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d-ISOMETRIC LINEAR MAPPINGS IN LINEAR d-NORMED BANACH MODULES

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Abstract. We prove the Hyers–Ulam stability of linear d-isometries in linear d-normed Banach modules over a unital C^*-algebra and of linear isometries in Banach modules over a unital C^*-algebra. The main purpose of this paper is to investigate d-isometric C^*-algebra isomorphisms between linear d-normed C^*-algebras and isometric C^*-algebra isomorphisms between C^*-algebras, and d-isometric Poisson C^*-algebra isomorphisms between linear d-normed Poisson C^*-algebras and isometric Poisson C^*-algebra isomorphisms between Poisson C^*-algebras.

We moreover prove the Hyers–Ulam stability of their d-isometric homomorphisms and of their isometric homomorphisms.

1. Introduction and preliminaries

In 1940, S. M. Ulam [58] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $||\cdot||$, respectively. Hyers [20] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$||f(x+y) - f(x) - f(y)|| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x) - T(x)|| \le \epsilon$$

for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \ge 0$ and $p \in [0, 1)$ such that

$$||f(x+y) - f(x) - f(y)|| \le \epsilon(||x||^p + ||y||^p)$$

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for all $x, y \in X$. Th. M. Rassias [47] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in X$. This inequality has provided a lot of influence in the development of what we now call *Hyers–Ulam stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was taken up by a number of mathematicians (cf. [8, 9, 12, 19, 21, 22, 23, 24, 32, 33, 35, 45, 50, 51, 52, 54, 56]). Th. M. Rassias [48] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [14] following the same approach as in Th. M. Rassias [47], gave an affirmative solution to this question for $p > 1$.

Găvruta [15] generalized the Rassias' result in the following form: Let G be an abelian group and X a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k}\varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to X$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$.

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies

$$d_Y\big(f(x), f(y)\big) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(\cdot, \cdot)$ and $d_Y(\cdot, \cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; i.e., for all x, y in X with $d_X(x, y) = r$, we have $d_Y\big(f(x), f(y)\big) = r$. Then r is called a conservative(or preserved) distance for the mapping f. Aleksandrov [1] posed the following problem:

Remark (Aleksandrov problem). Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.

The isometric problems have been investigated in several papers (see [2, 7, 11, 16, 18, 27, 28, 29, 49, 53, 57, 60]). Th. M. Rassias and P. Šemrl [55] proved the following theorem for mappings satisfying the strong distance one preserving property (SDOPP), i.e., for every $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$ and conversely.

Theorem 1.1 ([55]). *Let X and Y be real normed linear spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a Lipschitz mapping with Lipschitz constant $\kappa \leq 1$. Assume that f is a surjective mapping satisfying* (SDOPP). *Then f is an isometry.*

Definition 1.1 ([5]). Let X be a real linear space with $\dim X \geq d$ and $\|\cdot, \dots, \cdot\| : X^d \to \mathbb{R}$ a function. Then $(X, \|\cdot, \dots, \cdot\|)$ is called a *linear d-normed space* if

(dN$_1$) $\|x_1, \dots, x_d\| = 0 \Longleftrightarrow x_1, \dots, x_d$ are linearly dependent

(dN$_2$) $\|x_1, \dots, x_d\| = \|x_{j_1}, \dots, x_{j_d}\|$ for every permutation $(j_1, \dots, j_d)$ of $(1, \dots, d)$

(dN$_3$) $\|\alpha x_1, \dots, x_d\| = |\alpha| \|x_1, \dots, x_d\|$

(dN$_4$) $\|x + y, x_2, \dots, x_d\| \leq \|x, x_2, \dots, x_d\| + \|y, x_2, \dots, x_d\|$

for all $\alpha \in \mathbb{R}$ and all $x, y, x_1, \dots, x_d \in X$. The function $\|\cdot, \dots, \cdot\|$ is called the *d-norm on X*.

In [46], the author defined the notion of d-isometry and proved the Rassias and Šemrl's theorem in linear d-normed spaces.

Definition 1.2 ([46]). We call $f : X \to Y$ a *d-Lipschitz mapping* if there is a $\kappa \geq 0$ such that

$$\|f(x_1) - f(y_1), \dots, f(x_d) - f(y_d)\| \leq \kappa \|x_1 - y_1, \dots, x_d - y_d\|$$

for all $x_1, \dots, x_d, y_1, \dots, y_d \in X$. The smallest such κ is called the *d-Lipschitz constant.*

Definition 1.3 ([46]). Let X and Y be linear d-normed spaces and $f : X \to Y$ a mapping. We call f a *d-isometry* if

$$\|x_1 - y_1, \dots, x_d - y_d\| = \|f(x_1) - f(y_1), \dots, f(x_d) - f(y_d)\|$$

for all $x_1, \dots, x_d, y_1, \dots, y_d \in X$.

For a map $f : X \to Y$, consider the following condition which is called the *d-distance one preserving property* : For $x_1, \dots, x_d, y_1, \dots, y_d \in X$ with $\|x_1 - y_1, \dots, x_d - y_d\| = 1$, $\|f(x_1) - f(y_1), \dots, f(x_d) - f(y_d)\| = 1$.

