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TOPOLOGICAL COMPLEXITY OF SEMIGROUP ACTIONS

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Abstract. In this paper, we study the complexity of semigroup actions using complexity functions of open covers. The main results are as follows: (1) A dynamical system is equicontinuous if and only if any open cover has bounded complexity; (2) Weak-mixing implies scattering; (3) We get a criterion for the scattering property.

1. Introduction

It is well known that topological entropy is an invariant of topological conjugacy. If the topological entropy is positive, the system is complex and chaotic. If the entropy is zero, the system is rather simple. However from the theory and application, there still exists relatively complex and chaotic behavior. Therefore, for more general research on complexity of a system, one can study the complexity function of a system. This idea was firstly introduced in the research of ergodic theory (see [1]), and then in symbolic dynamical systems (see [2]) by Ferenczi. Recently, Blanchard, Host and Maass used open covers to define a complexity function for a continuous map on a compact metric space, and discussed the equicontinuity and scattering properties (see [3]). We study topological complexity of semigroup actions. In §2 a dynamical system is equicontinuous if and only if any open cover has bounded complexity; In §3 we prove that weak-mixing implies scattering; In §4 we get a criterion for the scattering property.

Let X be a compact metric space and T a topological semigroup. Denote the set of all finite subsets of T by $F(T)$.

• Suppose X is a topological space, T is a topological semigroup, if a map

$$\pi : X \times T \to X$$

satisfies

$$\pi(\pi(x,t),s) = \pi(x,ts), \forall x \in X,\ \forall t,s \in T,$$

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then we call π is right action of T on X; If the right action π is continuous, then (X, T, π) is called a semi-dynamical system (abbreviation: (X, T)). Often we write $\pi(x, t) = xt$.

• If there is a point $x \in X$ such that $\overline{xT} = X$ (where $xT = \{xt \mid t \in T\}$), then (X, T) is called topologically transitive; If for any non-empty open sets U, V, there exists $t \in T$ such that $\pi_t^{-1}U \cap V \neq \emptyset$, then (X, T) is called topologically ergodic; If $(X \times X, T)$ is topologically ergodic, then (X, T) is weak-mixing.

• If for any $\varepsilon > 0$, there is $\eta > 0$ such that if $x, y \in X$ with $d(x, y) < \eta$, then for all $t \in T$, one has $d(xt, yt) < \varepsilon$, we say that (X, T) is equicontinuous.

2. Equicontinuity and topological complexity function

If α, β are open covers of X, define $\alpha \vee \beta = \{U \cap V | U \in \alpha, V \in \beta\}$, then $\alpha \vee \beta$ is an open cover of X. For $A \in F(T)$, denote $\alpha_0^A = \bigvee_{t \in A} (\pi_t)^{-1}\alpha$. Let $r_0(T, \alpha, A)$ denote the number of sets in a finite subcover of α_0^A with smallest cardinality. We get a map $r_0(T, \alpha, \cdot) : F(T) \to Z^+$, $A \mapsto r_0(T, \alpha, A)$. $r_0(T, \alpha, \cdot)$ is said to be the topological complexity function of the cover α of (X, T), we often call it complexity function of α.

Given a finite cover $\alpha = \{U_1, \ldots, U_k\}$. Put $E = \{1, \ldots, k\}$. One defines a map $\omega : T \to E, t \mapsto \omega(t)$. Denote $J^*(\omega) = \bigcap_{t \in T} \pi_t^{-1} U_{\omega(t)}$, $J_A^*(\omega) = \bigcap_{t \in A} \pi_t^{-1} U_{\omega(t)}$, for $A \subset T$. Let $M(T, E)$ be the set of maps from T to E and $M(A, E)$ the set of maps from A to E.

