{i=0}^{n-1}p_iq_i$, where $p=(p_0,p_1,\cdots,p_{n-1})$ and $q=(q_0,q_1,\cdots,q_{n-1})\in Z_4^n$.

Definition 2. *The dual code of a linear code C over $M_{n\times s}(Z_4)$* is defined by

$$C^{\perp}=\{Q\in M_{n\times s}(Z_4)|\langle P,Q\rangle=0,\forall P\in C\}$$

and $C^{\perp}$ is also a linear code over $M_{n\times s}(Z_4)$.

For the purpose of make computations easier in the proof of the following lemmas and theorems, we define the following map. Let

$$\varphi: M_{1\times s}(Z_4)\to Z_4[x]/(x^s)$$

$$p=(p_0,p_1,\ldots,p_{s-1})\mapsto p(x)=p_0+p_1x+\ldots+p_{s-1}x^{s-1},$$

where $p=(p_0,p_1,\ldots,p_{s-1})\in M_{1\times s}(Z_4)$ and $p(x)=p_0+p_1x+\ldots+p_{s-1}x^{s-1}\in Z_4[x]/(x^s)$. The map is a Z_4- module isomorphism from a Z_4- code C to $\varphi(C)$

It can be extended naturally as follows. Let

$$\varphi: M_{n\times s}(Z_4)\to M_{n\times 1}(Z_4[x]/(x^s))$$

$$P\mapsto(p_{10}+p_{11}x+\ldots+p_{1,s-1}x^{s-1},\cdots,p_{n0}+p_{n1}x+\ldots+p_{n,s-1}x^{s-1})^T,$$

where $P=(P_1,P_2,\cdots,P_n)^T\in M_{n\times s}(Z_4)$ and $P_i=(p_{i0},p_{i1},\cdots,p_{i,s-1})$, $1\le i\le n$. The map is also denoted by φ, and it is also a Z_4-module isomorphism from a code C over $M_{n\times s}(Z_4)$ to $\varphi(C)$.

Let $p(x)=p_0+p_1x+\ldots+p_{s-1}x^{s-1}\in Z_4[x]/(x^s)$, the $l^{th}(0\le l\le s-1)$ coefficient of $p(x)$ is defined by $c_l(p(x))=p_l$. Let $p(x),q(x)\in Z_4[x]/(x^s)$, the inner product of them in terms of polynomials then becomes

$$\langle p(x),q(x)\rangle=c_{s-1}(p(x)q(x)).$$

It can be extended to matrices as follows. Let $P(x)=(P_1(x),P_2(x),\cdots,P_n(x))^T,Q(x)=(Q_1(x),Q_2(x),\cdots,Q_n(x))^T\in M_{n\times 1}(Z_4[x]/(x^s))$, where $P_i(x)=p_{i0}+p_{i1}x+\ldots+p_{i,s-1}x^{s-1}$,$Q_i(x)=q_{i0}+q_{i1}x+\ldots+q_{i,s-1}x^{s-1}\in Z_4[x]/(x^s)$, $1\le i\le n$, and the inner product of $P(x)$ and $Q(x)$ then becomes

$$\langle P(x),Q(x)\rangle=\sum_{i=1}^{n}\langle P_i(x),Q_i(x)\rangle=\sum_{i=1}^{n}c_{s-1}(P_i(x)Q_i(x)).$$

The Hamming weight of an element $a \in Z_4$ is defined by

$$w(a) = \begin{cases} 0, & a = 0 \\ 1, & a \neq 0 \end{cases}.$$

Then, the weight of p is defined by $w(p) = \sum_{i=0}^{n-1} w(p_i)$, where $p = (p_0, p_1, \cdots, p_{n-1}) \in Z_4^n$.

Definition 3. Let $p = (p_0, p_1, \cdots, p_{n-1}) \in Z_4^n$ and $Y = (y_1, y_2, \cdots, y_n)$, we define the *complete ρ weight enumerator of a Z_4−code C* by

$$W_C(Y) = \sum_{P \in C} y_1^{w(p_0)} y_2^{w(p_1)} \cdots y_n^{w(p_{n-1})}.$$

Definition 4. Let $P = (p_{ij})_{n \times s}$ and $Y_{ns} = (y_{10}, \cdots, y_{1,s-1}, \cdots, y_{n0}, \cdots, y_{n,s-1})$, where $1 \leq i \leq n, 0 \leq j \leq s-1$. We define the *complete ρ weight enumerator of a code C over $M_{n \times s}(Z_4)$* by

$$W_C(Y_{ns}) = \sum_{P \in C} y_{10}^{w(p_{10})} \cdots y_{1,s-1}^{w(p_{1,s-1})} \cdots y_{n0}^{w(p_{n0})} \cdots y_{n,s-1}^{w(p_{n,s-1})}.$$

In the above definition, if we let $n = 1, s = n$, and arrange the subscripts properly, then we easily obtain the definition 3. Note that it is a polynomial of ns variables. Further, it is possible to obtain the ρ weight enumerator of C through a proper transformation.