Definition 1.4 ([6]). The points $x, y, z \in X$ are said to be *colinear* if $x - y$ and $x - z$ are linearly dependent.

Theorem 1.2 ([46, Theorem 2.7]). *Let $f : X \to Y$ be a d-Lipschitz mapping with d-Lipschitz constant $\kappa \leq 1$. Assume that if x, y, z are colinear then $f(x)$, $f(y), f(z)$ are colinear, and that if $x_1 - y_1, \dots, x_d - y_d$ are linearly dependent then $f(x_1) - f(y_1), \dots, f(x_d) - f(y_d)$ are linearly dependent. If f satisfies the d-distance one preserving property, then f is a d-isometry.*

We define the notion of linear d-normed Banach space.

Definition 1.5. A linear d-normed and normed space X with d-norm

$$\|\cdot, \dots, \cdot\|_X$$

and norm $\|\cdot\|$ is called a *linear d-normed Banach space* if $(X, \|\cdot\|)$ is a Banach space.

A Banach $*$-algebra $\mathcal{A}$ is a Banach algebra with an involution $* : \mathcal{A} \to \mathcal{A}$ such that $(x^*)^* = x$ holds for all $x \in \mathcal{A}$. A C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra whose norm $\|\cdot\|$ satisfies $\|x^*\| = \|x\|$ and $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. An algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called a *C^*-algebra homomorphism* if $H(x^*) = H(x)^*$ holds for all $x \in \mathcal{A}$ (see [3, 4, 10, 13, 26, 32, 33, 34, 35, 37, 41, 43]).

Definition 1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be linear d-normed unital C^*-algebras. A C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called a *d-isometric C^*-algebra homomorphism* if the mapping $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$\|H(x_1) - H(y_1), \dots, H(x_d) - H(y_d)\|_{\mathcal{B}} = \|x_1 - y_1, \dots, x_d - y_d\|_{\mathcal{A}}$$

for all $x_1, \dots, x_d, y_1, \dots, y_d \in \mathcal{A}$. If, in addition, the C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is bijective, then the C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called a *d-isometric C^*-algebra isomorphism.*

Definition 1.7. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. A C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called an *isometric C^*-algebra homomorphism* if the mapping $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$\|H(x) - H(y)\| = \|x - y\|$$

for all $x, y \in \mathcal{A}$. If, in addition, the C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is bijective, then the C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called an *isometric C^*-algebra isomorphism.*

A *Poisson C^*-algebra* $\mathcal{A}$ is a C^*-algebra with a $\mathbb{C}$-bilinear map $\{\cdot,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called a *Poisson bracket*, such that $(\mathcal{A}, \{\cdot,\cdot\})$ is a complex Lie algebra and

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

for all $a, b, c \in \mathcal{A}$. Poisson algebras have played an important role in many mathematical areas and have been studied to find sympletic leaves of the corresponding Poisson varieties. It is also important to find or construct a Poisson bracket in the theory of Poisson algebra (see [17, 30, 31, 38]).

Definition 1.8. A C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called a *Poisson C^*-algebra homomorphism* if $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$\{H(x), H(y)\} = \{x, y\}$$

for all $x, y \in \mathcal{A}$.

Definition 1.9. A Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called a *d-isometric Poisson C^*-algebra homomorphism* if $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$||H(x_1) - H(y_1), \dots, H(x_d) - H(y_d)||_{\mathcal{B}} = ||x_1 - y_1, \dots, x_d - y_d||_{\mathcal{A}}$$

for all $x_1, \dots, x_d, y_1, \dots, y_d \in \mathcal{A}$. If, in addition, the Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is bijective, then the Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called a *d-isometric Poisson C^*-algebra isomorphism.*

Definition 1.10. A Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called an *isometric Poisson C^*-algebra homomorphism* if $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$||H(x) - H(y)|| = ||x - y||$$

for all $x, y \in \mathcal{A}$. If, in addition, the Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is bijective, then the Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called an *isometric Poisson C^*-algebra isomorphism.*

The paper is organized as follows: In Section 2, we prove the Hyers–Ulam stability of linear d-isometries in linear d-normed Banach modules over a unital C^*-algebra and of linear isometries in Banach modules over a unital C^*-algebra.

In Section 3, we investigate d-isometric C^*-algebra isomorphisms between linear d-normed unital C^*-algebras and isometric C^*-algebra isomorphisms between unital C^*-algebras, and prove the Hyers–Ulam stability of d-isometric C^*-algebra homomorphisms in linear d-normed unital C^*-algebras and of isometric C^*-algebra homomorphisms in unital C^*-algebras.

In Section 4, we investigate d-isometric Poisson C^*-algebra isomorphisms between linear d-normed unital Poisson C^*-algebras and isometric Poisson C^*-algebra isomorphisms between unital Poisson C^*-algebras, and prove the Hyers–Ulam stability of d-isometric Poisson C^*-algebra homomorphisms in linear d-normed unital Poisson C^*-algebras and of isometric Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras.