Lemma 2.1. *Suppose T is countable, a finite cover $\alpha = (U_1, \ldots, U_k)$ has complexity bounded by m if and only if there exist $\omega_1, \ldots, \omega_m \in M(T, E)$ such that $\bigcup_{i=1}^m J^*(\omega_i) = X$.*

Proof. Since T is countable, suppose $T = \{t_1, t_2, \ldots, t_n, \ldots\}$. Take $A_n = \{t_1, \ldots, t_n\}$, then $r_0(T, \alpha, A_n) \leq m$.

Denote $H(n)$ the set of m-tuples $(v_1, \ldots, v_m)$ of elements of $M(T, E)$ such that $(J_{A_n}^*(v_1), \ldots, J_{A_n}^*(v_m))$ covers X, the set $H(n)$ is non-empty and a closed subset of $M(T, E)^m$. If $(J_{A_n}^*(v_1), \ldots, J_{A_n}^*(v_m))$ covers X, then $(J_{A_{n-1}}^*(v_1), \ldots, J_{A_{n-1}}^*(v_m))$ covers X too, hence $H(n) \subseteq H(n-1)$, the intersection $H = \cap_{n=0}^\infty H(n)$ is non-empty, so there is $\omega = (\omega_1, \ldots, \omega_m) \in H$. Obviously,

$$\cup_{i=1}^m J^*(\omega_i) = \lim_{n \to \infty} \cup_{i=1}^m J_{A_n}^*(\omega_i) = X.$$

□

Theorem 2.1. *(X, T) is equicontinuous if and only if for any finite open cover α, $r_0(T, \alpha, \cdot)$ is bounded.*

Proof. $\Rightarrow$. Let ε be a Lebesgue number of the cover α, since (X, T) is equicontinuous, there is $\eta > 0$, $\eta \leq \varepsilon$ such that if $d(x, y) < \eta$, then for any $t \in T$, one has $d(xt, yt) < \varepsilon$. Let $x_1, \ldots, x_k$ be such that the open balls $\{B(x_i, \eta) | 1 \leq i \leq K\}$ cover X. By equicontinuity, for any $t \in T$, we have $B(x_i, \eta)t \subset B(x_i t, \varepsilon)$.

Since ε is a Lebesgue number of α, there exists $U_{j(i,t)} \in \alpha$ such that $B(x_i t, \varepsilon) \subset U_{j(i,t)}$. Therefore, for any $A \in F(T)$ one has

$$B(x_i, \eta) \subset \bigcap_{t \in A} (\pi_t)^{-1} U_{j(i,t)}.$$

This means that $\bigcap_{t \in A} (\pi_t)^{-1} U_{j(i,t)}$ is a subcover of α_0^A. Hence $r_0(T, \alpha, A) \leq k$.

$\Leftarrow$. Suppose (X, T) is not equicontinuous, then there are $\varepsilon > 0$ and $x \in X$ such that for any $\eta > 0$, there exist $y \in X$ with $d(x, y) < \eta$ and $t \in T$ such that $d(xt, yt) > \varepsilon$. In the following we consider a finite cover α by open balls with radius $\frac{\varepsilon}{4}$. Let $\bar{\alpha} = (U_1, \ldots, U_k)$ be a cover made up of the closures of the elements of α. Suppose that $\eta_n > 0$ go to zero, choose $y_n \in X$ with $d(x, y_n) < \eta_n$, there is t_n such that $d(xt_n, yt_n) > \varepsilon$. Let $A_n = \{t_1, \ldots, t_n\}$ $(\forall n \in N)$, so $A_{n-1} \subset A_n$ and $r_0(T, \alpha, A_n)$ is bounded. By Lemma 2.1, there is a closed cover $(D_1, \ldots, D_m)$ of X, where

$$D_i = \bigcap_{j=1}^{\infty} (\pi_{t_j})^{-1} U_{l(i,j)}, \; U_{l(i,j)} \in \bar{\alpha}.$$

Without loss of generality, let $y_n \in D_i$, $1 \leq i \leq k$, then $x \in D_i$, hence $d(xt_i, yt_i) \leq \frac{\varepsilon}{2}$, which contradicts the assumption. □

3. Mixing, scattering

Definition 3.1. (X, T) is called scattering, if for any finite cover α by non-dense open sets, $r_0(T, \alpha, \cdot)$ is unbounded.

