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WEAK LAWS OF LARGE NUMBERS FOR ARRAYS UNDER A CONDITION OF UNIFORM INTEGRABILITY

Soo Hak Sung, Supranee Lisawadi, and Andrei Volodin

ABSTRACT. For an array of dependent random variables satisfying a new notion of uniform integrability, weak laws of large numbers are obtained. Our results extend and sharpen the known results in the literature.

1. Introduction

The notion of the uniform integrability plays the central role in establishing weak laws of large numbers. In this paper we introduce the new notion of integrability and prove some weak laws of large numbers under this condition.

The classical notion of *uniform integrability* of a sequence $\{X_n, n \geq 1\}$ of integrable random variables is defined through the condition

$$\lim_{a\to\infty} \sup_{n\geq 1} E|X_n|I(|X_n| > a) = 0.$$

Landers and Rogge [8] prove that the uniform integrability condition is sufficient in order that a sequence of pairwise independent random variables verifies the weak law of large numbers.

Chandra [1] obtains the weak law of large numbers under a new condition which is weaker than uniform integrability: the condition of Cesàro uniform integrability. A sequence $\{X_n, n \geq 1\}$ of integrable random variables is said to be *Cesàro uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\geq 1} \frac{1}{k_n} \sum_{i=1}^{k_n} E|X_i|I(|X_i| > a) = 0,$$

where $\{k_n, n \geq 1\}$ is a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$.

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Ordóñez Cabrera [10], by studying the weak convergence for weighted sums of random variables, introduces the condition of uniform integrability concerning the weights, which is weaker than uniform integrability, and leads to Cesàro uniform integrability as a special case.

In the following let $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n \geq 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Let $\{k_n, n \geq 1\}$ be a sequence of positive numbers such that $k_n \to \infty$ as $n \to \infty$.

Definition 1.1. Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $\{a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ an array of constants with $\sum_{i=u_n}^{v_n} |a_{ni}| \leq C$ for all $n \in N$ and some constant $C > 0$. The array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is $\{a_{ni}\}$*-uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\geq 1} \sum_{i=u_n}^{v_n} |a_{ni}| E|X_{ni}| I(|X_{ni}| > a) = 0.$$

Under the condition of $\{a_{ni}\}$-uniform integrability, Ordóñez Cabrera [10] obtains the weak law of large numbers for weighted sums of pairwise independent random variables; the condition of pairwise independence can be even dropped, at the price of slightly strengthening the conditions on the weights.

Sung [13] introduces the concept of Cesàro type uniform integrability with exponent r.

Definition 1.2. Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $r > 0$. The array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is said to be *Cesàro type uniformly integrable with exponent* r if

$$\sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r < \infty \text{ and } \lim_{a\to\infty} \sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > a) = 0.$$

Note that the conditions of Cesàro uniform integrability and Cesàro type uniform integrability with exponent r are equivalent when $u_n = 1, v_n = k_n, n \geq 1$, and $r = 1$. Sung [13] obtains the weak law of large numbers for an array $\{X_{ni}\}$ satisfying Cesàro type uniform integrability with exponent r for some $0 < r < 2$.

Chandra and Goswami [2] introduce the concept of Cesàro α-integrability ($\alpha > 0$), and show that Cesàro α-integrability for any $\alpha > 0$ is weaker than Cesàro uniform integrability.

Definition 1.3. Let $\alpha > 0$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *Cesàro α-integrable* if

$$\sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} E|X_i| < \infty \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} E|X_i| I(|X_i| > i^\alpha) = 0.$$

Under the Cesàro α-integrability condition for some $\alpha > \frac{1}{2}$, Chandra and Goswami [2] obtain the weak law of large numbers for a sequence of pairwise independent random variables. They also prove that Cesàro α-integrability for appropriate α is also sufficient for the weak law of large numbers to hold for certain special dependent sequences of random variables.

Ordóñez Cabrera and Volodin [11] introduce the notion of h-integrability for an array of random variables concerning an array of constant weights, and prove that this concept is weaker than Cesàro uniform integrability, $\{a_{ni}\}$-uniform integrability and Cesàro α-integrability.