In the sequel of this paper, the definitions of the Lee and the exact complete ρ weight enumerator of a code C over $M_{n \times s}(Z_4)$ are given at first. Then, the MacWilliams identities for the two weight enumerators of a linear code C over $M_{n \times s}(Z_4)$ are proved, and some corollaries are obtained at last.

3. The Lee complete ρ weight enumerator

Definition 5. The *Lee weight of an element $a \in Z_4$* is defined by

$$w_L(a) = \begin{cases} 0, & a = 0 \\ 1, & a = 1 \text{ or } 3 \\ 2, & a = 2 \end{cases}$$

and then the Lee weight of p is defined by $w_L(p) = \sum_{i=0}^{n-1} w_L(p_i)$, where $p = (p_0, p_1, \cdots, p_{n-1}) \in Z_4^n$.

Definition 6. Let $p = (p_0, p_1, \cdots, p_{n-1}) \in Z_4^n$ and $Y = (y_1, y_2, \cdots, y_n)$. We define the Lee complete ρ weight enumerator of a Z_4−code C by

$$Lee_C(Y) = \sum_{P \in C} y_1^{w_L(p_0)} y_2^{w_L(p_1)} \cdots y_n^{w_L(p_{n-1})}.$$

Definition 7. Let $P=(p_{ij})_{n\times s}$ and $Y_{ns}=(y_{10},\cdots,y_{1,s-1},\cdots,y_{n0},\cdots,y_{n,s-1})$, where $1\le i\le n, 0\le j\le s-1$. We define the *Lee complete ρ weight enumerator of a code C over $M_{n\times s}(Z_4)$* by

$$Lee_C(Y_{ns})=\sum_{P\in C} y_{10}^{w_L(p_{10})}\cdots y_{1,s-1}^{w_L(p_{1,s-1})}\cdots y_{n0}^{w_L(p_{n0})}\cdots y_{n,s-1}^{w_L(p_{n,s-1})}.$$

In the above definition, if we let $n=1, s=n$, and arrange the subscripts properly , then we can obtain the definition 6 easily. Note that it is a polynomial of ns variables. Further, it is possible to obtain the complete ρ weight enumerator of C through a proper transformation.

If we let $s=1$, then the ρ metric defined in this paper is just the usual Hamming metric, and the inner product of P and Q defined above becomes the usual inner product. If we arrange the subscripts properly, then we obtain a new weight enumerator

$$Lee_C^*(Y)=\sum_{P\in C} y_1^{w_L(p_0)}y_2^{w_L(p_1)}\cdots y_n^{w_L(p_{n-1})},$$

where $p=(p_0,p_1,\cdots,p_{n-1})\in Z_4^n$ and $Y=(y_1,y_2,\cdots,y_n)$, which is called the *Lee complete weight enumerator of a Z_4-code.*

Lemma 1.[12] *Let $\xi(a)=i^a$ for all $a\in Z_4$, where $i^2=-1$. Let $H\neq\{0\}$ be a subgroup of Z_4. Then*

$$\sum_{\alpha\in H}\xi(\alpha)=\begin{cases}1, & H=\{0\}\\ 0, & otherwise\end{cases}.$$

Lemma 2. *Let $C\subset M_{n\times s}(Z_4)$ be a linear code, and $P(x),Q(x)\in M_{n\times 1}$ $(Z_4[x]/(x^s))$. Then*

$$\sum_{P(x)\in C}\xi(\langle P(x),Q(x)\rangle)=\begin{cases}0, & Q(x)\notin C^\perp\\ |C|, & Q(x)\in C^\perp\end{cases}.$$