2. d-isometric linear mappings in linear d-normed Banach modules over a C^*-algebra

Throughout this section, let $\mathcal{A}$ be a unital C^*-algebra with norm $|\cdot|$, unit e and unitary group $\mathcal{U}(\mathcal{A}) := \{u \in \mathcal{A} | uu^* = u^*u = e\}$. Assume that X is a left (linear d-normed) Banach modules over $\mathcal{A}$ with d-norm $||\cdot, \dots, \cdot||_X$ and norm $||\cdot||$ and that Y is a left (linear d-normed) Banach modules over $\mathcal{A}$ with d-norm $||\cdot, \dots, \cdot||_Y$ and norm $||\cdot||$.

Definition 2.1. An $\mathcal{A}$-linear mapping $L : X \to Y$ is called a *d-isometric $\mathcal{A}$-linear mapping* if the mapping $L : X \to Y$ satisfies

$$||L(x_1) - L(y_1), \dots, L(x_d) - L(y_d)||_Y = ||x_1 - y_1, \dots, x_d - y_d||_X$$

for all $x_1, \dots, x_d, y_1, \dots, y_d \in X$.

Theorem 2.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^d \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \dots, x_d) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \dots, 2^j x_d) < \infty, \tag{2.i}$$

$$\|f(ux_1 + ux_2) - uf(x_1) - uf(x_2)\| \le \varphi(x_1, x_2, \underbrace{0, \dots, 0}_{d-2 \text{ times}}), \tag{2.ii}$$

$$|\ \|f(x_1), \dots, f(x_d)\|_Y - \|x_1, \dots, x_d\|_X\ | \le \varphi(x_1, \dots, x_d) \tag{2.iii}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_d \in X$. Then there exists a unique d-isometric $\mathcal{A}$-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x, \underbrace{0, \dots, 0}_{d-2 \text{ times}}) \tag{2.iv}$$

for all $x \in X$.

Proof. Let $u = 1 \in \mathcal{U}(\mathcal{A})$. By the Găvruta's theorem [18], it follows from (2.i) and (2.ii) that there exists a unique additive mapping $L : X \to Y$ satisfying (2.iv). The additive mapping $L : X \to Y$ is given by

$$L(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \tag{2.1}$$

for all $x \in X$.

By the same reasoning as in the proof of [32] and [37], one can show that the additive mapping $L : X \to Y$ is an $\mathcal{A}$-linear mapping.

It follows from (2.iii) that

$$\begin{aligned} &|\ \|\frac{1}{2^n} f(2^n x_1), \dots, \frac{1}{2^n} f(2^n x_d)\|_Y - \|x_1, \dots, x_d\|_X\ | \\ &= \frac{1}{2^{dn}} |\ \|f(2^n x_1), \dots, f(2^n x_d)\|_Y - \|2^n x_1, \dots, 2^n x_d\|_X\ | \\ &\le \frac{1}{2^{dn}} \varphi(2^n x_1, \dots, 2^n x_d) \le \frac{1}{2^n} \varphi(2^n x_1, \dots, 2^n x_d), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x_1, \dots, x_d \in X$ by (2.i). So by (2.1)

$$\begin{aligned} \|L(x_1), \dots, L(x_d)\|_Y &= \lim_{n \to \infty} \|\frac{1}{2^n} f(2^n x_1), \dots, \frac{1}{2^n} f(2^n x_d)\|_Y \\ &= \|x_1, \dots, x_d\|_X \end{aligned}$$

for all $x_1, \dots, x_d \in X$. Hence

$$\begin{aligned} \|L(x_1) - L(y_1), \dots, L(x_d) - L(y_d)\|_Y &= \|L(x_1 - y_1), \dots, L(x_d - y_d)\|_Y \\ &= \|x_1 - y_1, \dots, x_d - y_d\|_X \end{aligned}$$

for all $x_1, \dots, x_d, y_1, \dots, y_d \in X$. So the mapping $L : X \to Y$ is a d-isometry, as desired. □

Corollary 2.2. *Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$||f(ux_1 + ux_2) - uf(x_1) - uf(x_2)|| \le \theta(||x_1||^p + ||x_2||^p),$$

$$|\ ||f(x_1), \dots, f(x_d)||_Y - ||x_1, \dots, x_d||_X\ | \le \theta \sum_{j=1}^{d} ||x_j||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_d \in X$. Then there exists a unique d-isometric $\mathcal{A}$-linear mapping $L : X \to Y$ such that

$$||f(x) - L(x)|| \le \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \dots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$ and apply Theorem 2.1. □

Definition 2.2. An $\mathcal{A}$-linear mapping $L : X \to Y$ is called an *isometric $\mathcal{A}$-linear mapping* if the mapping $L : X \to Y$ satisfies

$$||L(x) - L(y)|| = ||x - y||$$

for all $x, y \in X$.

Theorem 2.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^d \to [0, \infty)$ satisfying* (2.i) *and* (2.ii) *such that*

$$\text{(2.v)} \qquad |\ ||f(x)|| - ||x||\ | \le \varphi(\underbrace{x, \dots, x}_{d \text{ times}})$$

for all $x \in X$. Then there exists a unique isometric $\mathcal{A}$-linear mapping $L : X \to Y$ satisfying (2.iv).

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathcal{A}$-linear mapping $L : X \to Y$ satisfying (2.iv). The $\mathcal{A}$-linear mapping $L : X \to Y$ is given by

$$\text{(2.2)} \qquad L(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$.