Definition 1.4. Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $\{a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ an array of constants with $\sum_{i=u_n}^{v_n} |a_{ni}| \leq C$ for all $n \in N$ and some constant $C > 0$. Let moreover $\{h(n),\ n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. The array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is said to be *h-integrable with respect to the array of constants* $\{a_{ni}\}$ if

$$\sup_{n\geq 1} \sum_{i=u_n}^{v_n} |a_{ni}|E|X_{ni}| < \infty \text{ and } \lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}|E|X_{ni}|I(|X_{ni}| > h(n)) = 0.$$

Under appropriate conditions on the weights, Ordóñez Cabrera and Volodin [11] prove that h-integrability concerning the weights is sufficient for the weak law of large numbers to hold for weighted sums of an array of random variables, when these random variables are subject to some special kind of rowwise dependence, and, of course, when the array of random variables is pairwise independent.

The main idea of notions of $\{a_{ni}\}$-uniform integrability introduced in Ordóñez Cabrera [10] and h-integrability with respect to the array of constants $\{a_{ni}\}$ introduced in Ordóñez Cabrera and Volodin [11] is to deal with *weighted sums* of random variables. We now introduce a new concept of integrability which deals with usual *normed sums* of random variables.

Definition 1.5. Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $r > 0$. Let moreover $\{h(n),\ n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. The array $\{X_{ni}\}$ is said to be *h-integrable with exponent r* if

$$\sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r < \infty \text{ and } \lim_{n\to\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > h(n)) = 0.$$

Note that the notion of h-integrability with exponent r is strictly weaker than the notion of Cesàro type uniform integrability with exponent r (cf. Lemma 2.1 and Remark 2.1 below).

One of the most interesting applications of all these concepts of integrability is connected with Gut's [4] general weak law of large numbers. In order to formulate this result we need the following notations.

Consider an array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Set $\mathcal{F}_{nj} = \sigma\{X_{ni}, u_n \leq i \leq j\}, u_n \leq j \leq v_n, n \geq 1$, and $\mathcal{F}_{n,u_n-1} = \{\emptyset, \Omega\}, n \geq 1$.

When $u_n = 1, v_n = k_n, n \geq 1$, weak laws of large numbers for the array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ have been established by several authors (see, Gut [4] and Hong and Oh [5]). Gut [4] proved that, for some $0 < r < 2$

$$\frac{\sum_{i=1}^{k_n}(X_{ni} - a_{ni})}{k_n^{1/r}} \to 0 \text{ in } L^r$$

if $\{|X_{ni}|^r, 1 \leq i \leq k_n, n \geq 1\}$ is an array of Cesàro uniformly integrable random variables, where $a_{ni} = 0$ if $0 < r < 1$ and $a_{ni} = E(X_{ni}|\mathcal{F}_{n,i-1})$ if $1 \leq r < 2$.

For the more general array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ of random variables, weak laws of large numbers have been established by many authors (see, Ordóñez Cabrera and Volodin [11], Sung [13], and Sung et al. [14]).

In this paper, we obtain weak laws of large numbers for the array of dependent random variables (martingale difference sequence or negatively associated random variables) satisfying the condition of h-integrability with exponent r. Our results extend and sharpen the results of Sung [13] and Ordóñez Cabrera and Volodin [11] connected with Gut's general weak law of large numbers.

2. Preliminary lemmas

The following lemma shows that the notion of h-integrability with exponent r is weaker than the notion of Cesàro type uniform integrability with exponent r.

Lemma 2.1. *If the array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ satisfies the condition of Cesàro type uniform integrability with exponent $r > 0$, then it satisfies the condition of h-integrability with exponent r.*

Proof. Note that the first condition of the Cesàro type uniform integrability with exponent r and the first condition of the h-integrability with exponent r are same. Hence it suffices to show that the second condition of Cesàro type uniform integrability with exponent r implies the second condition of h-integrability with exponent r. If $\{X_{ni}\}$ satisfies the second condition of Cesàro type uniform integrability with exponent r, then there exists $A > 0$ such that $\sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > a) < \epsilon$ if $a > A$. Since $h(m) \uparrow \infty$ as $m \uparrow \infty$, there exists M such that $h(m) > A$ if $m > M$. For $m > M$,

$$\frac{1}{k_m} \sum_{i=u_m}^{v_m} E|X_{mi}|^r I(|X_{mi}|^r > h(m)) \leq \sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > h(m)) < \epsilon.$$

Hence the second condition of h-integrability with exponent r is satisfied. □

Remark 2.1. The concept of h-integrability with exponent r is strictly weaker than the concept of Cesàro type uniform integrability with exponent r, i.e., there exists an array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ which is h-integrable with exponent r, but not Cesàro type uniform integrability with exponent r. This can be obtained by a simple modification of the example in Remark 2 of Ordóñez Cabrera and Volodin [11], which is a modification of Example 2.2 from Chandra and Goswami [2].