Remark 3.1. (X, T) is scattering if and only if for any non-trivial closed cover α, $r_0(T, \alpha, \cdot)$ is unbounded.

Theorem 3.1. *Weak-mixing implies scattering.*

Proof. For any non-trivial closed cover $\alpha = (W_1, \ldots, W_n)$ of X. Since (X, T) is weak-mixing, for any open sets U, V, there is $t \in T$ such that

$$U \cap \pi_t^{-1} U \neq \emptyset, \; U \cap \pi_t^{-1} V \neq \emptyset, \tag{3.1}$$

here we take U, V such that $U \cap V = \emptyset$ and U, V do not simultaneously belong to any element of α.

Now suppose $U \subset W_1$, $V \subset W_2$. By (3.1), there are $x_1, x_1' \in U$ and $t_1 \in T$ such that

$$x_1 t_1 \in U, \; x_1' t_1 \in V.$$

So one takes $A = \{t_1\}$, then $r_0(T, \alpha, A) \geq 2$. By the continuity of π, there exists a neighborhood $U_1 \subset U$ of x_1' such that $U_1 t_1 \subset V$ and there are $x_2, x_2' \in U_1$ and $t_2 \in T$, such that

$$x_2 t_1 \in V, \; x_2' t_1 \in V, \; x_2 t_2 \in V, \; x_2' t_2 \in U.$$

By the continuity of π, there exists a neighborhood U_2 of x_2', $x_3, x_3' \in U_2$ and $t_3 \in T$ such that

$$x_3t_1, x_3't_1 \in V,\ x_3t_2 \in U, x_3't_2 \in U,\ x_3t_3 \in U, x_3't_3 \in V.$$

Using similar arguments repeatedly, we can get an infinite sequence

$$\{x_1, x_2, \ldots, x_n, \ldots\} \text{ and } \{t_1, \ldots, t_n, \ldots\} \text{ satisfy}$$
$$x_n \in U,\ i = 1, 2, \ldots$$
$$x_1t_1 \in U,\ x_it_1 \in V,\ i = 2, 3, \ldots$$
$$x_2t_2 \in V,\ x_it_2 \in U,\ i = 3, 4, \ldots$$
$$x_3t_3 \in U, x_it_3 \in V,\ i = 4, 5, \ldots$$
$$\vdots$$

for any $N \geq 1$, choose $A_N = \{t_1, t_2, \ldots, t_{N-1}\}$, then $r_0(T, \alpha, A_N) \geq N$. □

Proposition 3.1. *Suppose T is a topological group, then scattering implies topological ergodicity.*

Proof. If (X, T) is not topologically ergodic, then there are non-empty open sets U, V, for any $t \in T$ we have $U \cap Vt = \emptyset$. Assume that $U \cap V = \emptyset$ (if $U \cap V \neq \emptyset$, take $U_1 \subset U$, $V_1 \subset V$ and $U_1 \cap V_1 = \emptyset$, we can use U_1, V_1 to replace U, V respectively). Now choose $\alpha = (U^c, V^c)$, for any $A = (t_1, \ldots, t_n) \in F(T)$, since $\{\bigcap_{t_i \in A}(\pi_{t_i})^{-1}U^c, \bigcap_{t_i \in A}(\pi_{t_i})^{-1}V^c\}$ is a subcover of α_0^A, we have $r_0(T, \alpha, A) \leq 2$. □

Lemma 3.1 ([4]). *Suppose that T is an abelian topological group, then (X, T) is not weak-mixing if and only if there exist two non-empty open sets U and V such that*

$$\text{(3.2)} \qquad \textit{either } U \cap Ut = \emptyset \textit{ or } U \cap Vt = \emptyset, \forall t \in T.$$