Proof. If $Q(x)\in C^\perp$, then it is clear. If $Q(x)\notin C^\perp$, then there exists $P(x)\in C$ such that $\langle P(x),Q(x)\rangle\neq 0$. Then

$$\begin{aligned}\sum_{P(x)\in C}\xi(\langle P(x),Q(x)\rangle) &= \sum_{P(x)\in C}\xi\left(\sum_{i=1}^{n}c_{s-1}(P_i(x)Q_i(x))\right)\\ &= \sum_{P(x)\in C}\prod_{i=1}^{n}\xi(c_{s-1}(P_i(x)Q_i(x)))\\ &= \sum_{P(x)\in C}\prod_{i=1}^{n}\xi\left(\sum_{j=0}^{s-1}p_{ij}q_{i,s-1-j}\right)\end{aligned}$$

$$= \sum_{P(x)\in C}\prod_{i=1}^{n}\prod_{j=0}^{s-1}\xi(p_{ij}q_{i,s-1-j})$$
$$= \prod_{i=1}^{n}\prod_{j=0}^{s-1}\sum_{p_{ij}\in Z_4}\xi(p_{ij}q_{i,s-1-j}).$$

(from lemma 1.) □

Lemma 3. *Let $\beta \in Z_4$ and i, j be fixed, and let $P_i(x) = p_{i0} + p_{i1}x + \ldots + p_{i,s-1}x^{s-1} \in Z_4[x]/(x^s)$. Then*

(1) $\sum_{\alpha\in Z_4}\xi(\beta\alpha)y_{ij}^{w_L(\alpha)} = (1+y_{ij})^{2-w_L(\beta)}(1-y_{ij})^{w_L(\beta)}$;

(2) $\sum_{\alpha\in Z_4}\xi(\langle P_i(x),\alpha x^j\rangle)y_{ij}^{w_L(\alpha)} = (1+y_{ij})^{2-w_L(p_{i,s-1-j})}(1-y_{ij})^{w_L(p_{i,s-1-j})}$.

Proof. (1) From the definitions, we have

$$\sum_{\alpha\in Z_4}\xi(\beta\alpha)y_{ij}^{w_L(\alpha)} = \begin{cases}(1+y_{ij})^2, & \beta=0\\ (1+y_{ij})(1-y_{ij}), & \beta = 1or3\\ (1-y_{ij})^2, & \beta=2\end{cases}$$
$$= (1+y_{ij})^{2-w_L(\beta)}(1-y_{ij})^{w_L(\beta)}.$$

(2) From (1), we have

$$\sum_{\alpha\in Z_4}\xi(\langle P_i(x),\alpha x^j\rangle)y_{ij}^{w_L(\alpha)}$$
$$= \sum_{\alpha\in Z_4}\xi(c_{s-1}(P_i(x)(\alpha x^j)))y_{ij}^{w_L(\alpha)}$$
$$= \sum_{\alpha\in Z_4}\xi(p_{i,s-1-j}\alpha)y_{ij}^{w_L(\alpha)}$$
$$= \begin{cases}(1+y_{ij})^2, & p_{i,s-1-j}=0\\ (1+y_{ij})(1-y_{ij}), & p_{i,s-1-j} = 1or3\\ (1-y_{ij})^2, & p_{i,s-1-j}=2\end{cases}$$
$$= (1+y_{ij})^{2-w_L(p_{i,s-1-j})}(1-y_{ij})^{w_L(p_{i,s-1-j})}.$$

□

Lemma 4. *Let $f : M_{n\times 1}(Z_4[x]/(x^s)) \to C[y_{10},\cdots,y_{1,s-1},\cdots,y_{n0},\cdots,y_{n,s-1}]$. Then*

$$\sum_{Q(x)\in C^\perp} f(Q(x)) = \frac{1}{|C|}\sum_{P(x)\in C}\hat{f}(P(x))$$

where $\hat{f}(P(x)) = \sum_{Q(x)\in M_{n\times 1}(Z_4[x]/(x^s))}\xi(\langle P(x),Q(x)\rangle)f(Q(x))$.

Proof. We take $R = Z_4$ in lemma 2.3 in [10], then the result is obtained. □

Now we obtain a MacWilliams identity for the Lee complete ρ weight enumerator of a linear code C over $M_{n\times s}(Z_4)$ as follows:

Theorem 1. *Let C be a linear code over $M_{n\times s}(Z_4)$. Then*

$$\sum_{Q(x)\in C^{\perp}} y_{10}^{w_L(q_{10})}\cdots y_{1,s-1}^{w_L(q_{1,s-1})}\cdots y_{n0}^{w_L(q_{n0})}\cdots y_{n,s-1}^{w_L(q_{n,s-1})}$$

$$=\frac{1}{|C|}\left(\prod_{i=1}^{n}\prod_{j=0}^{s-1} y_{ij}^2\right)\sum_{P(x)\in C}\prod_{k=1}^{n}\prod_{l=0}^{s-1}\left(\frac{1-y_{kl}}{1+y_{kl}}\right)^{w_L(p_{k,s-1-l})}.$$