It follows from (2.v) that

$$|\ ||\frac{1}{2^n} f(2^n x)|| - ||x||\ | = \frac{1}{2^n} |\ ||f(2^n x)|| - ||2^n x||\ | \le \frac{1}{2^n} \varphi(\underbrace{2^n x, \dots, 2^n x}_{d \text{ times}}),$$

which tends to zero as $n \to \infty$ for all $x \in X$ by (2.i). So by (2.2)

$$||L(x)|| = \lim_{n \to \infty} ||\frac{1}{2^n} f(2^n x)|| = ||x||$$

for all $x \in X$. Hence

$$||L(x) - L(y)|| = ||L(x - y)|| = ||x - y||$$

for all $x, y \in X$, as desired. □

Corollary 2.4. *Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$||f(ux_1 + ux_2) - uf(x_1) - uf(x_2)|| \le \theta(||x_1||^p + ||x_2||^p),$$
$$|\ ||f(x)|| - ||x||\ | \le d\theta||x||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x, x_1, \dots, x_d \in X$. Then there exists a unique isometric $\mathcal{A}$-linear mapping $L : X \to Y$ such that

$$||f(x) - L(x)|| \le \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \dots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$ and apply Theorem 2.3. □

3. d-isometric isomorphisms between linear d-normed unital C^*-algebras

Throughout this section, assume that $\mathcal{A}$ is a (linear d-normed) unital C^*-algebra with d-norm $||\cdot, \dots, \cdot||_{\mathcal{A}}$, norm $||\cdot||$, unit e and unitary group $\mathcal{U}(\mathcal{A})$ and that $\mathcal{B}$ is a (linear d-normed) unital C^*-algebra with d-norm $||\cdot, \dots, \cdot||_{\mathcal{B}}$ and norm $||\cdot||$.

We investigate d-isometric C^*-algebra isomorphisms between linear d-normed unital C^*-algebras.

Theorem 3.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there is a function $\varphi : \mathcal{A}^d \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \dots, x_d) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x_1, \dots, 2^j x_d) < \infty, \tag{3.i}$$

$$||h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)|| \le \varphi(x_1, x_2, \underbrace{0, \dots, 0}_{d-2 \text{ times}}), \tag{3.ii}$$

$$||h(2^n u^*) - h(2^n u)^*|| \le \varphi(\underbrace{2^n u, \dots, 2^n u}_{d \text{ times}}), \tag{3.iii}$$

$$|\ ||h(x_1), \dots, h(x_d)||_{\mathcal{B}} - ||x_1, \dots, x_d||_{\mathcal{A}}\ | \le \varphi(x_1, \dots, x_d) \tag{3.iv}$$

for all $u \in \mathcal{U}(\mathcal{A})$, all $x_1, \dots, x_d \in \mathcal{A}$, all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $n \in \mathbb{Z}$. Assume that

$$\lim_{n \to \infty} \frac{h(2^n e)}{2^n} \text{ is invertible.} \tag{3.v}$$

Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a d-isometric C^*-algebra isomorphism.

Proof. We can consider a C^*-algebra as a Banach module over the unital C^*-algebra $\mathbb{C}$. By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}) \tag{3.vi}$$

for all $x \in \mathcal{A}$. The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{3.1}$$

for all $x \in \mathcal{A}$.

By (3.i) and (3.iii), we get

$$H(u^*) = \lim_{n\to\infty} \frac{h(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{h(2^n u)^*}{2^n} = (\lim_{n\to\infty} \frac{h(2^n u)}{2^n})^* = H(u)^*$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [33, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* \\ &= (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Since $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u)h(y) = H(u)h(y) \tag{3.2}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (3.2),

$$2^n H(uy) = H(2^n uy) = H(u(2^n y)) = H(u)h(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(uy) = \frac{1}{2^n} H(u)h(2^n y) = H(u) \cdot \frac{1}{2^n} h(2^n y) \tag{3.3}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (3.3) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{3.4}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in$

$\mathcal{U}(\mathcal{A})$), it follows from (3.4) that

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^{m} \lambda_j u_j y) = \sum_{j=1}^{m} \lambda_j H(u_j y) = \sum_{j=1}^{m} \lambda_j H(u_j)H(y) \\ &= H(\sum_{j=1}^{m} \lambda_j u_j)H(y) = H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (3.2) and (3.4),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

It follows from (3.iv) that

$$\begin{aligned} &| \, \|\frac{1}{2^n}h(2^n x_1), \ldots, \frac{1}{2^n}h(2^n x_d)\|_{\mathcal{B}} - \|x_1, \ldots, x_d\|_{\mathcal{A}} \, | \\ &= \frac{1}{2^{dn}} | \, \|h(2^n x_1), \ldots, h(2^n x_d)\|_{\mathcal{B}} - \|2^n x_1, \ldots, 2^n x_d\|_{\mathcal{A}} \, | \\ &\le \frac{1}{2^{dn}} \varphi(2^n x_1, \ldots, 2^n x_d) \le \frac{1}{2^n} \varphi(2^n x_1, \ldots, 2^n x_d), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x_1, \ldots, x_d \in \mathcal{A}$ by (3.i). By (3.1),