The following lemma is needed to prove our main results.

Lemma 2.2. *Suppose that* $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ *be an array of* h*-integrable with exponent* r *random variables for some* $r > 0, k_n \to \infty, h(n) \uparrow \infty$, *and* $h(n)/k_n \to 0$. *Then the following statements hold.*

(i) $\sum_{i=u_n}^{v_n} E|X_{ni}|^\alpha I(|X_{ni}|^r > k_n) = o(k_n^{\alpha/r})$ *if* $0 < \alpha \le r$,

(ii) $\sum_{i=u_n}^{v_n} E|X_{ni}|^\beta I(|X_{ni}|^r \le k_n) = o(k_n^{\beta/r})$ *if* $r < \beta$.

Proof. The proof is similar to that of Sung [13]. Since $h(n)/k_n \to 0$ as $n \to \infty$, there exists N such that $h(n) \le k_n$ if $n > N$. If $0 < \alpha \le r$, then for $n > N$

$$\frac{1}{k_n^{\alpha/r}} \sum_{i=u_n}^{v_n} E|X_{ni}|^\alpha I(|X_{ni}|^r > k_n) \le \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > k_n)$$
$$\le \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > h(n)),$$

hence (i) holds by the condition of h-integrability with exponent r.

Now we prove that (ii) holds. From the proof of Lemma 1 of Sung [13]

$$\frac{1}{k_n{}^{\beta/r}} \sum_{i=u_n}^{v_n} E|X_{ni}|^\beta I(|X_{ni}|^r \le k_n)$$
$$\le \frac{1}{k_n{}^{\beta/r}} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > 0)$$
$$+ \frac{1}{k_n{}^{\beta/r}} \sum_{i=u_n}^{v_n} \sum_{j=1}^{k_n-1} ((j+1)^{(\beta/r)-1} - j^{(\beta/r)-1}) E|X_{ni}|^r I(|X_{ni}|^r > j)$$
$$=: A_n + B_n.$$

For A_n we have

$$A_n \le \frac{1}{k_n{}^{(\beta/r)-1}} \cdot \sup_{n\ge1} \left\{ \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r \right\} \to 0$$

by $\beta/r > 1$ and the first condition of h-integrability with exponent r.

For B_n, the second condition of h-integrability with exponent r implies that there exists N such that $k_n^{-1} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > h(n)) < \epsilon$ if $n > N$. For $n > N$ we obtain

$$B_n$$
$$= \frac{1}{k_n{}^{\beta/r}} \sum_{i=u_n}^{v_n} \sum_{j=1}^{[h(n)]} ((j+1)^{(\beta/r)-1} - j^{(\beta/r)-1}) E|X_{ni}|^r I(|X_{ni}|^r > j)$$

$$+\frac{1}{{k_n}^{\beta/r}}\sum_{i=u_n}^{v_n}\sum_{j=[h(n)]+1}^{k_n-1}((j+1)^{(\beta/r)-1}-j^{(\beta/r)-1})E|X_{ni}|^r I(|X_{ni}|^r>j)$$

$$\leq \frac{1}{{k_n}^{\beta/r}}\sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r>1)(([h(n)]+1)^{(\beta/r)-1}-1)$$

$$+\frac{1}{{k_n}^{\beta/r}}\sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r>[h(n)]+1)({k_n}^{(\beta/r)-1}-([h(n)]+1)^{(\beta/r)-1})$$

$$\leq \left(\frac{[h(n)]+1}{k_n}\right)^{(\beta/r)-1}\cdot\sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|^r+\epsilon,$$

where $[a]$ denotes the integer part of a. Thus $\limsup_{n\to\infty} B_n \leq \epsilon$ by $h(n)/k_n \to 0$ and the first condition of h-integrability with exponent r. Since $\epsilon > 0$ is arbitrary, $B_n \to 0$ as $n \to \infty$. □

Mention that in Theorem 3.1 below, each row of the array $\{X_{ni} - a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ forms a martingale difference sequence. This is the main idea behind the proof. In order to consider the weak law of large numbers for an array of random variables satisfying other dependent conditions, we will need the following definitions, cf. Joag-Dev and Proschan [6].