Proposition 3.2. *Suppose that T is an abelian topological group, for any standard cover α, there is $A \in F(T)$ such that $r_0(T, \alpha, A) > |A| + 1$, then (X, T) is weak-mixing, where $|A|$ denote the number of elements of A.*

Proof. Suppose (X, T) is not weak-mixing, by Lemma 3.1, there are non-empty open sets U and V satisfy (3.2) without loss of generality, suppose $U \cap V = \emptyset$. Let $\alpha' = \{U', V'\}$ is a standard cover of X such that $V^c \subset U'$ and $U^c \subset V'$. For any $t \in T$, either $U \cap Ut \neq \emptyset$, then $U \subset (Vt)^c \subset U't$; or $U \cap Ut = \emptyset$ then $U \subset V't$. This means for any $A = \{t_1, \ldots, t_n\} \in F(T)$, there is $W_{l(i,j)} = U'$ or V', we have

$$U \subset U' \bigcap_{1 \leq i,j \leq n} W_{l(i,j)} t_j^{-1} t_i.$$

For any $x \in X$, if there is $t_i \in A$ such that $xt_i \in U$, then $x \in (\pi_{t_i}^{-1})U = Ut_i^{-1}$, therefore,

$$x \in \bigcap_{1 \leq j \leq n} W_{l(i,j)} t_j^{-1}.$$

If for any $t_i \in A$, $xt_i \notin U$, then $xt_i \in V'$. in this case, $x \in \bigcap_{1\leq i\leq n} V't_i^{-1}$, hence $r_0(T, \alpha', A) \leq |A| + 1$, $\forall A \in F(T)$. □

4. Criterion for the scattering property

Definition 4.1. (X, T) and (Y, T) are weak disjoint, if $(X \times Y, T)$ is topologically ergodic.

If J is a closed invariant set of $X \times Y$, write

$$J^*(y) = \{x \in X | (x, y) \in J\},$$
$$J_*(x) = \{y \in Y | (x, y) \in J\}.$$

In [4], K. Petersen concluded that if T is an abelian topological group, then for any minimal system (Y, T), (X, T) and (Y, T) are weak disjoint. We prove the following theorem:

Proposition 4.1. *Suppose that T is an abelian topological group, if (X, T) is scattering, then for any minimal system (Y, T), (X, T) and (Y, T) are weak disjoint.*

Proof. Assume the assertion is not true, then there is a minimal system (Y, T) such that $(X \times Y, T)$ is not topologically ergodic. So there is a non-empty open invariant set $U \subset X \times Y$ with $\bar{U} \neq X$, put $J = \bar{U}$.

Suppose the projection of U to X is U_1, there is a transitive point $x_1 \in U_1$, $J_*(x_1)$ is closed and has non-empty interior in the minimal set Y, so there is $A \in F(T)$ such that $\bigcup_{t\in A} J_*(x_1)t = Y$. Let $K = \bigcup_{t\in A} J(Id \times \pi_t)$, then K is a closed invariant set. Obviously, K is closed. We need to prove it is invariant, since for any $(x, y) \in K$, there is $s \in A$ such that $(x, y) = (x, y_1 s)$ and $(x, y_1) \in J$. For any $t \in T$, $(xt, yt) = (xt, y_1 st) = (xt, y_1 ts)$, furthermore, $(xt, y_1 t) \in J$, hence, $(xt, yt) \in K$. Then $K(x_1) = Y$. Since x_1 is a transitive point, then for any $x \in X$, we have $K(x) = Y$ and $K = X \times Y$.

Fix $y_0 \in Y$, since $J \neq X \times Y$, and (Y, T) is minimal, we know $J^*(y_0) \neq X$. This implies there is a closed neighborhood U of y_0 such that $J^*(U) \neq X$. Put $F = J^*(U)$, we have $\bigcup_{t\in A} J^*(y_0)t = X$, assume $A = \{t_1, \ldots, t_n\}$, therefore $\alpha = (Ft_1, \ldots, Ft_n)$ is a non-trivial closed cover of X.