$$\begin{aligned} \|H(x_1), \ldots, H(x_d)\|_{\mathcal{B}} &= \lim_{n\to\infty} \|\frac{1}{2^n}h(2^n x_1), \ldots, \frac{1}{2^n}h(2^n x_d)\|_{\mathcal{B}} \\ &= \|x_1, \ldots, x_d\|_{\mathcal{A}} \end{aligned}$$

for all $x_1, \ldots, x_d \in \mathcal{A}$. Since $H : \mathcal{A} \to \mathcal{B}$ is additive,

$$\begin{aligned} \|H(x_1) - H(y_1), \ldots, H(x_d) - H(y_d)\|_{\mathcal{B}} &= \|H(x_1 - y_1), \ldots, H(x_d - y_d)\|_{\mathcal{B}} \\ &= \|x_1 - y_1, \ldots, x_d - y_d\|_{\mathcal{A}} \end{aligned}$$

for all $x_1, \ldots, x_d, y_1, \ldots, y_d \in \mathcal{A}$. So the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a d-isometry.

Therefore, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a d-isometric C^*-algebra isomorphism, as desired. □

Corollary 3.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)\| \le \theta(\|x_1\|^p + \|x_2\|^p),$$

$$\|h(2^n u^*) - h(2^n u)^*\| \le 2^{np} d\theta,$$

$$| \, \|h(x_1), \ldots, h(x_d)\|_{\mathcal{B}} - \|x_1, \ldots, x_d\|_{\mathcal{A}} \, | \le \theta \sum_{j=1}^{d} \|x_j\|^p$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in \mathcal{U}(\mathcal{A})$, *all* $n \in \mathbb{Z}$ *and all* $x_1, \dots, x_d \in \mathcal{A}$. *Assume that* $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ *is invertible. Then the bijective mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a d-isometric* C^**-algebra isomorphism.*

Proof. Define $\varphi(x_1, \dots, x_d) = \theta \sum_{j=1}^d ||x_j||^p$ and apply Theorem 3.1. □

Theorem 3.3. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a bijective mapping satisfying* $h(0) = 0$ *and* $h(2^n uy) = h(2^n u)h(y)$ *for all* $u \in \mathcal{U}(\mathcal{A})$, *all* $y \in \mathcal{A}$ *and all* $n \in \mathbb{Z}$, *for which there is a function* $\varphi : \mathcal{A}^d \to [0, \infty)$ *satisfying* (3.i), (3.iii), (3.iv) *and* (3.v) *such that*

$$||h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)|| \le \varphi(x_1, x_2, \underbrace{0, \dots, 0}_{d-2 \text{ times}}), \tag{3.vii}$$

for all $x_1, x_2 \in \mathcal{A}$ *and* $\mu = 1, i$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then the bijective mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a d-isometric* C^**-algebra isomorphism.*

Proof. Put $\mu = 1$ in (3.vii). By the same reasoning as in the proof of Theorem 3.1, there exists a unique d-isometric involutive additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.vi). The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$. By the same reasoning as in the proof of [47, Theorem], the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (3.vii). By the same method as in the proof of Theorem 3.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{ih(2^n x)}{2^n} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$ and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

Since $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u)h(y) = H(u)h(y) \tag{3.5}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (3.5),

$$2^n H(uy) = H(2^n uy) = H(u(2^n y)) = H(u)h(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(uy) = \frac{1}{2^n} H(u)h(2^n y) = H(u)\frac{1}{2^n} h(2^n y) \tag{3.6}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (3.6) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{3.7}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (3.7) that

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^{m} \lambda_j u_j y) = \sum_{j=1}^{m} \lambda_j H(u_j y) = \sum_{j=1}^{m} \lambda_j H(u_j)H(y) \\ &= H(\sum_{j=1}^{m} \lambda_j u_j)H(y) = H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (3.5) and (3.7),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a d-isometric C^*-algebra isomorphism, as desired. □

Corollary 3.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$||h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)|| \leq \theta(||x_1||^p + ||x_2||^p),$$

$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$

$$|\, ||h(x_1), \ldots, h(x_d)||_{\mathcal{B}} - ||x_1, \ldots, x_d||_{\mathcal{A}} \,| \leq \theta \sum_{j=1}^{d} ||x_j||^p$$

for $\mu = 1, i$, all $u \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$ and all $x_1, \ldots, x_d \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ is invertible. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a d-isometric C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$ and apply Theorem 3.3. □

Now we prove the Hyers–Ulam stability of d-isometric C^*-algebra homomorphisms in linear d-normed unital C^*-algebras.

Theorem 3.5. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (3.i), (3.ii), (3.iii) *and* (3.iv) *such that*

$$\|h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)\| \leq \varphi(2^n u, 2^n v, \underbrace{0, \dots, 0}_{d-2 \text{ times}}) \tag{3.viii}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ and all $n \in \mathbb{Z}$. Then there exists a unique d-isometric C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.vi).

Proof. By the same reasoning as in the proof of Theorem 3.1, there exists a unique d-isometric $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.vi). The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{3.8}$$

for all $x \in \mathcal{A}$.

