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COMMON FIXED POINTS FOR COMPATIBLE MAPPINGS OF TYPE (P) AND AN APPLICATION IN DYNAMIC PROGRAMMING

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ABSTRACT. In this paper common fixed point theorems dealing with compatible mappings of type (P) are established. As a application, the existence and uniqueness of common solution for a system of functional equations arising in dynamic programming is given. The results presented in this paper improve, generalize and unify the corresponding results in this field.

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1. Introduction

It is known that the concepts of compatible mappings and compatible mappings of type (P) are generalizations of commuting mappings and weekly commuting mappings and the concepts of compatible mappings and compatible mappings of type (P) are equivalent under some conditions. There are a few researchers including Chang [2,3], Hadžić [5], Jungck [6,7], Liu [10,11,12], Liu and Kim [18], Pathak, Cho, Kang and Lee [19] and others, who proved some common fixed point theorems concerning the compatible mappings and compatible mappings of type (P) and established the existence and uniqueness of solution and common solutions for some classes of functional equations and systems of functional equations arising in dynamic programming. For example, Pathak, Cho, Kang and Lee [19] studied the existence of common fixed point for the compatible mappings of type (P)

$$d(Ax, By) \le \varphi\Big(\max\big\{d(Sx,Tx), d(Sx,Ax), d(Ty,By), \frac{1}{2}[d(Sx,By) + d(Ty,Ax)]\big\}\Big)$$

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for all $x, y \in X$. For more details, it is referred to [1-19].

Inspired by the results in [1-19], in this paper we show that two common fixed point theorems for the compatible mappings of type (P) dealing with contractive mappings. As a application, we give the existence and uniqueness of common solution for a system of functional equations arising in dynamic programming, which improve and generalize the corresponding results in [1,2,10,15,19].

Let $\mathbb{N}$ denote the set of positive integers. Now we recall the following definitions and propositions.

Definition 1.1. [7] Let (X, d) be a metric space and $S, T : X \to X$ be mappings. S and T are called to be *compatible* if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, where $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to} Tx_n = z$ for some $z \in X$.

Definition 1.2. [19] Let (X, d) be a metric space and $S, T : X \to X$ be mappings. S and T are called to be *compatible of type* (P) if $\lim_{n\to\infty} d(SSx_n, TTx_n)$ $= 0$, where $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to} Tx_n = z$ for some $z \in X$.

Proposition 1.1. [19] *Let (X, d) be a metric space and $S, T : X \to X$ be continuous mappings. Then S and T are compatible if and only if S and T are compatible of type (P).*

Proposition 1.2. [19] *Let (X, d) be a metric space and $S, T : X \to X$ be compatible mappings of type (P) and $Sx_n, Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then*

(1) $\lim_{n\to\infty} TTx_n = Sz$ *if S is continuous at z;*

(2) $\lim_{n\to\infty} SSx_n = Tz$ *if T is continuous at z;*

(3) $STz = TSz$ *and* $Sz = Tz$ *if S and T are continuous at z.*

2. Common fixed point theorems

In this section, two common fixed point theorems dealing with compatible mappings of type (P) in metric spaces are presented. Define

$$\Phi = \{\varphi : \varphi : [0, +\infty) \to [0, +\infty) \text{ is a nondecreasing, upper semicontinuous function and } \varphi(t) < t \text{ for all } t > 0\}.$$

Theorem 2.1. *Let (X, d) be a complete metric space and $A, B, S, T : X \to X$ be mappings. Suppose that S and T are continuous mappings such that*

(a) *there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying*

$$Tx_{2n-1} = Ax_{2n-2} \quad \textit{and} \quad Sx_{2n} = Bx_{2n-1}, \quad \forall n \in \mathbb{N}; \tag{2.1}$$

(b) *the pairs* $\{A, S\}$ *and* $\{B, T\}$ *are compatible of type* (P);

(c)
$$d(Ax,By) - \min\{d(Sx,By), d(Ty,Ax), \max\{d(Sx,Ax), d(Ty,By)\}\}$$
$$\le \varphi\Big(\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By),$$
$$\frac{1}{2}[d(Sx,By)+d(Ty,Ax)], \frac{1}{2}[d(Ax,By)+d(Sx,Ty)], \tag{2.2}$$
$$\frac{1}{2}[d(Ax,By)+d(Sx,Ax)], \frac{1}{2}[d(Ax,By)+d(Ty,By)]\}\Big)$$

for all $x, y \in X$, *where* $\varphi \in \Phi$.