Take $M \in F(T)$ such that $\beta = \bigcup_{t\in M} Ut$ cover Y. Suppose $M = A$. There is a map $\chi : T \to A$ such that $y_0 t \in U\chi(t)$, for any $x \in X$, there is $t_i \in A$ such that $x \in J^*(y_0 t_i)$. For any $t_i \in A$ define a map $\chi_{t_i} : T \to A$ such that $\chi_{t_i}(t) = \chi(t_i t)$, for any $t \in T$ we have

$$xt \in J^*(y_0 t_i t) \subset J^*(U\chi(t_i t)) = F\chi(t_i t) = F\chi_{t_i}(t).$$

For any $B = (b_1, \ldots, b_k) \in F(T)$, we know

$$\{\bigcap_{b_i\in B} \pi_{b_i}^{-1} F\chi_{t_j}(b_i) | 1 \leq j \leq n\}$$

is a subcover of $\bigvee_{b_i\in B} \pi_{b_i}^{-1}\alpha$. Therefore, the complexity function of α is bounded. □

In the following we suppose T is a topological group satisfying the second axiom of countability. Let $\mathcal{K}(X)$ be the class of all non-empty and closed subset of X. for $x \in X$, we denote by $B_d(x, \varepsilon)$ the ball centered in x and radius ε in metric d. for any $A, B \in \mathcal{K}(X)$, write

$$\rho(A,B) = \max\{\sup\ d(a,B) : a \in A, \sup\ d(b,A) : b \in B\}$$

where $d(x,A) = \inf_{a\in A} d(x,a)$. It is well known that ρ is a metric on $\mathcal{K}(X)$ and $(\mathcal{K}(X),\rho)$ is a compact metric space if and only if (X,d) is compact metric space. For $A \in \mathcal{K}(X)$, Denote by $B_\rho(A,\varepsilon)$ the ball centered in A and radius ε in $\mathcal{K}(X)$.

Define a map

$$\begin{aligned} \bar{\pi} : \mathcal{K}(X) \times T &\to \mathcal{K}(X) \\ (A,t) &\mapsto At\ . \end{aligned}$$

Proposition 4.2. *$\bar{\pi}$ is a continuous action of T on $\mathcal{K}(X)$.*

Proof. It is easy to check $\bar{\pi}$ is a right action of T on $\mathcal{K}(X)$. We only need to proof $\bar{\pi}$ is continuous. For any $t \in T$ and $A \in \mathcal{K}(X)$, for any $a \in A$, by the continuity of $\pi : X \times T \to X$ at point (a,t), for any $\varepsilon > 0$, there exist a positive number η_a and a neighborhood U_a of t, such that for any $x \in B_d(a,\eta_a)$ and $s \in U_a$, we have

$$d(xs, at) < \varepsilon. \tag{4.1}$$

Since A is a compact set, there are finite elements $a_1, \ldots, a_N$ of A such that

$$A \subseteq \bigcup_{i=1}^{N} B_d(a_i, \frac{\eta_{a_i}}{2}).$$

Put $\eta = \inf_{1\le i\le N} \eta_{a_i}$, then we consider the neighborhood $B_\rho(A, \frac{\eta}{2})$ of A and $U = U_{a_1} \cap \cdots \cap U_{a_N}$ of t, for any $E \in B_\rho(A, \frac{\eta}{2})$ and any $s \in U$, we want to proof $\rho(Es, At) < 2\varepsilon$. For any $x \in E$, there is $a \in A$ such that $d(x,a) < \frac{\eta}{2}$, and there is $a_i, 1 \le i \le N$ such that $d(a, a_i) < \frac{\eta_{a_i}}{2}$, thus $d(x, a_i) < \eta_{a_i}$, by (4.1), $d(xs, a_i t) < \varepsilon$. Hence $d(xs, At) < \varepsilon$, Therefore $\sup_{xs\in Es} d(xs, At) < \varepsilon$.