By (3.viii),

$$\begin{aligned} \frac{1}{2^{2n}} \|h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)\| &\leq \frac{1}{2^{2n}} \varphi(2^n u, 2^n v, \underbrace{0, \dots, 0}_{d-2 \text{ times}}) \\ &\leq \frac{1}{2^n} \varphi(2^n u, 2^n v, \underbrace{0, \dots, 0}_{d-2 \text{ times}}), \end{aligned}$$

which tends to zero as $n \to \infty$ by (3.i). By (3.8),

$$\begin{aligned} H(uv) &= \lim_{n\to\infty} \frac{h(2^n u \cdot 2^n v)}{2^{2n}} = \lim_{n\to\infty} \frac{h(2^n u)h(2^n v)}{2^{2n}} = \lim_{n\to\infty} \frac{h(2^n u)}{2^n} \frac{h(2^n v)}{2^n} \\ &= H(u)H(v) \end{aligned}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(xv) &= H(\sum_{j=1}^m \lambda_j u_j v) = \sum_{j=1}^m \lambda_j H(u_j v) = \sum_{j=1}^m \lambda_j H(u_j)H(v) \\ &= H(\sum_{j=1}^m \lambda_j u_j)H(v) = H(x)H(v) \end{aligned}$$

for all $x \in \mathcal{A}$ and all $v \in \mathcal{U}(\mathcal{A})$. By the same method as given above, one can obtain that

$$H(xy) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$. So the mapping $H : \mathcal{A} \to \mathcal{B}$ is a d-isometric C^*-algebra homomorphism, as desired. $\square$

Corollary 3.6. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that*

$$||h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)|| \leq \theta(||x_1||^p + ||x_2||^p),$$
$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$
$$||h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)|| \leq 2^{np+1}\theta,$$
$$|\ ||h(x_1), \ldots, h(x_d)||_{\mathcal{B}} - ||x_1, \ldots, x_d||_{\mathcal{A}}\ | \leq \theta \sum_{j=1}^{d} ||x_j||^p$$

for all $\mu \in \mathbb{T}^1$, all $u, v \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$ and all $x_1, \ldots, x_d \in \mathcal{A}$. Then there exists a unique d-isometric C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$||h(x) - H(x)|| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$ and apply Theorem 3.5. □

We investigate isometric C^*-algebra isomorphisms between unital C^*-algebras.

Theorem 3.7. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there is a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (3.i), (3.ii), (3.iii) *and* (3.v) *such that*

$$(3.\text{ix}) \qquad |\ ||h(x)|| - ||x||\ | \leq \varphi(\underbrace{x, \ldots, x}_{d\ times})$$

for all $x \in \mathcal{A}$. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is an isometric C^*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 3.1, there exists a unique C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.vi), and $H(x) = h(x)$ for all $x \in \mathcal{A}$. The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$(3.9) \qquad H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$.

It follows from (3.ix) that

$$|\ \frac{1}{2^n}||h((2^n x)|| - ||x||\ | = \frac{1}{2^n}|\ ||h(2^n x)|| - ||2^n x||\ | \leq \frac{1}{2^n}\varphi(\underbrace{2^n x, \ldots, 2^n x}_{d\text{ times}}),$$

which tends to zero as $n \to \infty$ for all $x \in \mathcal{A}$ by (3.i). By (3.9),

$$||H(x)|| = \lim_{n\to\infty} ||\frac{1}{2^n} h(2^n x)|| = ||x||$$

for all $x \in \mathcal{A}$. Since $H : \mathcal{A} \to \mathcal{B}$ is additive,

$$||H(x) - H(y)|| = ||H(x - y)|| = ||x - y||$$

for all $x, y \in \mathcal{A}$. So the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is an isometry.

Therefore, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is an isometric C^*-algebra isomorphism, as desired. □

Corollary 3.8. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$||h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)|| \leq \theta(||x_1||^p + ||x_2||^p),$$
$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$
$$|\ ||h(x)|| - ||x||\ | \leq d\theta ||x||^p$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$ and all $x, x_1, x_2 \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ is invertible. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is an isometric C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$ and apply Theorem 3.7. □

Theorem 3.9. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there is a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (3.i), (3.iii), (3.v), (3.vii) *and* (3.ix) *such that If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is an isometric C^*-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 3.1, 3.3 and 3.7. □

Corollary 3.10. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$||h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)|| \leq \theta(||x_1||^p + ||x_2||^p),$$
$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$
$$|\ ||h(x)|| - ||x||\ | \leq d\theta ||x||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$, all $x, x_1, x_2 \in \mathcal{A}$ and $\mu = 1, i$. Assume

$$\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$$

is invertible. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is an isometric C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$ and apply Theorem 3.9. □

Now we prove the Hyers–Ulam stability of isometric C^*-algebra homomorphisms in unital C^*-algebras.

Theorem 3.11. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (3.i), (3.ii), (3.iii), (3.viii) *and* (3.ix). *Then there exists a unique isometric C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying* (3.vi).