Two random variables X and Y are said to be *negatively quadrant dependent* (NQD) or *lower case negatively dependent* (LCND) if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y) \text{ for all } x \text{ and } y.$$

A finite family $\{X_i, 1 \leq i \leq n\}$ is said to be *negatively associated* (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$

$$Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$$

whenever f and g are coordinatewise increasing.

An infinite family of random variables is NA if every finite subfamily is NA.

The following lemma can be easily obtained by the definition of NA. Note that, for a pair of random variables, NQD is equivalent to NA.

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of NA (pairwise NQD, resp.) random variables. Let $\{f_n, n \geq 1\}$ be a sequence of increasing functions. Then $\{f_n(X_n), n \geq 1\}$ is a sequence of NA (pairwise NQD, resp.) random variables.*

The following lemma was proved by Shao [12].

Lemma 2.4. *Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of NA random variables with mean zero and $E|X_i|^p < \infty (1 \leq i \leq n)$ for some $1 < p \leq 2$. Then*

$$E\max_{1\leq k\leq n}|\sum_{i=1}^{k}X_i|^p \leq 2^{3-p}\sum_{i=1}^{n}E|X_i|^p.$$

3. On the Gut's general weak law of large numbers

Now, we state and prove one of our main results.

Theorem 3.1. *Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of h-integrable with exponent $0 < r < 2$ random variables, $k_n \to \infty, h(n) \uparrow \infty$, and $h(n)/k_n \to 0$. Then*

$$\frac{\sum_{i=u_n}^{v_n}(X_{ni} - a_{ni})}{k_n^{1/r}} \to 0$$

in L^r and, hence, in probability as $n \to \infty$, where $a_{ni} = 0$ if $0 < r < 1$ and $a_{ni} = E(X_{ni}|\mathcal{F}_{n,i-1})$ if $1 \leq r < 2$.

The proof of Theorem 3.1 is similar to that of Theorem 1 of Sung [13] and is omitted.

The following corollary was proved by Sung [13].

Corollary 3.1. *Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of random variables satisfying the Cesàro type uniform integrability with exponent $0 < r < 2$ and $k_n \to \infty$. Then*

$$\frac{\sum_{i=u_n}^{v_n}(X_{ni} - a_{ni})}{k_n^{1/r}} \to 0$$

in L^r and, hence, in probability as $n \to \infty$, where $a_{ni} = 0$ if $0 < r < 1$ and $a_{ni} = E(X_{ni}|\mathcal{F}_{n,i-1})$ if $1 \leq r < 2$.

Proof. By Lemma 2.1, the condition of Cesàro type uniform integrability with exponent r implies the condition of h-integrability with exponent r, and so the result follows from Theorem 3.1. □

The following theorem shows that, for the case of $r = 1$, a sequence of martingale difference in Theorem 3.1 can be replaced by a sequence of pairwise NQD random variables.

Theorem 3.2. *Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of rowwise pairwise NQD h-integrable with exponent $r = 1$ random variables, i.e.,*

$$\sup_{n\geq 1} \frac{1}{k_n}\sum_{i=u_n}^{v_n} E|X_{ni}| < \infty \text{ and } \lim_{n\to\infty} \frac{1}{k_n}\sum_{i=u_n}^{v_n} E|X_{ni}|I(|X_{ni}| > h(n)) = 0$$

where $k_n \to \infty, h(n) \uparrow \infty$, and $h(n)/k_n \to 0$. Then

$$\frac{1}{k_n}\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni}) \to 0$$

in L^1 and, hence, in probability as $n \to \infty$.