Then the sequence $\{y_n\}_{n\in\mathbb{N}}$ *generated by*

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \quad and \quad y_{2n} = Sx_{2n} = Bx_{2n-1}, \quad \forall n \in \mathbb{N} \tag{2.3}$$

converges to a unique point common fixed point of A, B, S *and* T *in* X.

Proof. Define $d_n = d(y_n, y_{n+1})$ for $n \in \mathbb{N}$. It follows from (2.2) that for each $n \in \mathbb{N}$

$$\begin{aligned} d_{2n+1} &= d(Ax_{2n}, Bx_{2n+1}) - \min\{d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1}, Ax_{2n}),\\ &\qquad \max\{d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1})\}\}\\ &\le \varphi\Big(\max\Big\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}),\\ &\qquad \frac{1}{2}[d(Sx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, Ax_{2n})],\\ &\qquad \frac{1}{2}[d(Ax_{2n}, Bx_{2n+1}) + d(Sx_{2n}, Tx_{2n+1})],\\ &\qquad \frac{1}{2}[d(Ax_{2n}, Bx_{2n+1}) + d(Sx_{2n}, Ax_{2n})],\\ &\qquad \frac{1}{2}[d(Ax_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, Bx_{2n+1})]\Big\}\Big)\\ &= \varphi\Big(\max\Big\{d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2}[d(y_{2n}, y_{2n+2}) + 0], \frac{1}{2}[d_{2n+1} + d_{2n}],\\ &\qquad \frac{1}{2}[d_{2n+1} + d_{2n}], \frac{1}{2}[d_{2n+1} + d_{2n+1}]\Big\}\Big)\\ &\le \varphi(\max\{d_{2n}, d_{2n+1}\}). \end{aligned} \tag{2.4}$$

It is claimed that $d_{2n+1} \le d_{2n}$ for all $n \in \mathbb{N}$. In fact, if $d_{2n+1} > d_{2n}$ for some $n \in \mathbb{N}$, then (2.4) ensures that $d_{2n+1} \le \varphi(d_{2n+1}) < d_{2n+1}$, which is impossible. Similarly, it can be shown that $d_{2n+2} \le d_{2n+1}$ for all $n \in \mathbb{N}$. Therefore, $d_{n+1} \le d_n$ for all $n \in \mathbb{N}$. Since $\{d_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, it converges to 0 as $n \to \infty$.

In order to prove $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Assume that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence.

It follows that for some $\varepsilon_0 > 0$ and each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ such that $2m(k) > 2n(k) > k$ and $d(y_{2m(k)}, y_{2n(k)}) > \varepsilon_0$. Further, let $2m(k)$ denote the smallest even integer which satisfies $2m(k) > 2n(k) > k, d(y_{2m(k)}, y_{2n(k)}) > \varepsilon_0$ and $d(y_{2m(k)-2}, y_{2n(k)}) \leq \varepsilon_0$. It follows that

$$\varepsilon_0 < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}, \quad \forall n \in \mathbb{N},$$

which implies that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon_0. \tag{2.5}$$

From (2.5) and the following inequalities:

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1}, \quad \forall n \in \mathbb{N},$$

$$|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2n(k)}, \quad \forall n \in \mathbb{N}$$

and

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1} + d_{2n(k)}, \quad \forall n \in \mathbb{N},$$

it follows that

$$\begin{aligned}\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) \\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \varepsilon_0.\end{aligned} \tag{2.6}$$