Similarly, since $\rho(E,A) < \frac{\eta}{2}$, then for any $a \in A$ there is $x \in E$ such that $d(a,x) < \frac{\eta}{2}$, and there is a_i satisfies $d(a,a_i) < \frac{\eta_{a_i}}{2}$, thus $d(x,a_i) < \eta_{a_i}$, so $d(xs, a_i t) < \varepsilon$, then $d(a_i t, Es) < \varepsilon$. By $d(at, a_i t) < \varepsilon$, we have $d(at, Es) < 2\varepsilon$. Hence $\sup_{at\in At} d(xt, Es) < 2\varepsilon$. Therefore $\rho(Es, At) < \varepsilon$. ☐

Proposition 4.3. *If for any minimal system (Y,T), (X,T) and (Y,T) are weak disjoint, then the system (X,T) is scattering.*

Proof. Suppose the system (X,T) is not scattering, then there is a non-trivial closed cover $\alpha = \{U_1, \ldots, U_k\}$ of X with bounded complexity. Put $E = \{1, \ldots, k\}$. Suppose the countable dense subset of T is $Q = \{t_1, t_2, \ldots\}$, by

Lemma 2.1, there exist $\omega_1, \dots, \omega_m \in M(Q, E)$ such that

$$\bigcup_{j=1}^{m} J^*(\omega_j) = X.$$

For any $1 \le j \le m$, put $Z_j = J^*(\omega_j) \subset X$. Obviously, Z_j is closed. If $t \notin Q$, then there exist a net $\{t_i\} \in Q$ and $\{t_i\}$ converge to t. Hence there are a subnet $\{t_{N_i}\}$ of $\{t_i\}$ and $1 \le l(t) \le k$. For any $x \in Z_j$, we have $xt_{N_j} \in U_{l(t)}$, then $xt \in U_{l(t)}$. Define a map $\omega'_j : T \to E$ by

$$\omega'_j(t) = \begin{cases} \omega_j(t), & t \in Q \\ l(t), & x \notin Q. \end{cases}$$

Thus $Z_j \subseteq J^*(\omega'_j)$, for any $1 \le l \le m$. Therefore, there are $\omega'_1, \dots, \omega'_m \in M(T, E)$ such that

$$\bigcup_{j=1}^{m} J^*(\omega'_j) = X. \tag{4.2}$$

Call H the set of m-tuples $(Z_1, \dots, Z_m)$ satisfying:

(1) for any $1 \le i \le m$, Z_i is closed,

(2) $\bigcup_{i=1}^{m} Z_i = X$,

(3) there exist $\omega_1, \dots, \omega_m \in M(T, E)$, such that $Z_i \subseteq J^*(\omega_i)$, $\forall 1 \le i \le m$.

By (4.2), we know H is non-empty. In the following we want to prove H is a closed subset of $\mathcal{K}(X)^m$. Suppose the sequence $(Z_1^n, \dots, Z_m^n) \in H$, $n \in N$ converges to $(Z_1, \dots, Z_m)$. It is easy to see Z_i is closed for all $1 \le i \le m$. We say that $\bigcup_{i=1}^{m} Z_i = X$. If the assertion is not true, then there is $x \in X$, and for all $1 \le i \le m$, we have $x \notin Z_i$. Let $d(x, Z_i) = \varepsilon_i$, $\varepsilon_i > 0$, take $\varepsilon = \inf_{1 \le i \le m} \varepsilon_i$. There exists $k \in N$ such that $\rho(Z_i^k, Z_i) < \frac{\varepsilon_i}{2}$, for all $1 \le i \le m$. Hence $x \notin Z_i^k$ $(\forall 1 \le i \le m)$, which contradicts $\bigcup_{i=1}^{m} Z_i^k = X$.