Proof. The proof is similar to the proofs of Theorems 3.1, 3.5 and 3.7. □

Corollary 3.12. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$||h(\mu x_1 + \mu x_2) - \mu h(x_1) - \mu h(x_2)|| \leq \theta(||x_1||^p + ||x_2||^p),$$
$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$
$$||h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)|| \leq 2^{np+1}\theta,$$
$$|\ ||h(x)|| - ||x||\ | \leq d\theta ||x||^p$$

for all $\mu \in \mathbb{T}^1$, all $u, v \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$ and all $x, x_1, x_2 \in \mathcal{A}$. Then there exists a unique isometric C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$||h(x) - H(x)|| \leq \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$ and apply Theorem 3.11. □

4. d-isometric homomorphisms between linear d-normed Poisson C^*-algebras

Throughout this section, let $\mathcal{A}$ be a (linear d-normed) unital Poisson C^*-algebra with d-norm $||\cdot, \ldots, \cdot||_{\mathcal{A}}$, norm $||\cdot||$, unit e and unitary group $\mathcal{U}(\mathcal{A})$, and $\mathcal{B}$ a (linear d-normed) unital Poisson C^*-algebra with d-norm $||\cdot, \ldots, \cdot||_{\mathcal{B}}$ and norm $||\cdot||$.

We are going to investigate d-isometric Poisson C^*-algebra isomorphisms between linear d-normed unital Poisson C^*-algebras.

Theorem 4.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n u y) = h(2^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \ldots, x_d, z, w) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \ldots, 2^j x_d, 2^j z, 2^j w) < \infty, \tag{4.i}$$

$$\begin{aligned} &||h(\mu x_1 + \mu x_2 + \{z, w\}) - \mu h(x_1) - \mu h(x_2) - \{h(z), h(w)\}|| \\ &\leq \varphi(x_1, x_2, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}, z, w), \end{aligned} \tag{4.ii}$$

$$||h(2^n u^*) - h(2^n u)^*|| \leq \varphi(\underbrace{2^n u, \ldots, 2^n u}_{d \text{ times}}, 0, 0), \tag{4.iii}$$

$$\text{(4.iv)} \qquad |\ \|h(x_1), \dots, h(x_d)\|_{\mathcal{B}} - \|x_1, \dots, x_d\|_{\mathcal{A}}\ | \le \varphi(x_1, \dots, x_d, 0, 0)$$

for all $u \in \mathcal{U}(\mathcal{A})$, all $x_1, \dots, x_d, z, w \in \mathcal{A}$, all $\mu \in \mathbb{T}^1$ and all $n \in \mathbb{Z}$. Assume that

$$\text{(4.v)} \qquad \lim_{n\to\infty} \frac{h(2^n e)}{2^n} \textit{ is invertible.}$$

Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a d-isometric Poisson C^*-algebra isomorphism.

Proof. Let $z = w = 0$ in (4.ii). By the same reasoning as in the proof of Theorem 3.1, there exists a unique d-isometric C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\text{(4.vi)} \qquad \|h(x) - H(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x, \underbrace{0, \dots, 0}_{d \text{ times}})$$

for all $x \in \mathcal{A}$, and $H(x) = h(x)$ for all $x \in \mathcal{A}$. The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$\text{(4.1)} \qquad H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$.

It follows from (4.1) that

$$\text{(4.2)} \qquad H(x) = \lim_{n\to\infty} \frac{h(2^{2n} x)}{2^{2n}}$$

for all $x \in \mathcal{A}$. Let $x_1 = x_2 = 0$ in (4.ii). Then we get

$$\|h(\{z, w\}) - \{h(z), h(w)\}\| \le \varphi(\underbrace{0, \dots, 0}_{d \text{ times}}, z, w)$$

for all $z, w \in \mathcal{A}$. So

$$\text{(4.3)} \qquad \begin{aligned} \frac{1}{2^{2n}}\|h(\{2^n z, 2^n w\}) - \{h(2^n z), h(2^n w)\}\| &\le \frac{1}{2^{2n}}\varphi(\underbrace{0, \dots, 0}_{d \text{ times}}, 2^n z, 2^n w) \\ &\le \frac{1}{2^n}\varphi(\underbrace{0, \dots, 0}_{d \text{ times}}, 2^n z, 2^n w) \end{aligned}$$

for all $z, w \in \mathcal{A}$. By (4.i), (4.2) and (4.3),

$$\begin{aligned} H(\{z, w\}) = \lim_{n\to\infty} \frac{h(2^{2n}\{z, w\})}{2^{2n}} &= \lim_{n\to\infty} \frac{h(\{2^n z, 2^n w\})}{2^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{2^{2n}}\{h(2^n z), h(2^n w)\} \\ &= \lim_{n\to\infty} \{\frac{h(2^n z)}{2^n}, \frac{h(2^n w)}{2^n}\} = \{H(z), H(w)\} \end{aligned}$$

for all $z, w \in \mathcal{A}$.

Therefore, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a d-isometric Poisson C^*-algebra isomorphism, as desired. □

Corollary 4.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that*

$$\begin{aligned} ||h(\mu x_1 + \mu x_2 + \{z,w\}) - \mu h(x_1) - n\mu h(x_2) - \{h(z), h(w)\}|| \\ \leq \theta(||x_1||^p + ||x_2||^p + ||z||^p + ||w||^p), \end{aligned}$$
$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$
$$|\ ||h(x_1), \ldots, h(x_d)||_{\mathcal{B}} - ||x_1, \ldots, x_d||_{\mathcal{A}}\ | \leq \theta \sum_{j=1}^{d} ||x_j||^p$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$ and all $x_1, \ldots, x_d, z, w \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ is invertible. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a d-isometric Poisson C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p)$ and apply Theorem 4.1. □

Now we prove the Hyers–Ulam stability of d-isometric Poisson C^*-algebra homomorphisms in linear d-normed unital Poisson C^*-algebras.