Proof. We take $f((Q_1(x),\cdots,Q_n(x))^T)=\prod_{i=1}^{n}\prod_{j=0}^{s-1} y_{ij}^{w_L(q_{ij})}$ in lemma 4. Then

$$\begin{aligned}\hat{f}(P(x)) &= \sum_{Q(x)\in M_{n\times 1}(Z_4[x]/(x^s))}\xi(\langle P(x),Q(x)\rangle)\prod_{i=1}^{n}\prod_{j=0}^{s-1} y_{ij}^{w_L(q_{ij})}\\ &= \sum_{q_{10}\in Z_4}\xi(\langle P_1(x),q_{10}\rangle)y_{10}^{w_L(q_{10})}\cdots\\ &\quad \sum_{q_{1,s-1}\in Z_4}\xi(\langle P_1(x),q_{1,s-1}x^{s-1}\rangle)y_{1,s-1}^{w_L(q_{1,s-1})}\\ &\quad \cdots\cdots\\ &\quad \sum_{q_{n0}\in Z_4}\xi(\langle P_n(x),q_{n0}\rangle)y_{n0}^{w_L(q_{n0})}\cdots\\ &\quad \sum_{q_{n,s-1}\in Z_4}\xi(\langle P_n(x),q_{n,s-1}x^{s-1}\rangle)y_{n,s-1}^{w_L(q_{n,s-1})}.\end{aligned}$$

Applying lemma 3,

$$\begin{aligned}\hat{f}(P(x)) &= (1+y_{10})^{2-w_L(p_{1,s-1})}(1-y_{10})^{w_L(p_{1,s-1})}\cdots\\ &\quad (1+y_{1,s-1})^{2-w_L(p_{10})}(1-y_{1,s-1})^{w_L(p_{10})}\\ &\quad \cdots\cdots\\ &\quad (1+y_{n0})^{2-w_L(p_{n,s-1})}(1-y_{n0})^{w_L(p_{n,s-1})}\cdots\\ &\quad (1+y_{n,s-1})^{2-w_L(p_{n0})}(1-y_{n,s-1})^{w_L(p_{n0})}\\ &= \prod_{i=1}^{n}\prod_{j=0}^{s-1}\left((1+y_{ij})^{2-w_L(p_{i,s-1-j})}(1-y_{ij})^{w_L(p_{i,s-1-j})}\right)\\ &= \left(\prod_{i=1}^{n}\prod_{j=0}^{s-1} y_{ij}^2\right)\sum_{P(x)\in C}\prod_{i=1}^{n}\prod_{j=0}^{s-1}\left(\frac{1-y_{kl}}{1+y_{kl}}\right)^{w_L(p_{k,s-1-l})}.\end{aligned}$$

Finally by applying lemma 4, we obtain the result. □

In the above theorem, if we let $n=1, s=n$ and arrange the subscripts properly, then we can easily obtain the following corollary which is called a

MacWilliams identity for the Lee complete ρ weight enumerator of a linear code C over Z_4.

Corollary 1. *Let C be a linear code overZ_4 , $q(x) = q_0 + q_1x + \cdots + q_{n-1}x^{n-1}$ and $p(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} \in Z_4[x]/(x^n)$. Then*

$$\sum_{q(x)\in C^\perp} y_1^{w_L(q_0)} y_2^{w_L(q_1)} \cdots y_n^{w_L(q_{n-1})}$$
$$= \frac{1}{|C|}\left(\prod_{i=1}^{n} y_i^2\right) \sum_{p(x)\in C} \prod_{k=1}^{n}\left(\frac{1-y_k}{1+y_k}\right)^{w_L(p_{n-k})}.$$

If we let $s = 1$ in theorem 1 and arrange the subscripts properly, then we obtain another corollary as follows, which is called a MacWilliams identity for the Lee complete weight enumerator of a linear code over Z_4.