By (2.2), (2.5) and (2.6), one derives that

$$\begin{aligned}
& d(y_{2n(k)}, y_{2m(k)}) \\
& \leq d_{2n(k)} + d(Ax_{2n(k)}, Bx_{2m(k)-1}) \\
& \leq d_{2n(k)} + \varphi\Big(\max \Big\{ d(Sx_{2n(k)}, Tx_{2m(k)-1}), \\
& \qquad d(Sx_{2n(k)}, Ax_{2n(k)}), d(Tx_{2m(k)-1}, Bx_{2m(k)-1}), \\
& \qquad \frac{1}{2}\big[d(Sx_{2n(k)}, Bx_{2m(k)-1}) + d(Tx_{2m(k)-1}, Ax_{2n(k)})\big], \\
& \qquad \frac{1}{2}\big[d(Ax_{2n(k)}, Bx_{2m(k)-1}) + d(Sx_{2n(k)}, Tx_{2m(k)-1})\big], \\
& \qquad \frac{1}{2}\big[d(Ax_{2n(k)}, Bx_{2m(k)-1}) + d(Sx_{2n(k)}, Ax_{2n(k)})\big], \\
& \qquad \frac{1}{2}\big[d(Ax_{2n(k)}, Bx_{2m(k)-1}) + d(Tx_{2m(k)-1}, Bx_{2m(k)-1})\big] \Big\}\Big) \\
& \quad + \min \big\{ d(Sx_{2n(k)}, Bx_{2m(k)-1}), d(Tx_{2m(k)-1}, Ax_{2n(k)}), \\
& \qquad \max\{d(Sx_{2n(k)}, Ax_{2n(k)}), d(Tx_{2m(k)-1}, Bx_{2m(k)-1})\}\big\}
\end{aligned}$$

$$\begin{aligned}
&\le d_{2n(k)} + \varphi\Big(\max\Big\{ d(y_{2n(k)}, y_{2m(k)-1}), d_{2n(k)}, d_{2m(k)-1}, \\
&\quad \frac{1}{2}\big[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)-1}, y_{2n(k)+1})\big], \\
&\quad \frac{1}{2}\big[d(y_{2n(k)+1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2m(k)-1})\big], \\
&\quad \frac{1}{2}\big[d(y_{2n(k)+1}, y_{2m(k)}) + d_{2n(k)}\big], \\
&\quad \frac{1}{2}\big[d(y_{2n(k)+1}, y_{2m(k)}) + d_{2m(k)-1}\big] \Big\}\Big) \\
&\quad + \min\big\{ d(y_{2n(k)}, y_{2m(k)}), d(y_{2m(k)-1}, y_{2n(k)+1}), \\
&\quad \max\{d_{2n(k)}, d_{2m(k)-1}\}\big\}, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Let $k \to \infty$ in the inequalities. It is easy to gain that $\varepsilon_0 \le \varphi(\varepsilon_0) < \varepsilon_0$, a contradiction. Therefore $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence $\{y_n\}_{n\in\mathbb{N}}$ converges to a point $z \in X$ as $n \to \infty$ by completeness of X. Thus the subsequences $\{Ax_{2n-2}\}_{n\in\mathbb{N}}, \{Bx_{2n-1}\}_{n\in\mathbb{N}}, \{Sx_{2n}\}_{n\in\mathbb{N}}$ and $\{Tx_{2n-1}\}_{n\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ also converge to the point z as $n \to \infty$.

Since the pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type (P), it follows from the continuity of S and T, (2.3) and Proposition 1.2 that

$$\begin{aligned}
&Ty_{2n} \to Tz, \quad By_{2n} = BBx_{2n-1} \to Tz, \\
&Sy_{2n-1} \to Sz, \quad Ay_{2n-1} = AAx_{2n-2} \to Sz \quad \text{as } n \to \infty.
\end{aligned} \tag{2.7}$$

By (2.2) and (2.3), it is derived that

$$\begin{aligned}
&d(Ay_{2n-1}, By_{2n}) \\
&= d(Ay_{2n-1}, By_{2n}) - \min\big\{d(Sy_{2n-1}, By_{2n}), \\
&\quad d(Ty_{2n}, Ay_{2n-1}), \max\{d(Sy_{2n-1}, Ay_{2n-1}), d(Ty_{2n}, By_{2n})\}\big\} \\
&\le \varphi\Big(\max\Big\{ d(Sy_{2n-1}, Ty_{2n}), d(Sy_{2n-1}, Ay_{2n-1}), d(Ty_{2n}, By_{2n}), \\
&\quad \frac{1}{2}\big[d(Sy_{2n-1}, By_{2n}) + d(Ty_{2n}, Ay_{2n-1})\big], \\
&\quad \frac{1}{2}\big[d(Ay_{2n-1}, By_{2n}) + d(Sy_{2n-1}, Ty_{2n})\big], \\
&\quad \frac{1}{2}\big[d(Ay_{2n-1}, By_{2n}) + d(Sy_{2n-1}, Ay_{2n-1})\big], \\
&\quad \frac{1}{2}\big[d(Ay_{2n-1}, By_{2n}) + d(Ty_{2n}, By_{2n})\big] \Big\}\Big).
\end{aligned} \tag{2.8}$$