Theorem 4.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ satisfying* (4.i), (4.ii), (4.iii) *and* (4.iv) *such that*

$$\text{(4.vii)} \qquad ||h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)|| \leq \varphi(2^n u, 2^n v, \underbrace{0, \ldots, 0}_{d \text{ times}})$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ and all $n \in \mathbb{Z}$. Then there exists a unique d-isometric Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.vi).

Proof. The proof is similar to the proofs of Theorems 3.1, 3.5 and 4.1. □

Corollary 4.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that*

$$\begin{aligned} ||h(\mu x_1 + \mu x_2 + \{z,w\}) - \mu h(x_1) - \mu h(x_2) - \{h(z), h(w)\}|| \\ \leq \theta(||x_1||^p + ||x_2||^p + ||z||^p + ||w||^p), \end{aligned}$$
$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$
$$||h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)|| \leq 2^{np+1}\theta,$$
$$|\ ||h(x_1), \ldots, h(x_d)||_{\mathcal{B}} - ||x_1, \ldots, x_d||_{\mathcal{A}}\ | \leq \theta \sum_{j=1}^{d} ||x_j||^p$$

for all $\mu \in \mathbb{T}^1$, all $u, v \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$ and all $x_1, \ldots, x_d, z, w \in \mathcal{A}$. Then there exists a unique d-isometric Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$||h(x) - H(x)|| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p)$ and apply Theorem 4.3. □

We are going to investigate isometric Poisson C^*-algebra isomorphisms between unital Poisson C^*-algebras.

Theorem 4.5. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a bijective mapping satisfying* $h(0) = 0$ *and* $h(2^n uy) = h(2^n u)h(y)$ *for all* $y \in \mathcal{A}$, *all* $u \in \mathcal{U}(\mathcal{A})$ *and all* $n \in \mathbb{Z}$, *for which there exists a function* $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ *satisfying* (4.i), (4.ii), (4.iii) *and* (4.v) *such that*

$$\text{(4.viii)} \qquad | \, ||h(x)|| - ||x|| \, | \leq \varphi(\underbrace{x, \ldots, x}_{d \text{ times}}, 0, 0)$$

for all $x \in \mathcal{A}$. *Then the bijective mapping* $h : \mathcal{A} \to \mathcal{B}$ *is an isometric Poisson* C^**-algebra isomorphism.*

Proof. The proof is similar to the proofs of Theorems 3.1, 3.7 and 4.1. □

Corollary 4.6. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a bijective mapping satisfying* $h(0) = 0$ *and* $h(2^n uy) = h(2^n u)h(y)$ *for all* $u \in \mathcal{U}(\mathcal{A})$, *all* $y \in \mathcal{A}$ *and all* $n \in \mathbb{Z}$, *for which there exist constants* $\theta \geq 0$ *and* $p \in [0, 1)$ *such that*

$$\begin{aligned} ||h(\mu x_1 + \mu x_2 + \{z, w\}) - \mu h(x_1) - n\mu h(x_2) - \{h(z), h(w)\}|| \\ \leq \theta(||x_1||^p + ||x_2||^p + ||z||^p + ||w||^p), \\ ||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta, \\ | \, ||h(x)|| - ||x|| \, | \leq d\theta ||x||^p \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in \mathcal{U}(\mathcal{A})$, *all* $n \in \mathbb{Z}$ *and all* $x, x_1, x_2, z, w \in \mathcal{A}$. *Assume that* $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ *is invertible. Then the bijective mapping* $h : \mathcal{A} \to \mathcal{B}$ *is an isometric Poisson* C^**-algebra isomorphism.*

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p)$ and apply Theorem 4.5. □

Now we prove the Hyers–Ulam stability of isometric Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras.

Theorem 4.7. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a mapping satisfying* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ *satisfying* (4.i), (4.ii), (4.iii), (4.vii) *and* (4.viii). *Then there exists a unique isometric Poisson* C^**-algebra homomorphism* $H : \mathcal{A} \to \mathcal{B}$ *satisfying* (4.vi).

Proof. The proof is similar to the proofs of Theorems 3.1, 3.7 and 4.1. □

Corollary 4.8. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$||h(\mu x_1 + \mu x_2 + \{z, w\}) - \mu h(x_1) - \mu h(x_2) - \{h(z), h(w)\}|| \leq \theta(||x_1||^p + ||x_2||^p + ||z||^p + ||w||^p),$$

$$||h(2^n u^*) - h(2^n u)^*|| \leq 2^{np} d\theta,$$

$$||h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)|| \leq 2^{np+1}\theta,$$

$$|\ ||h(x)|| - ||x||\ | \leq d\theta ||x||^p$$

for all $\mu \in \mathbb{T}^1$, all $u, v \in \mathcal{U}(\mathcal{A})$, all $n \in \mathbb{Z}$ and all $x, x_1, x_2, z, w \in \mathcal{A}$. Then there exists a unique isometric Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$||h(x) - H(x)|| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p)$ and apply Theorem 4.7. □

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