Proof. Let $X'_{ni} = X_{ni}I(|X_{ni}| \le k_n) - k_nI(X_{ni} < -k_n) + k_nI(X_{ni} > k_n)$ and $X''_{ni} = X_{ni} - X'_{ni}$. Then $\{X'_{ni}, u_n \le i \le v_n, n \ge 1\}$ is an array of rowwise pairwise NQD by Lemma 2.3. We can write that

$$\begin{aligned}\frac{1}{k_n}\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni}) &= \frac{1}{k_n}\sum_{i=u_n}^{v_n}(X'_{ni} - EX'_{ni}) + \frac{1}{k_n}\sum_{i=u_n}^{v_n}(X''_{ni} - EX''_{ni}) \\ &=: A_n + B_n.\end{aligned}$$

For A_n we actually prove that $A_n \to 0$ in L^2 and hence in L^1. By Lemma 2.2 with $\alpha = r = 1$ and $\beta = 2$, we have

$$\begin{aligned}E|A_n|^2 &= \frac{1}{k_n^2}E|\sum_{i=u_n}^{v_n}(X'_{ni} - EX'_{ni})|^2 \\ &= \frac{1}{k_n^2}\sum_{i=u_n}^{v_n}E(X'_{ni} - EX'_{ni})^2 + \frac{1}{k_n^2}\sum_{i\ne j}Cov(X'_{ni}, X'_{nj}) \\ &\le \frac{1}{k_n^2}\sum_{i=u_n}^{v_n}E(X'_{ni} - EX'_{ni})^2 \\ &\le \frac{1}{k_n^2}\sum_{i=u_n}^{v_n}E|X'_{ni}|^2 \\ &= \frac{1}{k_n^2}\sum_{i=u_n}^{v_n}\left\{E|X_{ni}|^2I(|X_{ni}| \le k_n) + k_n^2P(|X_{ni}| > k_n)\right\} \\ &\le \frac{1}{k_n^2}\sum_{i=u_n}^{v_n}E|X_{ni}|^2I(|X_{ni}| \le k_n) + \frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|I(|X_{ni}| > k_n) \to 0.\end{aligned}$$

Noting that $|X''_{ni}| \le |X_{ni}|I(|X_{ni}| > k_n)$, we also have by Lemma 2.2 that

$$\begin{aligned}E|B_n| &= \frac{1}{k_n}E|\sum_{i=u_n}^{v_n}(X''_{ni} - EX''_{ni})| \\ &\le \frac{2}{k_n}\sum_{i=u_n}^{v_n}E|X''_{ni}| \\ &\le \frac{2}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|I(|X_{ni}| > k_n) \to 0.\end{aligned}$$

□

Corollary 3.2. *Suppose that $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is an array of rowwise pairwise NQD random variables. Let $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ is an array of constants. Assume that the following conditions hold.*

(i) *$\{X_{ni}\}$ is h-integrable concerning the array $\{a_{ni}\}$,*

(ii) $h(n) \sup_{u_n \le i \le v_n} |a_{ni}| \to 0$.

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0$$

in L^1 and, hence, in probability as $n \to \infty$.

Proof. Let $k_n = 1/\sup_{u_n \le i \le v_n} |a_{ni}|$. Then $k_n \to \infty$ and $h(n)/k_n \to 0$ by (ii). We can write that

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) = \sum_{i=u_n}^{v_n} a_{ni}^+(X_{ni} - EX_{ni}) - \sum_{i=u_n}^{v_n} a_{ni}^-(X_{ni} - EX_{ni})$$
$$=: A_n + B_n,$$

where $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$. Since $\{X_{ni}\}$ is an array of rowwise pairwise NQD random variables, both $\{k_n a_{ni}^+ X_{ni}\}$ and $\{k_n a_{ni}^- X_{ni}\}$ are arrays of rowwise pairwise NQD by Lemma 2.3.

Take $k_n a_{ni}^+ X_{ni}$ instead of X_{ni} in Theorem 3.2. Then we have by (i) that

$$\sup_{n \ge 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|k_n a_{ni}^+ X_{ni}| \le \sup_{n \ge 1} \sum_{i=u_n}^{v_n} |a_{ni}| E|X_{ni}| < \infty.$$

Since $k_n |a_{ni}^+| \le 1$, we also have by (i) that

$$\frac{1}{k_n} \sum_{i=u_n}^{v_n} E|k_n a_{ni}^+ X_{ni}| I(|k_n a_{ni}^+ X_{ni}| > h(n))$$
$$\le \sum_{i=u_n}^{v_n} |a_{ni}| E|X_{ni}| I(|X_{ni}| > h(n)) \to 0.$$

Thus $A_n \to 0$ in L^1 by Theorem 3.2. Similarly, we have that $B_n \to 0$ in L^1. □

Remark 3.1. Ordóñez Cabrera and Volodin [11] proved Corollary 3.2 when $\{a_{ni}\}$ is an array of non-negative constants satisfying (i) and $\sum_{i=u_n}^{v_n} a_{ni}^2 h(n)^2 \to 0$ as $n \to \infty$. Since the condition $\sum_{i=u_n}^{v_n} a_{ni}^2 h(n)^2 \to 0$ is stronger than (ii), Corollary 3.2 extends and sharpens the result of Ordóñez Cabrera and Volodin [11].