From (2.3), (2.7), (2.8) and the upper semicontinuity of φ, it follows that

$$\begin{aligned}
d(Sz, Tz) &\le \varphi\Big(\max\Big\{ d(Sz, Tz), 0, 0, \frac{1}{2}\big[d(Sz, Tz) + d(Tz, Sz)\big], \\
&\quad \frac{1}{2}\big[d(Sz, Tz) + d(Sz, Tz)\big], \frac{1}{2}\big[d(Sz, Tz) + 0\big], \frac{1}{2}\big[d(Sz, Tz) + 0\big] \Big\}\Big) \\
&\le \varphi(d(Sz, Tz)),
\end{aligned}$$

which implies that $Sz = Tz$. Similarly, it can be shown that $Sz = Bz$ and $Tz = Az$. Therefore

$$Az = Bz = Sz = Tz. \tag{2.9}$$

From (2.2) and (2.3) it follows that

$$\begin{aligned}
&d(Ax_{2n}, Bz) - \min\big\{d(Sx_{2n}, Bz), d(Tz, Ax_{2n}),\\
&\quad \max\{d(Sx_{2n}, Ax_{2n}), d(Tz, Bz)\}\big\}\\
&\le \varphi\Big(\max\Big\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz),\\
&\quad \frac{1}{2}\big[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})\big], \frac{1}{2}\big[d(Ax_{2n}, Bz) + d(Sx_{2n}, Tz)\big],\\
&\quad \frac{1}{2}\big[d(Ax_{2n}, Bz) + d(Sx_{2n}, Ax_{2n})\big], \frac{1}{2}\big[d(Ax_{2n}, Bz) + d(Tz, Bz)\big]\Big\}\Big).
\end{aligned}$$

Let $n \to \infty$. The above inequality yields that

$$\begin{aligned}
&d(z, Bz) - \min\big\{d(z, Bz), d(Tz, z), \max\{d(z, z), d(Tz, Bz)\}\big\}\\
&\le \varphi\Big(\max\Big\{d(z, Tz), d(z, z), d(Tz, Bz),\\
&\quad \frac{1}{2}\big[d(z, Bz) + d(Tz, z)\big], \frac{1}{2}\big[d(z, Bz) + d(z, Tz)\big],\\
&\quad \frac{1}{2}\big[d(z, Bz) + d(z, z)\big], \frac{1}{2}\big[d(z, Bz) + d(Tz, Bz)\big]\Big\}\Big)\\
&= \varphi\big(d(z, Bz)\big),
\end{aligned}$$

which implies that $z = Bz$. Thus $z = Az = Bz = Sz = Tz$. Suppose that $w \in X$ is another common fixed point of A, B, S and T different from z. It follows from (2.2) that

$$\begin{aligned}
&d(Az, Bw) - \min\big\{d(Sz, Bw), d(Tw, Az), \max\{d(Sz, Az), d(Tw, Bw)\}\big\}\\
&\le \varphi\Big(\max\Big\{d(Sz, Tw), d(Sz, Az), d(Tw, Bw),\\
&\quad \frac{1}{2}\big[d(Sz, Bw) + d(Tw, Az)\big], \frac{1}{2}\big[d(Az, Bw) + d(Sz, Tw)\big],\\
&\quad \frac{1}{2}\big[d(Az, Bw) + d(Sz, Az)\big], \frac{1}{2}\big[d(Az, Bw) + d(Tw, Bw)\big]\Big\}\Big)\\
&= \varphi\big(d(z, w)\big),
\end{aligned}$$

which implies that $z = w$. Therefore, z is the unique common fixed point of A, B, S and T. The proof is completed. □