Corollary 2. *Let C be a linear code overZ_4, $q(x) = q_0 + q_1x + \cdots + q_{n-1}x^{n-1}$ and $p(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} \in Z_4[x]/(x^n)$. Then*

$$\sum_{q(x)\in C^\perp} y_1^{w_L(q_0)} y_2^{w_L(q_1)} \cdots y_n^{w_L(q_{n-1})}$$
$$= \frac{1}{|C|}\left(\prod_{i=1}^{n} y_i^2\right) \sum_{p(x)\in C} \prod_{k=1}^{n}\left(\frac{1-y_k}{1+y_k}\right)^{w_L(p_{k-1})}.$$

The following transformation will play an important role in the proof of the last corollary in this part. Let

$$\begin{aligned} A &= \{y_1^{w_L(a_0)} \cdots y_n^{w_L(a_{n-1})} | a = (a_0, \cdots, a_{n-1}) \in Z_4^n\}, \\ B &= \{X^{2-w_L(a_0)} Y^{w_L(a_0)} \ldots X^{2-w_L(a_{n-1})} Y^{w_L(a_{n-1})} | a = (a_0, \cdots, a_{n-1}) \in Z_4^n\} \\ &= \{X^{2n-w_L(a)} Y^{w_L(a)} | a = (a_0, \cdots, a_{n-1}) \in Z_4^n\}. \end{aligned}$$

Define map $\phi : A \to B$

$$\begin{aligned} y_1^{w_L(a_0)} \cdots y_n^{w_L(a_{n-1})} \mapsto \quad & X^{2-w_L(a_0)} Y^{w_L(a_0)} \ldots X^{2-w_L(a_{n-1})} Y^{w_L(a_{n-1})} \\ = \quad & X^{2n-w_L(a)} Y^{w_L(a)} \end{aligned}$$

and ϕ is an additive group homomorphism from A to B.

Denote the left and the right of the formula in corollary 2 by LHS and RHS respectively. Then

(1) It is clear that $\phi(LHS) = Lee_{C^\perp}(X, Y)$;

(2) Because

$$RHS = \frac{1}{|C|} \sum_{p(x)\in C} \prod_{k=1}^{n} (1+y_k)^{2-w_L(p_{k-1})} (1-y_k)^{w_L(p_{k-1})}$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}\prod_{k=1}^{n}\left(\sum_{j=0}^{3} i^{p_{k-1}j}\right) y_k^{w_L(j)}$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}\sum_{j_1,\cdots,j_n\in Z_4} i^{p_0j_1+\cdots+p_{n-1}j_n} y_1^{w_L(j_1)}\cdots y_n^{w_L(j_n)},$$

so that

$$\phi(RHS) = \frac{1}{|C|}\sum_{p(x)\in C}\sum_{j_1,\cdots,j_n\in Z_4} i^{p_0j_1+\cdots+p_{n-1}j_n} X^{2-w_L(j_1)}Y^{w_L(j_1)}\cdots X^{2-w_L(j_n)}Y^{w_L(j_n)}$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}\prod_{k=1}^{n}\left(\sum_{j=0}^{3} i^{p_{k-1}j}X^{2-w_L(j)}Y^{w_L(j)}\right)$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}\prod_{k=1}^{n}(X+Y)^{2-w_L(p_{k-1})}(X-Y)^{w_L(p_{k-1})}$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}(X+Y)^{\sum_{k=1}^{n}(2-w_L(p_{k-1}))}(X-Y)^{\sum_{k=1}^{n}w_L(p_{k-1})}$$

$$= \frac{1}{|C|}\sum_{p(x)\in C}(X+Y)^{2n-w_L(p)}(X-Y)^{w_L(p)}$$

$$= \frac{1}{|C|}Lee_C(X+Y,X-Y).$$

Then we have:

Corollary 3.[12] *Let C be a linear code over Z_4. Then*

$$Lee_{C^\perp}(X,Y) = \frac{1}{|C|}Lee_C(X+Y,X-Y).$$

4. The exact complete ρ weight enumerator

The definition of the exact complete ρ weight enumerator of a code C over $M_{n\times s}(Z_4)$ will be given at first.

We can determine a code from its exact complete ρ weight enumerator. For example, let

$$\mathbf{c} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

be a codeword of C, where C is a code over $M_{n\times s}(Z_4)$. Then, we use the polynomial $y_{10}^1 y_{11}^0 y_{12}^2 y_{20}^3 y_{21}^1 y_{22}^0 y_{30}^0 y_{31}^0 y_{32}^1 y_{40}^1 y_{41}^0 y_{42}^0 = y_{10}y_{12}^2 y_{20}^3 y_{21}y_{32}y_{40}$ to denote the codeword c.