Since $(Z_1^n, \dots, Z_m^n) \in H$, satisfies (3) for all $n \in N$, then there is $\omega_1^n, \dots, \omega_m^n \in M(T, E)$, such that $Z_i^n \subseteq J^*(\omega_i^n)$ $(\forall 1 \le i \le m)$. For any $t \in T$, there is a sequence $\omega_i^{n_k}(t)$ of $\omega_i^n(t)$ and $1 \le l(t) \le m$ such that $\omega_i^{n_k}(t) = l(t)$. Define $\omega_i \in M(T, E)$ by $\omega_i(t) = l(t)$, $\forall t \in T$, then $Z_i \subset J^*(\omega_i)\, 1 \le i \le m$ (this is because for any $x \in Z_i$, there is $x_n \in Z_i^n$ such that the sequence x_n converges to x, then there is a subsequence x_{n_k} of x_n and $1 \le l(t) \le m$ such that $x_{n_k}t \in U_{l(t)}$. Hence $xt \in U_{l(t)}$).

Define $S = \overline{\pi} \times \cdots \times \overline{\pi}$, we want to prove $S_t H \subset H$ for all $t \in T$, that is for all $(Z_1, \dots, Z_m) \in H$, we prove $(Z_1 t, \dots, Z_m t) \in H$. First, for all $1 \le i \le m$, $Z_i t$ is a closed set. By $\pi_t X = X$, we get $\bigcup_{i=1}^{m} Z_i t = X$. Since there is $\omega_i \in M(T, E)$ such that $Z_i \subset J^*(\omega_i)$, then for any $x \in Z_i$, we have $xts \in U_{\omega_i(ts)}$, $\forall s \in T$, thus $xt \in \pi_s^{-1} C_{\omega_i(ts)}$. Hence we define $\omega'_i(s) = \omega_i(ts)$, $\forall s \in T$, then $\omega'_i \in M(T, E)$, therefore $Z_i t \in J^*(\omega'_i)$. Thus one has $(Z_1 t, \dots, Z_m t) \in H$.

There is $(Z_1, \dots, Z_m) \in H$ such that the closure of the orbit of $(Z_1, \dots, Z_m)$ Σ is a minimal subset. Put $K_i = \{((Z_1, \dots, Z_m), x) \in \Sigma \times X | x \in Z_i\}$. One proves K_i is a closed invariant subset. Suppose $((Z_1^n, \dots, Z_m^n), x_n) \in K_i \forall n \in$

N, and converge to $((Z_1, \dots, Z_m), x)$. If $x \notin Z_i$, since Z_i is closed set, let $d(x, Z_i) = \varepsilon > 0$, then there is N_1 such that if $n > N_1$, we have $d(x_n, Z_i) > \frac{\varepsilon}{2}$, because there is N_2, when $n > N_2$, one has $Z_i^n \in B_\rho(Z_i, \frac{\varepsilon}{2})$. Hence it contradicts $x_n \in Z_i^n$. So $x \in Z_i$. In the following one proves K_i is a invariant set, assume $((Z_1, \dots, Z_m), x) \in K_i$, because of $x \in Z_i$, then $xt \in Z_i t$. Thus $((Z_1 t, \dots, Z_m t), xt) \in K_i$.

Since $\bigcup_{i=1}^m K_i = \Sigma \times X$, then there is $1 \le i \le m$ such that the interior of K_i is non-empty.

For any $(Z_1, \dots, Z_m) \in \Sigma$, there is ω_i such that $Z_i \subset J^*(\omega_i)$. Take $x \notin U_{\omega_i(e)}$, thus $K_i \neq \Sigma \times X$. Therefore $(\Sigma \times X, T)$ is not topologically ergodic. □

By Proposition 4.1 and Proposition 4.3, we have

Theorem 4.1. *Suppose T is an abelian topological group satisfying the second axiom of countability, then (X, T) is scattering if and only if for any minimal system (Y, T), (X, T) and (Y, T) are weak disjoint.*

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