Theorem 2.2. *Let (X, d) be a complete metric space and $A, B, S, T : X \to X$ be mappings. Suppose that S and T are continuous mappings satisfying* (b) *and* (c) *in Theorem* 2.1 *and* (d) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$. *Then A, B, S and T have a unique common fixed point in X.*

Proof. Let x_0 be any point in X. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, one can choose a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that $Sx_{2n} = Bx_{2n-1}$ and $Tx_{2n-1} = Ax_{2n-2}$ for each $n \in \mathbb{N}$. Thus Theorem 2.2 follows from Theorem 2.1. The proof is completed. □

Remark 2.1. Theorem 2.2 improves and generalizes Theorem 1 of Chang [2], Theorem 2.1 of Liu [10] and Theorem 3.1 of Pathak, Cho, Kang and Lee [19].

3. An application in dynamic programming

In this section, let $\mathbb{R} = (-\infty, +\infty)$, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. $B(S)$ denotes the set of all bounded real-value functions on S and $d(f, g) = \sup\{|f(x) - g(x)| : x \in S\}$. It is clear that $(B(S), d)$ is a complete metric space.

It is well know that the existence and uniqueness problems of solutions of various functional equations arising in dynamic programming are of both theoretical and practical interest. In the past 20 years or so, many authors, including Bhakta and Mitra [1], Chang [3], Kang, Guan, Liu and Shim [8], Liu [10,11,12], Liu and Ume [13], Liu, Agarwal and Kang [14], Liu and Kang [15], Liu, Ume and Kang [16], Liu, Xu, Ume and Kang [17], Liu and Kim [18] and Pathak, Cho, Kang and Lee [19] and others, by using various fixed point, common fixed point and coincidence point theorems, studied and investigated the existence or uniqueness of solutions, common solutions or coincidence solutions for several classes of functional equations and systems of functional equations arising in dynamic programming. For example, in 1984, Bhakta and Mitra [1] established an existence and uniqueness of solution for the following functional equation

$$f(x) = \sup_{y \in D} \{r(x, y) + f(c(x, y))\}, \quad \forall x \in S.$$

In 1995, Pathak, Cho, Kang and Lee [19] investigated an existence and uniqueness of common solution for the following system of functional equations

$$\begin{aligned} f_i(x) &= \sup_{y \in D} H_i(x, y, f_i(T(x, y))), \\ g_i(x) &= \sup_{y \in D} F_i(x, y, g_i(T(x, y))), \quad \forall x \in S, i \in \{1, 2, 3, 4\}. \end{aligned}$$

In 1999, Liu [10] obtained the existence and uniqueness of common solution for the following system of functional equations

$$f_i(x) = \sup_{y\in D} \{u(x,y) + H_i(x,y,f_i(T(x,y)))\}, \quad \forall x \in S, i \in \{1,2,3,4\}.$$

In 2001, Liu [12] gained the existence and uniqueness of nonnegative solution for the following functional equation

$$f(x) = \inf_{y\in D} \max \{p(x,y) + f(a(x,y))\}, \quad \forall x \in S.$$

In 2003, Liu and Ume [13] presented some sufficient conditions which ensure the existence and uniqueness of solution for the following functional equation

$$f(x) = opt_{y\in D}\{u[p(x,y)+f(a(x,y))]+v\cdot opt_{y\in D}\{q(x,y),f(b(x,y))\}\}, \quad \forall x \in S,$$

where u and v are nonnegative constants with $u+v=1$. In 2006, Liu and Kang [15] studied the following functional equation

$$\begin{aligned} f(x) = opt_{y\in D}\{&u(x,y)\max\{p(x,y),f(a(x,y))\} \\ &+ v(x,y)\min\{q(x,y),f(b(x,y))\} \\ &+ w(x,y)[r(x,y)+f(c(x,y))]\}, \quad \forall x \in S, \end{aligned}$$

where *opt* denotes sup or inf. The above works motive us to investigate the following system of functional equations arising in dynamic programming:

$$\begin{aligned} f_i(x) = opt_{y\in D}\{&u(x,y)\max\{p(x,y),H_i(x,y,f_i(a(x,y)))\} \\ &+ v(x,y)\min\{q(x,y),H_i(x,y,f_i(b(x,y)))\} \\ &+ w(x,y)[r(x,y)+H_i(x,y,f_i(c(x,y)))]\} + k(x), \\ &\forall x \in S, i \in \{1,2,3,4\}, \end{aligned} \tag{3.1}$$

where *opt* denotes sup or inf, x and y denote the state and decision vectors, respectively, $a,b,c : S\times D \to S$ denote the transformations of the processes, $f_i(x)$ denote the optimal return function with the initial state x and $H_i : S\times D\times\mathbb{R} \to \mathbb{R}$ for $i \in \{1,2,3,4\}$, $u,v,w,p,q,r : S\times D \to \mathbb{R}$ and $k \in B(S)$.

The purpose of this section is to establish the existence and uniqueness of common solution for the system of functional equations (3.1), by using Theorem 2.2.

Lemma 3.1 [13]. *Let a,b,c,d be in $\mathbb{R}$. Then*

$$|opt\{a,b\} - opt\{c,d\}| \le \max\{|a-c|, |b-d|\}.$$

Theorem 3.1. *Suppose that the following conditions hold:*

(a) $k \in B(S)$, p, q, r *and* H_i *are bounded for* $i \in \{1,2,3,4\}$, *and* u, v, w *are nonnegative and* $u(x,y) + v(x,y) + w(x,y) \le 1$ *for all* $(x,y) \in S \times D$;

(b) $|H_1(x,y,g(t)) - H_2(x,y,h(t))|$

$$\le \varphi\Big(\max \Big\{ d(A_3g, A_4h), d(A_3g, A_1g), d(A_4h, A_2h), \frac{1}{2}\big[d(A_3g, A_2h) + d(A_4h, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_4h)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_4h, A_2h)\big] \Big\}\Big) + \min \big\{ d(A_3g, A_2h), d(A_4h, A_1g), \max\{d(A_3g, A_1g), d(A_4h, A_2h)\}\big\}$$

for all $(x,y) \in S \times D, g, h \in B(S)$ *and* $t \in S$, *where* $\varphi \in \Phi$ *and the mappings* A_i *are defined as follows:*

$$A_ig(x) = opt_{y\in D}\{u(x,y)\max\{p(x,y), H_i(x,y,g(a(x,y)))\} + v(x,y)\min\{q(x,y), H_i(x,y,g(b(x,y)))\} + w(x,y)[r(x,y) + H_i(x,y,g(c(x,y)))]\} + k(x)$$

for all $(g,x) \in B(S) \times S$, $i \in \{1,2,3,4\}$;

(c) $A_1(B(S)) \subseteq A_4(B(S))$ *and* $A_2(B(S)) \subseteq A_3(B(S))$;

(d) *For any sequence* $\{h_n\}_{n\in\mathbb{N}} \subseteq B(S)$ *and* $h \in B(S)$,

$$\lim_{n\to\infty} \sup_{x\in S} |h_n(x) - h(x)| = 0 \Rightarrow \lim_{n\to\infty} \sup_{x\in S} |A_ih_n(x) - A_ih(x)| = 0, \quad i \in \{3,4\};$$

(e) *For any sequence* $\{h_n\}_{n\in\mathbb{N}} \subseteq B(S)$, *if there exists* $h \in B(S)$ *such that*

$$\lim_{n\to\infty} \sup_{x\in S} |A_ih_n(x) - h(x)| = \lim_{n\to\infty} \sup_{x\in S} |A_{i+2}h_n(x) - h(x)| = 0, \quad i \in \{1,2\},$$

then

$$\lim_{n\to\infty} \sup_{x\in S} |A_{i+2}A_{i+2}h_n(x) - A_iA_ih_n(x)| = 0, \quad i \in \{1,2\}.$$

Then the system of functional equations (3.1) *has a unique common solution in* $B(S)$.