In general, we use y_{ij}^{a} (where $a \in Z_4$) to denote that it is just the element a located at the row i and column j in a matrix of C, and use the polynomial $y_{10}^{c_{10}} \cdots y_{1,s-1}^{c_{1,s-1}} \cdots y_{n0}^{c_{n0}} \cdots y_{n,s-1}^{c_{n,s-1}}$ to denote a matrix

$$\mathbf{c} = \begin{pmatrix} c_{10} & \cdots & c_{1,s-1} \\ \cdots & \cdots & \cdots \\ c_{n0} & \cdots & c_{n,s-1} \end{pmatrix}.$$

So c is decided by the polynomial singly, and the ρ weight of c can be seen from the polynomial directly. Then we have:

Definition 8. The *exact weight of an element* $a \in Z_4$ is defined by

$$w_e(a) = a.$$

Definition 9. Let $p = (p_0, p_1, \cdots, p_{n-1}) \in Z_4^n$ and $Y = (y_1, y_2, \cdots, y_n)$. We define the *exact complete* ρ *weight enumerator of a* Z_4*−code* C by

$$E_C(Y) = \sum_{P \in C} y_1^{w_e(p_0)} y_2^{w_e(p_1)} \cdots y_n^{w_e(p_{n-1})} = \sum_{P \in C} y_1^{p_0} y_2^{p_1} \cdots y_n^{p_{n-1}}.$$

Definition 10. Let $P = (p_{ij})_{n \times s} \in M_{n \times s}(Z_4)$ and $Y_{ns} = (y_{10}, \cdots, y_{1,s-1}, \cdots, y_{n0}, \cdots, y_{n,s-1})$, where $1 \le i \le n, 0 \le j \le s-1$. We define the *exact complete* ρ *weight enumerator of a code* C *over* $M_{n \times s}(Z_4)$ by

$$\begin{aligned} E_C(Y_{ns}) &= \sum_{P \in C} y_{10}^{w_e(p_{10})} \cdots y_{1,s-1}^{w_e(p_{1,s-1})} \cdots y_{n0}^{w_e(p_{n0})} \cdots y_{n,s-1}^{w_e(p_{n,s-1})} \\ &= \sum_{P \in C} y_{10}^{p_{10}} \cdots y_{1,s-1}^{p_{1,s-1}} \cdots y_{n0}^{p_{n0}} \cdots y_{n,s-1}^{p_{n,s-1}}. \end{aligned}$$

In the above definition, if we let $n = 1, s = n$, and arrange the subscripts properly, then we can obtain the definition 9 easily. Further, it is possible to obtain the Lee complete ρ weight enumerator of C through proper transformations.

If we let $s = 1$, and arrange the subscripts properly, then we obtain a new weight enumerator

$$E_C^*(Y) = \sum_{P \in C} y_1^{p_0} y_2^{p_1} \cdots y_n^{p_{n-1}}$$

as before, where $p = (p_0, p_1, \cdots, p_{n-1}) \in Z_4^n$ and $Y = (y_1, y_2, \cdots, y_n)$, which is called the *exact weight enumerator of a* Z_4*−code.*

Lemma 5. *Let* $\beta \in Z_4$ *and* i, j *be fixed, and let* $P_i(x) = p_{i0} + p_{i1}x + \ldots + p_{i,s-1}x^{s-1} \in Z_4[x]/(x^s)$. *Then*

(1) $\sum_{\alpha \in Z_4} \xi(\beta\alpha) y_{ij}^{\alpha} = (1 + \xi(\beta) y_{ij})(1 + (\xi(\beta) y_{ij})^2)$;

(2) $\sum_{\alpha \in Z_4} \xi(\langle P_i(x), \alpha x^j \rangle) y_{ij}^{\alpha} = (1 + \xi(p_{i,s-1-j}) y_{ij})(1 + (\xi(p_{i,s-1-j}) y_{ij})^2)$.

Proof. (1) From the definitions, we have the result immediately. (2) From (1), we have

$$\begin{aligned}\sum_{\alpha\in Z_4}\xi(\langle P_i(x),\alpha x^j\rangle)y_{ij}^{\alpha} &= \sum_{\alpha\in Z_4}\xi(c_{s-1}(P_i(x)(\alpha x^j)))y_{ij}^{\alpha}\\ &= \sum_{\alpha\in Z_4}\xi(p_{i,s-1-j}\alpha)y_{ij}^{\alpha}\\ &= (1+\xi(p_{i,s-1-j})y_{ij})(1+(\xi(p_{i,s-1-j})y_{ij})^2).\end{aligned}$$

□

Now we obtain a MacWilliams identity for the exact complete ρ weight enumerator of a linear code C over $M_{n\times s}(Z_4)$ as follows:

Theorem 2. *Let C be a linear code over $M_{n\times s}(Z_4)$. Then*

$$\begin{aligned}&\sum_{Q(x)\in C^{\perp}} y_{10}^{q_{10}}\cdots y_{1,s-1}^{wq_{1,s-1}}\cdots y_{n0}^{q_{n0}}\cdots y_{n,s-1}^{wq_{n,s-1}}\\ &=\frac{1}{|C|}\sum_{P(x)\in C}\prod_{k=1}^{n}\prod_{l=0}^{s-1}\Big(1+\xi(p_{k,s-1-l})y_{kl})(1+(\xi(p_{k,s-1-l})y_{kl})^2\Big).\end{aligned}$$

Proof. We take $f((Q_1(x),\cdots,Q_n(x))^T)=\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{ij}^{q_{ij}}$ in lemma 4. Then

$$\begin{aligned}\hat{f}(P(x)) &= \sum_{Q(x)\in M_{n\times 1}(Z_4[x]/(x^s))}\xi(\langle P(x),Q(x)\rangle)\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{ij}^{q_{ij}}\\ &= \sum_{q_{10}\in Z_4}\xi(\langle P_1(x),q_{10}\rangle)y_{10}^{q_{10}}\cdots\sum_{q_{1,s-1}\in Z_4}\xi(\langle P_1(x),q_{1,s-1}x^{s-1}\rangle)y_{1,s-1}^{q_{1,s-1}}\\ &\quad\cdots\cdots\\ &\quad\sum_{q_{n0}\in Z_4}\xi(\langle P_n(x),q_{n0}\rangle)y_{n0}^{q_{n0}}\cdots\sum_{q_{n,s-1}\in Z_4}\xi(\langle P_n(x),q_{n,s-1}x^{s-1}\rangle)y_{n,s-1}^{q_{n,s-1}}.\end{aligned}$$

Applying lemma 5,

$$\begin{aligned}\hat{f}(P(x)) &= (1+\xi(p_{1,s-1})y_{10})(1+(\xi(p_{1,s-1})y_{10})^2)\cdots\\ &\quad(1+\xi(p_{10})y_{1,s-1})(1+(\xi(p_{10})y_{1,s-1})^2)\\ &\quad\cdots\cdots\\ &\quad(1+\xi(p_{n,s-1})y_{n0})(1+(\xi(p_{n,s-1})y_{n0})^2)\cdots\\ &\quad(1+\xi(p_{n0})y_{n,s-1})(1+(\xi(p_{n0})y_{n,s-1})^2)\\ &= \prod_{k=1}^{n}\prod_{l=0}^{s-1}(1+\xi(p_{k,s-1-l})y_{kl})(1+(\xi(p_{k,s-1-l})y_{kl})^2).\end{aligned}$$

Finally by applying lemma 4, we obtain the result. □

In the above theorem, if we let $n = 1, s = n$ and arrange the subscripts properly, then we can easily obtain the following corollary which is called a MacWilliams identity for the exact complete ρ weight enumerator of a linear code C over Z_4.

Corollary 4. *Let C be a linear code overZ_4, $q(x) = q_0 + q_1x + \cdots + q_{n-1}x^{n-1}$ and $p(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} \in Z_4[x]/(x^n)$. Then*

$$\sum_{q(x)\in C^{\perp}} y_1^{q_0} y_2^{q_1} \cdots y_n^{q_{n-1}} = \frac{1}{|C|} \sum_{p(x)\in C} \prod_{k=1}^{n} ((1 + \xi(p_{n-k})y_k)(1 + (\xi(p_{n-k})y_k)^2)).$$

If we let $s = 1$ in theorem 2 and arrange the subscripts, then we will obtain another corollary as follows, which is called a MacWilliams identity for the exact weight enumerator of a linear code over Z_4.

Corollary 5. *Let C be a linear code over Z_4, $q(x) = q_0 + q_1x + \cdots + q_{n-1}x^{n-1}$ and $p(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} \in Z_4[x]/(x^n)$. Then*

$$\sum_{q(x)\in C^{\perp}} y_1^{q_0} y_2^{q_1} \cdots y_n^{q_{n-1}} = \frac{1}{|C|} \sum_{p(x)\in C} \prod_{k=1}^{n} \Big((1 + \xi(p_{k-1})y_k)(1 + (\xi(p_{k-1})y_k)^2)\Big).$$

The following transformation will play an important role in the proof of the last corollary in this part. Let

$$\begin{aligned} A &= \{y_1^{a_0} \cdots y_n^{a_{n-1}} | a = (a_0, \cdots, a_{n-1}) \in Z_4^n\}, \\ B &= \{X_{a_0} \cdots X_{a_{n-1}} | a = (a_0, \cdots, a_{n-1}) \in Z_4^n\} \\ &= \{X_0^{w_0(a)} X_1^{w_1(a)} X_2^{w_2(a)} X_3^{w_3(a)} | a = (a_0, \cdots, a_{n-1}) \in Z_4^n\}. \end{aligned}$$

Define map $\phi : A \to B$

$$y_1^{a_0} \cdots y_n^{a_{n-1}} \mapsto X_{a_0} \cdots X_{a_{n-1}} = X_0^{w_0(a)} X_1^{w_1(a)} X_2^{w_2(a)} X_3^{w_3(a)},$$

and ϕ is an additive group homomorphism from A to B.