It is interesting to consider whether Theorem 3.2 can be extended to the case $1 < r < 2$. But this is not a simple problem. Even if the condition of pairwise NQD in Theorem 3.2 is replaced by a stronger condition of pairwise i.i.d., it is still not known does Theorem 3.2 hold for the case $1 < r < 2$ or not. The problem can be formulated in the following way.

Open problem. Let $\{X_n, n \ge 1\}$ be a sequence of pairwise i.i.d. random variables with $E|X_1|^r < \infty$ for some $1 < r < 2$. Define $u_n = 1, v_n = n, k_n = n$ for $n \ge 1$, and $X_{ni} = X_i$ for $1 \le i \le n$ and $n \ge 1$. Then $\{X_{ni}, u_n \le i \le v_n, n \ge$

1} is an array of rowwise pairwise NQD h-integrable with exponent r random variables, and

$$\frac{\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni})}{k_n^{1/r}} = \frac{\sum_{i=1}^{n}(X_i - EX_i)}{n^{1/r}}.$$

Can we generalize the law of large numbers to the pairwise i.i.d. random variables with respect to L^r or almost sure convergence? For the case $r = 1$ the answer is positive, cf. the celebrated paper by Etemadi [3]. But for the case $1 < r < 2$, that is, for the case of Marcinkiewicz-Zygmund law of large numbers, the answer is still unknown.

We would like to pay the interested reader attention to two recent manuscripts by Kruglov [7] (that contains a new interesting technique that may lead to a solution of the problem mentioned above) and Li, Rosalsky, and Volodin [9] (for a generalization of Etemadi's [3] result on the case of pairwise NQD random variables).

The following theorem shows that if we replace pairwise NQD by NA, Theorem 3.2 holds for $1 < r < 2$.

Theorem 3.3. *Suppose that* $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ *is an array of rowwise NA h-integrable with exponent* $1 \le r < 2$ *random variables,* $k_n \to \infty$, $h(n) \uparrow \infty$, *and* $h(n)/k_n \to 0$. *Then*

$$\frac{\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni})}{k_n^{1/r}} \to 0$$

in L^r *and, hence, in probability as* $n \to \infty$.

Proof. The proof is similar to that of Theorem 3.2. Let

$$X'_{ni} = X_{ni}I(|X_{ni}| \le k_n^{1/r}) - k_n^{1/r}I(X_{ni} < -k_n^{1/r}) + k_n^{1/r}I(X_{ni} > k_n^{1/r})$$

and $X''_{ni} = X_{ni} - X'_{ni}$. By Lemma 2.3, both $\{X'_{ni}\}$ and $\{X''_{ni}\}$ are arrays of rowwise NA random variables. Observe that

$$\begin{aligned}\frac{1}{k_n^{1/r}}\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni}) &= \frac{1}{k_n^{1/r}}\sum_{i=u_n}^{v_n}(X'_{ni} - EX'_{ni}) + \frac{1}{k_n^{1/r}}\sum_{i=u_n}^{v_n}(X''_{ni} - EX''_{ni}) \\ &=: A_n + B_n.\end{aligned}$$

By Lemma 2.2 with $\alpha = r$ and $\beta = 2$, we have

$$\begin{aligned}E|A_n|^2 \le & \frac{1}{k_n^{2/r}}\sum_{i=u_n}^{v_n} E|X_{ni}|^2 I(|X_{ni}|^r \le k_n) \\ & + \frac{1}{k_n}\sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > k_n) \to 0.\end{aligned}$$

Using Lemma 2.4, the c_r-inequality, Jensen's inequality, and Lemma 2.2 with $\alpha = r$, we obtain

$$\begin{aligned} E|B_n|^r &\le \frac{2^{3-r}}{k_n} \sum_{i=u_n}^{v_n} E|X''_{ni} - EX''_{ni}|^r \\ &\le \frac{2^2}{k_n} \sum_{i=u_n}^{v_n} E|X''_{ni}|^r + |EX''_{ni}|^r \\ &\le \frac{2^3}{k_n} \sum_{i=u_n}^{v_n} E|X''_{ni}|^r \\ &\le \frac{2^3}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > k_n) \to 0. \end{aligned}$$