Proof. It follows from (a)-(e) that A_1, A_2, A_3 and A_4 are self mappings of $B(S)$, A_3 and A_4 are continuous, and the pairs of mappings $\{A_i, A_{i+2}\}$ are compatible of type of (P) for $i \in \{1,2\}$. Suppose that $opt = \sup$. For any $g, h \in B(S), x \in S$ and $\varepsilon > 0$, there exist $s, t \in D$ such that

$$A_1g(x) < u(x,s)\max\{p(x,s), H_1(x,s,g(a(x,s)))\} + v(x,s)\min\{q(x,s), H_1(x,s,g(b(x,s)))\} + w(x,s)[r(x,s) + H_1(x,s,g(c(x,s)))] + k(x) + \varepsilon \tag{3.2}$$

and

$$\begin{aligned} A_2h(x) < u(x,t)\max\{p(x,t), H_2(x,t,h(a(x,t)))\} \\ + v(x,t)\min\{q(x,t), H_2(x,t,h(b(x,t)))\} \\ + w(x,t)[r(x,t) + H_2(x,t,h(c(x,t)))] + k(x) + \varepsilon. \end{aligned} \tag{3.3}$$

Obviously,

$$\begin{aligned} A_1g(x) \geq u(x,t)\max\{p(x,t), H_1(x,t,g(a(x,t)))\} \\ + v(x,t)\min\{q(x,t), H_1(x,t,g(b(x,t)))\} \\ + w(x,t)[r(x,t) + H_1(x,t,g(c(x,t)))] + k(x) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} A_2h(x) \geq u(x,s)\max\{p(x,s), H_2(x,s,h(a(x,s)))\} \\ + v(x,s)\min\{q(x,s), H_2(x,s,h(b(x,s)))\} \\ + w(x,s)[r(x,s) + H_2(x,s,h(c(x,s)))] + k(x). \end{aligned} \tag{3.5}$$

In light of (3.2), (3.5), (b) and Lemma 3.1, it is deduced that

$$\begin{aligned} & A_1g(x) - A_2h(x) \\ & < u(x,s)\big[\max\{p(x,s), H_1(x,s,g(a(x,s)))\} \\ & \quad - \max\{p(x,s), H_2(x,s,h(a(x,s)))\}\big] \\ & \quad + v(x,s)\big[\min\{q(x,s), H_1(x,s,g(b(x,s)))\} \\ & \quad - \min\{q(x,s), H_2(x,s,h(b(x,s)))\}\big] \\ & \quad + w(x,s)\big[H_1(x,s,g(c(x,s))) - H_2(x,s,h(c(x,s)))\big] + \varepsilon \\ & \leq u(x,s)\max\big\{|p(x,s) - p(x,s)|, |H_1(x,s,g(a(x,s))) \\ & \quad - H_2(x,s,h(a(x,s)))|\big\} + v(x,s)\max\big\{|q(x,s) - q(x,s)|, \\ & \quad |H_1(x,s,g(b(x,s))) - H_2(x,s,h(b(x,s)))|\big\} \\ & \quad + w(x,s)\big|H_1(x,s,g(c(x,s))) - H_2(x,s,h(c(x,s)))\big| + \varepsilon \\ & \leq \big(u(x,s) + v(x,s) + w(x,s)\big)\max\big\{\big|H_1(x,s,g(a(x,s))) \\ & \quad - H_2(x,s,h(a(x,s)))\big|, \big|H_1(x,s,g(b(x,s))) - H_2(x,s,h(b(x,s)))\big|, \\ & \quad \big|H_1(x,s,g(c(x,s))) - H_2(x,s,h(c(x,s)))\big|\big\} + \varepsilon \\ & \leq \varphi\Big(\max\Big\{d(A_3g, A_4h), d(A_3g, A_1g), d(A_4h, A_2h), \\ & \quad \frac{1}{2}\big[d(A_3g, A_2h) + d(A_4h, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_4h)\big], \\ & \quad \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_4h, A_2h)\big]\Big\}\Big) \\ & \quad + \min\big\{d(A_3g, A_2h), d(A_4h, A_1g), \max\{d(A_3g, A_1g), d(A_4h, A_2h)\}\big\} \\ & \quad + \varepsilon. \end{aligned} \tag{3.6}$$