Denote the left and the right of the formula in corollary 5 by LHS and RHS respectively. Then

(1) It is clear that $\phi(LHS) = cwe_{C^{\perp}}(X, Y)$.

(2) Because

$$\begin{aligned} RHS &= \frac{1}{|C|} \sum_{p(x)\in C} \prod_{k=1}^{n} \sum_{j=0}^{3} (\xi(p_{k-1})y_k)^j \\ &= \frac{1}{|C|} \sum_{p(x)\in C} \sum_{k_1,\cdots,k_n\in Z_4} \prod_{j=1}^{n} (\xi(p_{j-1})y_j)^{k_j}, \end{aligned}$$

so that

$$\begin{aligned}
\phi(RHS) &= \frac{1}{|C|}\sum_{p(x)\in C}\sum_{k_1,\cdots,k_n\in Z_4}\prod_{j=1}^{n}\xi(p_{j-1}k_j)x_{k_j} \\
&= \frac{1}{|C|}\sum_{p(x)\in C}\sum_{k_1,\cdots,k_n\in Z_4}\xi(p_0k_1+\cdots p_{n-1}k_n)x_0^{\sum_{l=1}^{n}\delta_{0,k_l}}\cdots x_3^{\sum_{l=1}^{n}\delta_{3,k_l}} \\
&= \frac{1}{|C|}\sum_{p(x)\in C}\sum_{k_1\in Z_4}\xi(p_0k_1)x_0^{\delta_{0,k_l}}\cdots x_3^{\delta_{3,k_1}} \\
&\cdots\sum_{k_n\in Z_4}\xi(p_{n-1}k_n)x_0^{\delta_{0,k_n}}\cdots x_3^{\delta_{3,k_n}} \\
&= \frac{1}{|C|}\sum_{p(x)\in C}\left(\sum_{k=0}^{3}(\xi(p_0k)x_k)\cdots\sum_{k=0}^{3}(\xi(p_{n-1}k)x_k)\right) \\
&= \frac{1}{|C|}\sum_{p(x)\in C}\prod_{j=0}^{3}(\xi(jk)x_k)^{w_j(p)} \\
&= \frac{1}{|C|}\sum_{p(x)\in C}\left(\sum_{k=0}^{3}(\xi(0k)x_k)\right)^{w_0(p)}\cdots\left(\sum_{k=0}^{3}(\xi(3k)x_k)\right)^{w_3(p)} \\
&= \frac{1}{|C|}cwe_C\left(\sum_{k=0}^{3}\xi(0k)x_k,\cdots,\sum_{k=0}^{3}\xi(3k)x_k\right),
\end{aligned}$$

where

$$\delta_{i,j}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}.$$

Then we have:

Corollary 6.[12] *Let C be a linear code over Z_4. Then*

$$\begin{aligned}
cwe_{C^\perp}(x_0,x_1,x_2,x_3) &= \frac{1}{|C|}cwe_C(x_0+x_1+x_2+x_3, x_0+ix_1-x_2-ix_3, \\
&\quad x_0-x_1+x_2-x_3, x_0-ix_1-x_2+ix_3).
\end{aligned}$$

5. Conclusion

From this paper, we can recognize that the Lee complete ρ weight enumerator of a code over $M_{n\times s}(Z_4)$ is a generalization of the Lee complete weight enumerator and the Lee weight enumerator of a code over Z_4. Furthermore, the MacWilliams identity for the Lee complete ρ weight enumerator of a linear code over $M_{n\times s}(Z_4)$ is a generalization of the MacWilliams identitiy for the Lee complete weight enumerator and the Lee weight enumerator of a linear code over Z_4. Also, the exact complete ρ weight enumerator of a code over $M_{n\times s}(Z_4)$ is a generalization of the exact weight enumerator and the complete weight enumerator of a code over Z_4. Furthermore, the MacWilliams identity for the exact complete ρ weight enumerator of a linear code over $M_{n\times s}(Z_4)$ is also a generalization of the MacWilliams identity for the exact weight enumerator and the complete weight enumerator of a linear code overZ_4.

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