□

Corollary 3.3. *Let $1 \le r < 2$. Suppose that $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is an array of NA random variables. Let $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ is an array of constants. Assume that the following conditions hold.*

(i) *$\{|X_{ni}|^r\}$ is h-integrable concerning the array $\{|a_{ni}|^r\}$,*

(ii) $h(n) \sup_{u_n \le i \le v_n} |a_{ni}| \to 0.$

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0$$

in L^r and, hence, in probability as $n \to \infty$.

Proof. Let $k_n = 1/\sup_{u_n \le i \le v_n} |a_{ni}|$. Then $k_n \to \infty$ and $h(n)/k_n \to 0$ by (ii). Since $k_n \to \infty$ as $n \to \infty$, there exists N such that $\sup_{u_n \le i \le v_n} |a_{ni}| \le 1$ if $n > N$. It follows that for $n > N$

$$k_n |a_{ni}|^r = \frac{|a_{ni}|^r}{\sup_{u_i \le i \le v_n} |a_{ni}|} \le \frac{|a_{ni}|^r}{\sup_{u_i \le i \le v_n} |a_{ni}|^r} \le 1.$$

The rest of the proof is similar to that of Corollary 3.2 and is omitted. □

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References

[1] T. K. Chandra, *Uniform integrability in the Cesàro sense and the weak law of large numbers*, Sankhyā Ser. A **51** (1989), no. 3, 309–317.

[2] T. K. Chandra and A. Goswami, *Cesàro α-integrability and laws of large numbers I*, J. Theoret. Probab. **16** (2003), no. 3, 655–669.

[3] N. Etemadi, *An elementary proof of the strong law of large numbers*, Z. Wahrsch. Verw. Gebiete **55** (1981), no. 1, 119–122.

[4] A. Gut, *The weak law of large numbers for arrays*, Statist. Probab. Lett. **14** (1992), no. 1, 49–52.
[5] D. H. Hong and K. S. Oh, *On the weak law of large numbers for arrays*, Statist. Probab. Lett. **22** (1995), no. 1, 55–57.
[6] K. Joag-Dev and F. Proschan, *Negative association of random variables, with applications*, Ann. Statist. **11** (1983), no. 1, 286–295.
[7] V. Kruglov, *Growth of sums of pairwise-independent random variables with infinite means*, Teor. Veroyatn. Primen. **51** (2006), no. 2, 382–385.
[8] D. Landers and L. Rogge, *Laws of large numbers for pairwise independent uniformly integrable random variables*, Math. Nachr. **130** (1987), 189–192.
[9] D. Li, A. Rosalsky, and A. Volodin, *On the strong law of large numbers for sequences of pairwise negative quadrant dependent random variables*, Bull. Inst. Math. Acad. Sin. (N.S.) **1** (2006), no. 2, 281–305.
[10] M. Ordóñez Cabrera, *Convergence of weighted sums of random variables and uniform integrability concerning the weights*, Collect. Math. **45** (1994), no. 2, 121–132.
[11] M. Ordóñez Cabrera and A. Volodin, *Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability*, J. Math. Anal. Appl. **305** (2005), no. 2, 644–658.
[12] Q. M. Shao, *A comparison theorem on moment inequalities between negatively associated and independent random variables*, J. Theoret. Probab. **13** (2000), no. 2, 343–356.
[13] S. H. Sung, *Weak law of large numbers for arrays of random variables*, Statist. Probab. Lett. **42** (1999), no. 3, 293–298.
[14] S. H. Sung, T. C. Hu, and A. Volodin, *On the weak laws for arrays of random variables*, Statist. Probab. Lett. **72** (2005), no. 4, 291–298.

Soo Hak Sung
Department of Applied Mathematics
Pai Chai University
Taejon 302-735, Korea
E-mail address: sungsh@pcu.ac.kr

Supranee Lisawadi
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan S4S 0A2, Canada

Andrei Volodin
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan S4S 0A2, Canada
E-mail address: volodin@math.uregina.ca