Similarly, (3.3), (3.4), (b) and Lemma 3.1 yield that

$$
\begin{aligned}
&A_1g(x) - A_2h(x) \\
&> u(x,t)\big[\max\{p(x,t), H_2(x,t,g(a(x,t)))\} \\
&\qquad - \max\{p(x,t), H_1(x,t,h(a(x,t)))\}\big] \\
&\quad + v(x,t)\big[\min\{q(x,t), H_2(x,t,g(b(x,t)))\} \\
&\qquad - \min\{q(x,t), H_1(x,t,h(b(x,t)))\}\big] \\
&\quad + w(x,t)\big[H_2(x,t,g(c(x,t))) - H_1(x,t,h(c(x,t)))\big] - \varepsilon \\
&\geq -u(x,t)\max\big\{|p(x,t) - p(x,t)|, |H_1(x,t,g(a(x,t))) \\
&\qquad - H_2(x,t,h(a(x,t)))|\big\} - v(x,t)\max\big\{|q(x,t) - q(x,t)|, \\
&\qquad |H_1(x,t,g(b(x,t))) - H_2(x,t,h(b(x,t)))|\big\} \\
&\quad - w(x,t)\big|H_1(x,t,g(c(x,t))) - H_2(x,t,h(c(x,t)))\big| - \varepsilon \qquad (3.7) \\
&\geq -\big(u(x,t) + v(x,t) + w(x,t)\big)\max\big\{\big|H_1(x,t,g(a(x,t))) \\
&\qquad - H_2(x,t,h(a(x,t)))\big|, \big|H_1(x,t,g(b(x,t))) - H_2(x,t,h(b(x,t)))\big|, \\
&\qquad \big|H_1(x,t,g(c(x,t))) - H_2(x,t,h(c(x,t)))\big|\big\} - \varepsilon \\
&\geq -\varphi\Big(\max\Big\{d(A_3g, A_4h), d(A_3g, A_1g), d(A_4h, A_2h), \\
&\qquad \frac{1}{2}\big[d(A_3g, A_2h) + d(A_4h, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_4h)\big], \\
&\qquad \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_4h, A_2h)\big]\Big\}\Big) \\
&\quad - \min\big\{d(A_3g, A_2h), d(A_4h, A_1g), \max\{d(A_3g, A_1g), d(A_4h, A_2h)\}\big\} \\
&\quad - \varepsilon.
\end{aligned}
$$

It follows from (3.6) and (3.7) that

$$
\begin{aligned}
&d(A_1g, A_2h) \\
&= \sup_{x \in S}\big|A_1g(x) - A_2h(x)\big| \\
&\leq \varphi\Big(\max\Big\{d(A_3g, A_4h), d(A_3g, A_1g), d(A_4h, A_2h), \\
&\qquad \frac{1}{2}\big[d(A_3g, A_2h) + d(A_4h, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_4h)\big], \qquad (3.8) \\
&\qquad \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_4h, A_2h)\big]\Big\}\Big) \\
&\quad + \min\big\{d(A_3g, A_2h), d(A_4h, A_1g), \max\{d(A_3g, A_1g), d(A_4h, A_2h)\}\big\} \\
&\quad + \varepsilon.
\end{aligned}
$$

Thus (3.8) implies that

$$\begin{aligned} & d(A_1g, A_2h) - \min\big\{d(A_3g, A_2h), d(A_4h, A_1g), \\ & \qquad \max\{d(A_3g, A_1g), d(A_4h, A_2h)\}\big\} \\ & \le \varphi\Big(\max\Big\{d(A_3g, A_4h), d(A_3g, A_1g), d(A_4h, A_2h), \\ & \quad \frac{1}{2}\big[d(A_3g, A_2h) + d(A_4h, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_4h)\big], \\ & \quad \frac{1}{2}\big[d(A_1g, A_2h) + d(A_3g, A_1g)\big], \frac{1}{2}\big[d(A_1g, A_2h) + d(A_4h, A_2h)\big]\Big\}\Big). \end{aligned} \tag{3.9}$$

Suppose that *opt* = inf. Similarly, (3.9) holds also. It follows immediately from Theorem 2.2 that A_1, A_2, A_3 and A_4 have a unique common fixed point $z \in B(S)$, that is, z is a unique common solution of the functional equations (3.1). The proof is completed. □

Remak 3.1. Theorem 3.1 extends and improves Theorem 2.4 in [1], Theorem 2.1 in [15] and Theorem 5.1 in [19].

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